1. Multiple Choice Questions

(a) Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \mapsto \mathbb{C}$ a holomorphic function. Which of the following is NOT enough to conclude that f is constant

- ◯ $\Omega = \mathbb{C}$ and $|f(ix)| \leq 1$ for all $x \in \mathbb{R}$.
- ◯ $\Omega = \mathbb{C}$, and $f(\mathbb{C}) \cap D_1(0) = \emptyset$
- ◯ $\Omega = D_1(0)$ and $\Re(f)$ is constant.
- ◯ $\Omega = \mathbb{C}$ and $|f(z)| < |z|^{1/2}$ for $|z| > 2024$

Solution: (a) Take the function $\sin(iz)$. (b) is correct since $f(\mathbb{C}) \cap D_1(0) = \emptyset$ imply that $0 \notin f(\mathbb{C})$ and $1/f$ is holomorphic and bounded. By Liouville's theorem $1/f$ hence f is constant. (c) is correct since the disc is connected. (see also Exercise 1.5(c)) (d) Follows using Cauchy inequalities which in this case gives for the power series coefficients of f that $|a_n| < r^{1/2}/r^n$ for any $r > 2024$. Letting r go to ∞ shows that $a_n = 0$ for $n \geq 1$.

(b) Which of the following statements is correct?

- ◯ If f and g both have a pole at z_0 with non zero residues than the function fg has a pole at z_0 with non zero residue.
- ◯ If f and g both have a pole at z_0 with non zero residues than the function $f + g$ has a pole at z_0 with non zero residue.
- ◯ $f(z) = \frac{z^2+2023z}{\sin(z)}$ is bounded in a neighbourhood of 0.
- ◯ $f(z) = \frac{z^5+1}{z(z+1)^2}$ has simple pole at $z = 0$ and a double pole at $z = -1$.

Solution: (a) and (b) are false: take $f = g = 1/z$ and $f = -g$ respectively. (d) is false since $z^5 + 1$ also has a simple zero at $z = -1$. (c) is correct since the singularity at $z = 0$ is removable.

(c) Which formula holds true for ALL holomorphic functions $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ and ALL simple closed curves γ?

- ◯ $\int_\gamma \overline{f(z)}\, dz = 0$
- ◯ $\int_\gamma \frac{f'}{f}\, dz = 0$
- ◯ $\int_\gamma \frac{f(z)}{z}\, dz = 2\pi i f(0)$
- ◯ $\int_\gamma f''(z)\, dz = 2\pi i f'(0)$

Solution: The correct answer is (b) since f'/f is holomorphic. Counterexamples: (a) $f(z) = z$, $\gamma = \partial\mathbb{D}$, (c) when 0 not in the interior of γ and $f(0) \neq 0$, (d) f'' is holomorphic and hence the RHS is always equal to zero.

(d) Let $f, g, h : \mathbb{C} \to \mathbb{C}$ three holomorphic functions such that $f(0) = g(0) = h(0) = 0$. Then

- ◯ $\text{ord}_0(fg + h) \geq \max\{\text{ord}_0(f) + \text{ord}_0(g), \text{ord}_0(h)\}$.
- ◯ $\text{ord}_0(f^2gh) = 2\,\text{ord}_0(f) + \text{ord}_0(g) + \text{ord}_0(h)$
- ◯ $\text{ord}_0(f(1 + gh)) = \text{ord}_0(f)(1 + \text{ord}_0(g) + \text{ord}_0(h))$
- ◯ $\text{ord}_0(fgh) = \text{ord}_0(f)\,\text{ord}_0(g)\,\text{ord}_0(h)$

Solution: The correct solution is (b). Counterexamples: (a) f and g identically zero, $h = z$, (c) and (d) $f = g = h = z$

(e) Let $f(z) = \frac{e^z}{z-2}$. Which of the following statements is NOT correct. All circles are positively oriented.

- ◯ $\int_{|z|=1} f(z)dz = 0$.
- ◯ $\int_{|z|=3} f(z)dz = 2\pi i e^2$.
- ◯ $\int_{|z|=1} \frac{f'(z)}{f(z)}dz = 2\pi i$.
- ◯ $\int_{|z|=3} \frac{f(z)}{z-2}dz = 2\pi i e^2$.

Solution: The correct solution is (c). It should be 0 since f has no zeroes or poles inside $|z| = 1$

(f) Which of the following functions is NOT holomorphic?

- ◯ $f(z) = z^{2024} + 3$.
- ◯ $f(x + iy) = (\cos(x) + isin(x))e^{-y}$.
- ◯ $f(x + iy) = x^2 - y^2 + x + i(y + 2xy)$
- ◯ $f(x + iy) = x - iy + 2$.

Solution: The correct solution is (d), since $x - iy + 2 = \bar{z} + 2$.

2. Open question

(a) Let $\alpha \in \mathbb{C}$ be a fixed non-zero complex number. Construct a non-constant holomorphic function $f : \mathbb{C} \mapsto \mathbb{C}$ such that $f(z + \alpha) = f(z)$ for all $z \in \mathbb{C}$.

Hint: consider first the case $\alpha = 2\pi i$.

Solution: Consider the function $f(z) = e^z$. We check if $f(z + 2\pi i) = f(z)$:

$$f(z + 2\pi i) = e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z.$$

Since $e^{2\pi i} = 1$, it follows that $f(z + 2\pi i) = f(z)$. This satisfies the periodicity condition for $\alpha = 2\pi i$.

For any $\alpha \in \mathbb{C} \setminus \{0\}$, we can take the function:

$$f(z) = e^{\frac{2\pi i}{\alpha} z}.$$

Then,

$$f(z + \alpha) = e^{\frac{2\pi i}{\alpha}(z+\alpha)} = e^{\frac{2\pi i}{\alpha} z} \cdot e^{2\pi i} = e^{\frac{2\pi i}{\alpha} z} = f(z).$$

(b) Show that if a holomorphic function $f : \mathbb{C} \mapsto \mathbb{C}$ satisfies the relations $f(z + 1) = f(z)$ and $f(z + i) = f(z)$ for all $z \in \mathbb{C}$, then f is constant.

Solution: Let $Q = [-1, 1]^2$ be the closed square centered at 0 with side length 1. Since Q is compact, we have:

$$\sup_{z \in Q} |f(z)| =: B < \infty.$$

For any $z \in \mathbb{C}$, there exist integers $n, k \in \mathbb{Z}$ such that $z + n + ik \in Q$. Using the periodicity conditions $f(z + 1) = f(z)$ and $f(z + i) = f(z)$ repeatedly, we get:

$$f(z) = f(z + n + ik) \quad \Rightarrow \quad |f(z)| \leq B.$$

Hence, f is bounded on $\mathbb{C}$. By Liouville's theorem, which states that a bounded entire function must be constant, we conclude that f is constant.

3. Open question If f is holomorphic on $0 < |z| < 2$ and satisfies $f(\frac{1}{n}) = n^2$ and $f(\frac{-1}{n}) = n^3$ for all positive integers n, show that f has an essential singularity at 0.

Hint: show that f can have neither a removable singularity nor a pole at 0.

Solution: Since $f(1/n) = n^2$, f is unbounded near 0. hence by Riemann's theorem on removable singularities 0 is not removable. Assume on the contrary f has a pole of order $k \geq 1$ at 0. Then in a neighnourhood of zero, there is a holomorphic function g such that $g(0) \neq 0$ and

$$f(z) = z^{-k} g(z).$$

Let $z = 1/n$ then $n^2 = g(1/n)n^k$. Now letting n go to infinity gives $k = 2$. On the other hand using $z = -1/n$ gives $n^3 = g(-1/n)n^k$ and letting n go to infinity gives $k = 3$. Since the order of a pole is unique, this is a contradiction. Hence f does not have a pole either.

4. Open question Consider the function

$$f(z) = \frac{\sin z}{z(z-1)^2}.$$

(a) Find the zeros of f and their order.

Solution: The zeros of f are the zeros of the numerator $\sin z$ that are not canceled by zeros of the denominator. The zeros of $\sin z$ occur at

$$z = n\pi, \quad n \in \mathbb{Z}.$$

We need to consider these zeros except at points where the denominator also vanishes (which could potentially cancel the zero or create a singularity). The denominator $z(z-1)^2$ has zeros at $z = 0$ and $z = 1$.

- At $z = 0$:

- Numerator: $\sin 0 = 0$.
- Denominator: $z = 0$, so the denominator is zero.
- Therefore, both numerator and denominator vanish at $z = 0$.

- At $z = 1$:

- Numerator: $\sin 1 \neq 0$.
- Denominator: $(z-1)^2 = 0$, so the denominator has a zero of order 2.

To determine the order of zeros of f at $z = n\pi$, with $n \neq 0$ we note that since $\sin z$ has simple zeros at $z = n\pi$ $(n \neq 0)$, and the denominator does not vanish at these points, f has zeros of order 1 at these points.

At $z = 0$, both the numerator and the denominator vanish. To determine the behavior of $f(z)$ near $z = 0$, we can consider the limit:

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z}{z(z-1)^2} = 1.$$

To see this, one can use the Taylor series expansion $\sin z = z - \frac{z^3}{6} + \cdots$, near $z = 0$ to get

$$\frac{\sin z}{z(z-1)^2} = \frac{1}{(z-1)^2}(1 - z^2/6 + \cdots)$$

which goes to 1 as z goes to 0. Therefore, $f(z)$ has an analytic extention to $z = 0$. Hence, $z = 0$ is a removable singularity, not a zero or pole.

The zeros of f are at $z = n\pi$ for $n \in \mathbb{Z}$ with $z \neq 0$, and each zero is of order 1.

(b) Find the poles of f and their order.

Solution: The poles of f occur at the zeros of the denominator that are not canceled by zeros of the numerator.

From the denominator $z(z-1)^2$, we have:

- At $z = 0$: As before both the numerator and the denominator vanish at $z = 0$, and as shown earlier, $f(z)$ has a removable singularity at $z = 0$.

- At $z = 1$:

- Numerator: $\sin 1 \neq 0$.
- Denominator: $(z-1)^2 = 0$.
- The denominator has a zero of order 2, and the numerator does not vanish.

Therefore, $z = 1$ is a pole of order 2.

(c) Compute the integral

$$\int_\gamma f\, dz,$$

when γ is the circle of radius 2 centered in 0 positively oriented.

Solution: We will use the Residue Theorem, which states that if γ is a positively oriented simple closed contour enclosing a finite number of isolated singularities a_k of $f(z)$, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum \operatorname{Res}_{z=a_k} f(z),$$

where the sum is over all singularities a_k inside γ.

First, we identify the singularities inside the circle $|z| = 2$.

- The singularities of $f(z)$ are at $z = 1$ (pole of order 2) and $z = 0$ (removable singularity). - As established earlier, $z = 0$ is a removable singularity, so it does not contribute to the integral. - Therefore, the only pole inside γ is at $z = 1$.

We need to compute $\operatorname{Res}_{z=1} f(z)$.

Since the pole at $z = 1$ is of order 2, the residue is given by

$$\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right].$$

Compute $(z-1)^2 f(z)$:

$$\Phi(z) = (z-1)^2 f(z) = (z-1)^2 \cdot \frac{\sin z}{z(z-1)^2} = \frac{\sin z}{z}.$$

Now compute the derivative $\Phi'(z)$:

$$\Phi'(z) = \frac{d}{dz}\left(\frac{\sin z}{z}\right) = \frac{z\cos z - \sin z}{z^2}.$$

Then,

$$\operatorname{Res}_{z=1} f(z) = \Phi'(1) = \frac{1\cdot\cos 1 - \sin 1}{1^2} = \cos 1 - \sin 1.$$

Therefore, the integral is

$$\int_\gamma f(z)\,dz = 2\pi i \times \operatorname{Res}_{z=1} f(z) = 2\pi i(\cos 1 - \sin 1).$$

5. Open question Show that

$$\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \pi.$$

Solution: Let $f(z) = e^z/z$ and $\gamma(\theta) = e^{i\theta},\ \theta \in [0, 2\pi]$. Then

$$\begin{aligned}\int_\gamma \frac{e^z}{z} &= i\int_0^{2\pi} e^{\cos\theta + i\sin\theta} d\theta \\ &= i\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta) + i\sin(\sin\theta)]d\theta\end{aligned}$$

On the other hand, using the reside formula yields $\int_\gamma \frac{e^z}{z} = 2\pi i$ since $f(z)$ has only one simple pole inside the disc, specifically at $z = 0$ with residue equal to $e^0 = 1$. Hence by taking the imaginary part of both sides we obtain $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi$. Using the symmetry of the function we conclude $\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \pi$.