1. Multiple Choice Questions Exactly one answer to each MC question is correct.

(a) Which of the following is **not** enough to conclude that f is constant?

- ○ $\Omega = \mathbb{D}$ and $f(x) = \pi$ for all $x \in (-1, 1)$.
- ○ $\Omega = \mathbb{C}$ and $f(\mathbb{C}) \subset \mathbb{D}$.
- ○ $\Omega = \mathbb{C}$ and $|f(z)| < \log(1 + |z|)$ for $|z| > 2024$.
- ○ $\Omega = \mathbb{C} \setminus \{0\}$ and $f(1/n) = 0$ for all $n \in \mathbb{N}$.

Solution: (D) Take the function $\sin(\pi/z)$.

(b) Which of the following statements is **not** correct?

- ○ If f has an essential singularity at $z_0 = 0$, then for every $w \in \mathbb{C}$, there exists a sequence (z_n) in the image of f such that $f(z_n) \to 0$ and $z_n \to w$.
- ○ $\int_{|z|=1} \frac{1}{z(\cos(z))^2}\, dz = 2\pi i$.
- ○ $f(z) = \frac{(\sin(z))^3}{z^3(z+5)}$ has simple poles at $z = 0$ and $z = -5$.
- ○ If f and g both have a zero at z_0 of order 5, then the function fg has a zero at z_0 of order 10.

Solution: (C) Zero is a removable singularity of f.

(c) Which $\Omega \subset \mathbb{C}$ is **not** biholomorphic to $\mathbb{C} \setminus [0, +\infty)$?

- ○ $\Omega = \{z \in \mathbb{C} : \Im(z) > \Re(z)^2\}$.
- ○ $\Omega = \mathbb{C}$.
- ○ $\Omega = \{z \in \mathbb{C} : \Re(z) \in (-1, 1)\}$.
- ○ $\Omega = \{z \in \mathbb{C} : |z + i| < 2\}$.

Solution: (B) This follows from the Riemann mapping theorem.

(d) Let $f(z) = z^6 + 7z^3 - 2z^2 + 3$. How many zeroes does f have inside the open unit disk $\mathbb{D}$?

- ○ f has exactly 3 zeroes in $\mathbb{D}$.
- ○ f has exactly 4 zeroes in $\mathbb{D}$.
- ○ f has exactly 5 zeroes in $\mathbb{D}$.
- ○ f has exactly 6 zeroes in $\mathbb{D}$.

Solution: (A) By Rouché's theorem: On the unit circle, $|7z^3| = 7 > 6 \geq |z^6 - 2z^2 + 3|$.

(e) In which open set does the following map define a holomorphic function?

$$f(z) = \sum_{n \geq 1} \Big(2i\cos(\pi n)\Big)^{4n} z^{-2n}.$$

- ◯ $\{z \in \mathbb{C} : 4 < |z|\}$.
- ◯ $\{z \in \mathbb{C} : 2 < |z|\}$.
- ◯ $\{z \in \mathbb{C} : 0 < |z| < 2\}$.
- ◯ $\{z \in \mathbb{C} : 0 < |z| < \sqrt{2}\}$.

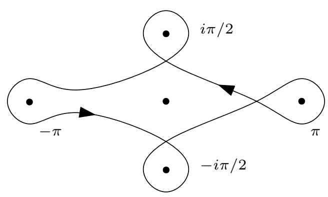
Solution: (A) The series converges on the annulus $r < |z| < R$, where $r = \limsup_n |a_{-n}|^{1/n}$ and $1/R = \limsup_n |a_n|^{1/n}$. Hence, $R = +\infty$ and $r = \limsup_n |(2i\cos(\pi n))^{4n}|^{1/2n} =$ 4.

(f) Which formula does **not** hold for every non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$?

- ◯ $\int_{|z|=1} \sin(f(z)^2)\, dz = 0$.
- ◯ $\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{f(z)z^2}\, dz = -\frac{f'(0)}{f(0)^2}$.
- ◯ $\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{f(z)-f(0)}\, dz = \mathrm{res}_{z=0} \frac{1}{f}$.
- ◯ $\int_{|z|=1} \frac{f'(z)}{f(z)}\, dz = 0$.

Solution: (C) Because $\mathrm{res}_{z=0} \frac{1}{f} = 0$, but the integral may differ from zero since $\frac{1}{f(z)-f(0)}$ has at least one pole.

2. Open Question Consider the meromorphic function $f(z) = \frac{z^2}{\sin(z)\cos(iz)}$, and the curve γ as in the following figure:



(a) Find **all** zeroes and poles of f and their order.

Solution: For every $k \in \mathbb{Z}$, $\sin(z)$ has a zero of order one at $z = k\pi$ and $\cos(iz)$ has a zero of order one at $z = i(\pi/2 + k\pi)$. On the other hand, z^2 has a zero of order two in $z = 0$. Hence, we conclude that f has a unique zero of order one in $z = 0$, and that all its poles are also of order one and are situated in $\{k\pi : k \in \mathbb{Z} \setminus \{0\}\} \cup \{i(\pi/2 + k\pi) : k \in \mathbb{Z}\}$.

(b) Compute the integral $\int_\gamma f\, dz$.

Solution: We compute the residues of f in $z = -\pi, \pi, -i\pi/2, i\pi/2$.

$$\text{res}_{z=\pi} f = \lim_{z\to\pi} \frac{(z-\pi)z^2}{\sin(z)\cos(iz)} = \frac{\pi^2}{\cos(i\pi)} \frac{1}{\sin'(\pi)} = -\frac{\pi^2}{\cos(i\pi)}, \tag{1}$$

$$\text{res}_{z=-\pi} f = \lim_{z\to-\pi} \frac{(z+\pi)z^2}{\sin(z)\cos(iz)} = \frac{\pi^2}{\cos(-i\pi)} \frac{1}{\sin'(-\pi)} = -\frac{\pi^2}{\cos(i\pi)}, \tag{2}$$

$$\text{res}_{z=i\pi/2} f = \lim_{z\to i\pi/2} \frac{(z-i\pi/2)z^2}{\sin(z)\cos(iz)} = \frac{-\pi^2}{4\sin(i\pi/2)} \frac{1}{\frac{d}{dz}\cos(iz)|_{z=i\pi/2}} = \frac{i\pi^2}{4\sin(i\pi/2)} \tag{3}$$

$$\text{res}_{z=-i\pi/2} f = \lim_{z\to -i\pi/2} \frac{(z+i\pi/2)z^2}{\sin(z)\cos(iz)} = \frac{-\pi^2}{4\sin(-i\pi/2)} \frac{1}{\frac{d}{dz}\cos(iz)|_{z=-i\pi/2}} = \frac{i\pi^2}{4\sin(i\pi/2)}. \tag{4}$$

The winding numbers associated to this 4 poles (following the above order) are: -1,1,-1,-1. Applying the Residue Theorem we get that

$$\int_\gamma f\, dz = \frac{\pi^3}{\sin(i\pi/2)}.$$

3. Open Question Show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos(\theta)}\, d\theta = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{(z+z^{-1})/2}}{z}\, dz = \sum_{n=0}^{+\infty} \left(\frac{1}{2^n n!}\right)^2.$$

Hint: take advantage of the series representation of the exponential: $e^w = \sum_{n=0}^{+\infty} \frac{w^n}{n!}$.

Solution: Taking advantage of the uniformly converging power series representation of the exponential $e^w = \sum_{n\geq 0} \frac{w^n}{n!}$ we rewrite the above integral as follows:

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{e^{(z+z^{-1})/2}}{z}\, dz = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{z/2}}{z} \sum_{n\geq 0} \frac{1}{2^n z^n n!}\, dz = \frac{1}{2\pi i}\int_{|z|=1} \sum_{n\geq 0} \frac{e^{z/2}}{2^n z^{n+1} n!}\, dz$$
$$= \sum_{n\geq 0} \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{z/2}}{2^n z^{n+1} n!}\, dz,$$

where in the last identity we exchanged integration with summation in virtue of Fubini's Theorem since the internal series converges uniformly in a neighbourhood of the unit circle. Fix $n \geq 0$. Then,

$$\frac{e^{z/2}}{2^n z^{n+1} n!} = \sum_{k \geq 0} \frac{z^{k-n-1}}{2^{k+n} n! k!},$$

and by definition its residue in zero is the coefficient of exponent equal to -1, that is when $k - n - 1 = -1 \Leftrightarrow k = n$:

$$\operatorname{res}_{z=0} \frac{e^{z/2}}{2^n z^{n+1} n!} = \frac{1}{2^{2n} n! n!} = \left(\frac{1}{2^n n!}\right)^2.$$

The conclusion follows by applying the Residue Theorem to each $n \geq 0$.

4. Open Question Let f be a holomorphic map of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself and such that $f(0) = 0$. Show that

$$|f(z) + f(-z)| \leq 2|z|^2,$$

for all $z \in \mathbb{D}$.

Recall Schwarz Lemma: Let $f : \mathbb{D} \to \mathbb{D}$ analytic, $f(0) = 0$. Then, $|f'(0)| \leq 1$, $|f(z)| \leq |z|$, $\forall z \in \mathbb{D}$.

Solution: By Schwarz's Lemma $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Let $g(z) := \frac{f(z)+f(-z)}{z}$. Developing f as $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$ for coefficients $(a_n) \subset \mathbb{C}$, we deduce that $a_0 = 0$ since by assumption $0 = f(0) = a_0$. Then

$$g(z) = a_2 z + a_4 z^3 + a_6 z^5 + \dots,$$

defines an analytic function also vanishing in zero. The elementary inequality

$$|g(z)| \leq \frac{|f(z)| + |f(-z)|}{2|z|} \leq 1,$$

implies that g satisfies the assumptions of Schwarz Lemma, and hence

$$|g(z)| \leq |z| \Rightarrow |f(z) + f(-z)| \leq 2|z|^2,$$

as wished.

5. Open Question Let $f : \mathbb{D} \to \hat{\mathbb{C}}$ be **any** function, and suppose that $g = f^2$ and $h = f^3$ are meromorphic functions. Let Z_f, Z_g, Z_h and P_f, P_g, P_h the set of zeros and poles of f, g, h respectively without taking into account their multiplicity.

(a) Show that f is meromorphic in $\mathbb{D}$.

Solution: The function f is equal to $\frac{h}{g}$. Being the quotient of two meromorphic functions, it is itself meromorphic.

(b) Determine all poles and zeros of f and their orders in terms of the poles and zeros of h and g. Show that $Z_f = Z_g = Z_h$ and $P_f = P_g = P_h$. What can you say about the orders of the poles and zeros?

Solution: By definition, g and h share the same zeros and poles. Let z_0 be a common zero. Then locally there exist two holomorphic functions h_1 and g_1 non-vanishing in z_0 and such that

$$h(z) = (z - z_0)^k h_1(z), \quad g(z) = (z - z_0)^m g_1(z),$$

where k and m are the orders of the zeros for h, g respectively. From $h^2(z) = f^6(z) = g^3(z)$ we deduce that $2k = 3m$, so in particular $k > m$, implying that $\mathrm{ord}_{z_0} h > \mathrm{ord}_{z_0} g$, and from

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} (z - z_0)^{k-m} \frac{h_1(z)}{g_1(z)} = 0,$$

we get that f shares the same zeros with h and g and can be extended to an analytic function in a neighbourhood of z_0. The same argument works also for the poles: let w_0 be a common pole of h and g. Then, locally there exists h_2 and g_2 holomorphic and such that

$$\frac{1}{h(z)} = (z - w_0)^\ell h_2(z), \quad \frac{1}{g(z)} = (z - w_0)^s h_1(z).$$

where ℓ and s are the orders of the zeros for h, g respectively. From $f^{-6} = g^{-3} = h^{-2}$ we get that $2\ell = 3s$, so in particular $\ell > s$ and

$$\lim_{z\to w_0} \frac{1}{f(z)} = \lim_{z\to w_0} (z - w_0)^{\ell - s} \frac{h_2(z)}{g_2(z)} = 0$$

showing that w_0 is also a pole of f.

(c) Show that if h is holomorphic in $\mathbb{D}$ the same holds for f.

Solution: By the previous exercise, if h is holomorphic, then the same holds for g, and f can be extended to an holomorphic function near all zeros of h and g.