1. Multiple Choice Questions Exactly one answer to each MC question is correct.

(a) Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \mapsto \mathbb{C}$ a holomorphic function. Which of the following is **NOT** enough to conclude that f is constant?

- ◯ $\Omega = \mathbb{C}$ and $|f(ix)| \leq 1$ for all $x \in \mathbb{R}$.
- ◯ $\Omega = \mathbb{C}$, and $f(\mathbb{C}) \cap D_1(0) = \emptyset$.
- ◯ $\Omega = D_1(0)$ and $\Re(f)$ is constant.
- ◯ $\Omega = \mathbb{C}$ and $|f(z)| < |z|^{1/2}$ for $|z| > 2024$.

(b) Which of the following statements is correct?

- ◯ If f and g both have a pole at z_0 with non zero residues than the function fg has a pole at z_0 with non zero residue.
- ◯ If f and g both have a pole at z_0 with non zero residues than the function $f + g$ has a pole at z_0 with non zero residue.
- ◯ $f(z) = \frac{z^2+2023z}{\sin(z)}$ is bounded in a neighbourhood of 0.
- ◯ $f(z) = \frac{z^5+1}{z(z+1)^2}$ has simple pole at $z = 0$ and a double pole at $z = -1$.

(c) Which formula holds true for ALL holomorphic functions $f\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ and ALL simple closed curves γ?

- ◯ $\int_\gamma \overline{f(z)}\, dz = 0$
- ◯ $\int_\gamma \frac{f(z)}{z}\, dz = 2\pi i f(0)$
- ◯ $\int_\gamma \frac{f'}{f}\, dz = 0$
- ◯ $\int_\gamma f''(z)\, dz = 2\pi i f'(0)$

(d) Let $f, g, h\colon \mathbb{C} \to \mathbb{C}$ be three holomorphic functions such that

$$f(0) = g(0) = h(0) = 0.$$

Which of the following is correct?

- ◯ $\mathrm{ord}_0(fg + h) \geq \max\{\mathrm{ord}_0(f) + \mathrm{ord}_0(g), \mathrm{ord}_0(h)\}$
- ◯ $\mathrm{ord}_0(f^2gh) = 2\,\mathrm{ord}_0(f) + \mathrm{ord}_0(g) + \mathrm{ord}_0(h)$
- ◯ $\mathrm{ord}_0(f(1 + gh)) = \mathrm{ord}_0(f)(1 + \mathrm{ord}_0(g) + \mathrm{ord}_0(h))$
- ◯ $\mathrm{ord}_0(fgh) = \mathrm{ord}_0(f)\,\mathrm{ord}_0(g)\,\mathrm{ord}_0(h)$

(e) Let $f(z) = \frac{e^z}{z-2}$. Which of the following equalities is **NOT** correct? All circles are positively oriented.

- ○ $\int_{|z|=1} f(z)dz = 0$
- ○ $\int_{|z|=1} \frac{f'(z)}{f(z)}dz = 2\pi i$
- ○ $\int_{|z|=3} f(z)dz = 2\pi i e^2$
- ○ $\int_{|z|=3} \frac{f(z)}{z-2}dz = 2\pi i e^2$

(f) Which of the following functions is **NOT** holomorphic?

- ○ $f(z) = z^{2024} + 3$
- ○ $f(x+iy) = (\cos(x) + isin(x))e^{-y}$
- ○ $f(x+iy) = x^2 - y^2 + x + i(y + 2xy)$
- ○ $f(x+iy) = x - iy + 2$

2. Open question

(a) Let $\alpha \in \mathbb{C}$ be a fixed non-zero complex number. Construct a non-constant holomorphic function $f\colon \mathbb{C} \mapsto \mathbb{C}$ such that $f(z+\alpha) = f(z)$ for all $z \in \mathbb{C}$.

Hint: first consider the case $\alpha = 2\pi i$.

(b) Show that if a holomorphic function $f : \mathbb{C} \mapsto \mathbb{C}$ satisfies the relations $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$, then f is constant.

3. Open question If f is holomorphic on $0 < |z| < 2$ and satisfies $f(\frac{1}{n}) = n^2$ and $f(\frac{-1}{n}) = n^3$ for all positive integers n, show that f has an essential singularity at 0.

Hint: show that f can have neither a removable singularity nor a pole at 0.

4. Open question Consider the function

$$f(z) = \frac{\sin z}{z(z-1)^2}.$$

(a) Find the zeros of f and their order.

(b) Find the poles of f and their order.

(c) Compute the integral $\int_\gamma f\, dz$, when γ is the circle of radius 2 centered in 0 positively oriented.

5. Open question Show that

$$\int_0^\pi e^{\cos\theta} \cos(\sin\theta) d\theta = \pi.$$