Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

3.1. MC Questions

(a) For which values of $z \in \mathbb{C}$ is $\cos(z)$ a real number?

A) Only for $z = x$ with $x \in \mathbb{R}$.

B) Only for $z = x + iy$ with $y = 0$ and $x = n\pi$ for some $n \in \mathbb{Z}$.

C) Only for $z = iy$ with $y \in \mathbb{R}$.

D) Only for $z = x + iy$ with $y = 0$ or $x = n\pi$ for some $n \in \mathbb{Z}$.

Solution: We use the equality

$$\cos(z) = \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y),$$

from which we deduce that the values of z for which $\cos(z)$ is real are:

$$z = x \text{ with } x \in \mathbb{R}, \quad \text{or} \quad z = n\pi + iy \text{ with } y \in \mathbb{R} \text{ and } n \in \mathbb{Z}.$$

(b) Let $a, b \in \mathbb{C}$ with $a \neq b$. Define $\nu = \frac{a-b}{|a-b|}$ and let γ be a parametrization of the line segment from a to b. Moreover, let $f\colon \mathbb{C} \to \mathbb{C}$ be a continuous function. Which of the following equalities holds?

A) $\int_\gamma f(z)dz = \int_0^{|b-a|} f(a + t\nu)dt$

B) $\int_\gamma f(z)dz = (b-a)\int_0^1 f(a(1-t) + bt)dt$

C) $\int_\gamma f(z)dz = \int_0^1 f(a(1-t) + bt)dt$

D) $\int_\gamma f(z)dz = \int_{|a|}^{|b|} f(t\nu)dt$

Solution: We can parametrize the segment from a to b as $\gamma(t) = a + t(b-a)$, $t \in [0, 1]$. Then, the differential dz is given by $dz = \gamma'(t)dt = (b-a)dt$.

Now, we compute the integral over γ:

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 f(a + t(b-a))(b-a)dt$$

Thus, the correct equality is $\int_\gamma f(z)dz = (b-a)\int_0^1 f(a(1-t) + bt)dt$

3.2. Complex line integrals

(a) Compute $\int_\gamma \cos(\Im(z))\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Solution: By parametrizing the unit circle using $\gamma(t) = e^{it}$, where $t \in [0, 2\pi]$, we have:

$$z = \gamma(t) = e^{it}, \quad dz = \gamma'(t)\,dt = ie^{it}\,dt = (-\sin(t) + i\cos(t))\,dt.$$

Also, the imaginary part of $z = e^{it}$ is $\Im(z) = \sin(t)$. Thus, the integral becomes:

$$\int_\gamma \cos(\Im(z))\,dz = \int_0^{2\pi} \cos(\sin(t))(-\sin(t) + i\cos(t))\,dt.$$

We can split this into the real and imaginary parts:

$$\int_0^{2\pi} \cos(\sin(t))(-\sin(t)+i\cos(t))\,dt = -\int_0^{2\pi} \cos(\sin(t))\sin(t)\,dt + i\int_0^{2\pi} \cos(\sin(t))\cos(t)\,dt.$$

Now, for the imaginary part:

$$i\int_0^{2\pi} \cos(\sin(t))\cos(t)\,dt = i\,[\sin(\sin(t))]_0^{2\pi} = i(\sin(\sin(2\pi)) - \sin(\sin(0))) = i(0-0) = 0.$$

For the real part, we use the symmetry of the integrand. The function $\cos(\sin(t))\sin(t)$ is odd on $[0, 2\pi]$ because:

$$\cos(\sin(\pi + t))\sin(\pi + t) = -\cos(\sin(t))\sin(t).$$

Thus, the real part integrates to zero:

$$-\int_0^{2\pi} \cos(\sin(t))\sin(t)\,dt = 0.$$

Therefore, the entire integral evaluates to:

$$\int_\gamma \cos(\Im(z))\,dz = 0.$$

(b) Compute $\int_\gamma (\bar{z})^k\,dz$ for any $k \in \mathbb{Z}$ and when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Solution: Notice that on the unit circle $\bar{z}$ is equal to z^{-1} because $z\bar{z} = |z|^2 = 1$. Hence, when $k \neq 1$, we have that

$$\int_\gamma (\bar{z})^k\,dz = \int_\gamma z^{-k}\,dz = 0,$$

since then z^{-k} admits the primitive $z^{-k+1}/(-k+1)$, and γ is a closed curve. When $k = 1$ then we have that

$$\int_\gamma \bar{z}\,dz = \int_\gamma z^{-1}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i.$$

(c) Compute $\int_\gamma (z^{2024} + \pi z^{13} + 1)\, dz$, when γ is the spiral $\{1 + te^{i\pi t} : t \in [0, 1]\}$.

Solution: The argument is a polynomial expression, and therefore we can easily find a primitive

$$F(z) = \frac{z^{2025}}{2025} + \frac{\pi z^{14}}{14} + z.$$

Hence, the integral over γ depends only on its end points:

$$\begin{aligned}\int_\gamma (z^{2024} + \pi z^{13} + 1)\, dz &= \int_\gamma F'\, dz = F(\gamma(1)) - F(\gamma(0)) = F(0) - F(1) \\ &= -F(1) = -\frac{1}{2025} - \frac{\pi}{14} - 1.\end{aligned}$$

3.3. Show that $exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. What's the image of $\cos\colon \mathbb{C} \to \mathbb{C}$?

Solution: Using the polar form $z = re^{i\theta}$, it is easy to see that $\mathbb{C} \setminus \{0\} \subseteq \exp(\mathbb{C})$. For the converse, we must show that there is no $\alpha \in \mathbb{C}$ such that $e^\alpha = 0$. We know:

$$|e^\alpha| = |e^{a+ib}| = |e^a| \cdot |e^{ib}| = |e^a| \neq 0,$$

so $e^\alpha \neq 0$.

We claim that $\cos(\mathbb{C}) = \mathbb{C}$. For any $\alpha \in \mathbb{C}$, the equation

$$\alpha = \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

is quadratic in $w = e^{iz}$. It is easy to see that this equation has a nonzero solution. Therefore, $\cos(\mathbb{C}) = \mathbb{C}$.

3.4. Let $MNPQ$ be a rectangle on the complex plane whose sides are parallel to the x-axis and y-axis. It is divided into smaller rectangles whose sides are parallel to the axes as well. It is known that each smaller rectangle has at least one side (horizontal or vertical) whose length belongs to the integers. Prove that $MNPQ$ also has at least one side of integer length.

Hint: $\int_a^b e^{2\pi i x} dx = 0 \iff b - a \in \mathbb{Z}$

Solution: Let us define a function F for any general rectangle on the complex plane whose sides are parallel to the x- and y-axes. For a rectangle with opposite corners at (x_1, y_1) and (x_2, y_2), define F as the product of two integrals:

$$F(x_1, x_2, y_1, y_2) = \left(\int_{x_1}^{x_2} e^{2\pi i x}\, dx \right) \left(\int_{y_1}^{y_2} e^{2\pi i y}\, dy \right).$$

Each integral is similar to the one in the hint:

$$\int_a^b e^{2\pi i x}\,dx = 0 \iff b - a \in \mathbb{Z}.$$

Thus, $F(x_1, x_2, y_1, y_2) = 0$ if and only if one of the two sides of the rectangle has an integer length.

Now, consider a situation where a large rectangle is divided into two smaller rectangles sharing an edge. We claim that the function F is additive over such divisions. Specifically, the value of F over the large rectangle is the sum of the F values over the smaller rectangles. This can be shown as follows: let the original rectangle have opposite corners at (x_1, y_1) and (x_2, y_2). Suppose WLOG it is divided along the x-axis at $x = x_3$. The function for the original rectangle is defined as:

$$F(x_1, x_2, y_1, y_2) = \left(\int_{x_1}^{x_2} e^{2\pi i x}\,dx\right)\left(\int_{y_1}^{y_2} e^{2\pi i y}\,dy\right).$$

Now, we divide the rectangle into two smaller rectangles: - Rectangle 1: From (x_1, y_1) to (x_3, y_2),

$$F(x_1, x_3, y_1, y_2) = \left(\int_{x_1}^{x_3} e^{2\pi i x}\,dx\right)\left(\int_{y_1}^{y_2} e^{2\pi i y}\,dy\right).$$

- Rectangle 2: From (x_3, y_1) to (x_2, y_2),

$$F(x_3, x_2, y_1, y_2) = \left(\int_{x_3}^{x_2} e^{2\pi i x}\,dx\right)\left(\int_{y_1}^{y_2} e^{2\pi i y}\,dy\right).$$

Summing the two smaller rectangles' contributions, we have:

$$F(x_1, x_3, y_1, y_2) + F(x_3, x_2, y_1, y_2) = \left(\int_{x_1}^{x_3} e^{2\pi i x}\,dx + \int_{x_3}^{x_2} e^{2\pi i x}\,dx\right)\left(\int_{y_1}^{y_2} e^{2\pi i y}\,dy\right).$$

Since:

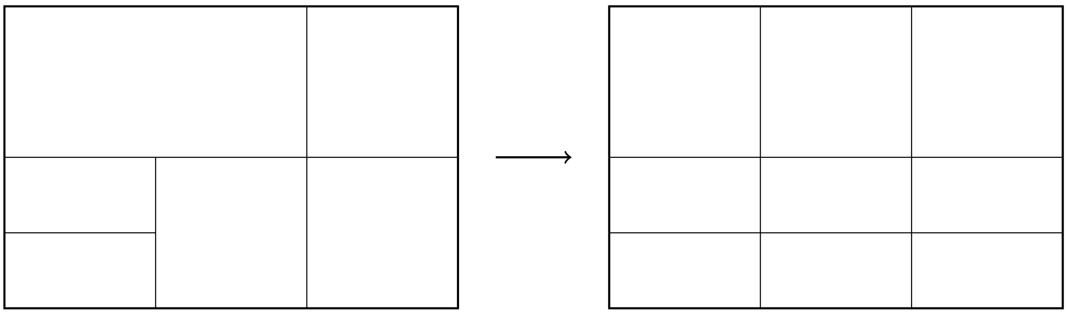
$$\int_{x_1}^{x_2} e^{2\pi i x}\,dx = \int_{x_1}^{x_3} e^{2\pi i x}\,dx + \int_{x_3}^{x_2} e^{2\pi i x}\,dx,$$

we conclude that:

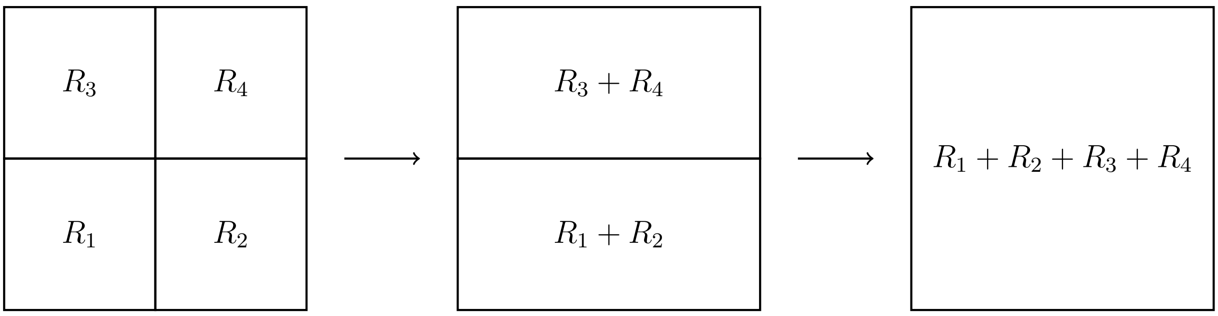
$$F(x_1, x_2, y_1, y_2) = F(x_1, x_3, y_1, y_2) + F(x_3, x_2, y_1, y_2).$$

Thus, F is additive when a rectangle is divided into two smaller rectangles.

Next, we consider a general decomposition of the rectangle $MNPQ$ into smaller rectangles. We can reduce the decomposition into a "grid" decomposition, as this is done by splitting all rectangles at every distinct x-coordinate and y-coordinate that appears in the decomposition. This ensures that all the rectangles in this new decomposition share a whole edge with each of their neighbour, and notice that the sum of F over all the rectangles remained unchanged (again by additivity). Therefore, the sum of the values F over all the smaller rectangles is also zero, which implies that F for the original rectangle is zero, as we can easily merge two rectangles sharing a full edge at the time and finally obtain the rectangle $MNPQ$.



Phase 1: General Decomposition   Phase 2: Grid Decomposition



Step 1: 4 small rectangles   Step 2: Merge   Step 3: Merge

3.5. ⋆ Harmonicity

(a) A real C^2-function $w = w(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if its Laplacian $\Delta w = \operatorname{div}(\nabla w) := \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ is equal to zero everywhere. Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. Denote with $u = \Re(f)$ and $v = \Im(f)$ the real part and imaginary part of f, so that $f(z) = u(z) + iv(z)$ for every $z \in \mathbb{C}$. Show that both u and v are harmonic functions by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

Solution: Differentiating the first Cauchy-Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ in the y-direction, interchanging the order of differentiation and taking advantage of the second Cauchy-Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ one gets that

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial^2 y},$$

which implies $\Delta u = 0$. The same works for v by starting with the second Cauchy-Riemann equation differentiated in the y direction.

(b) Let D be the unit disk centered at the origin and let $h\colon D \to \mathbb{R}$ be a C^∞ harmonic function. Show that there exists some holomorphic function $F\colon D \to \mathbb{C}$ such that $h = \Re(F)$.

Solution: In what follows, we use $h_x = \frac{\partial h}{\partial x}$ and similarly $h_y = \frac{\partial h}{\partial y}$. Using h we define the function

$$f(x+iy) = h_x(x,y) - i \cdot h_y(x,y)$$

with real part $u(x,y) = h_x(x,y)$ and imaginary part $v(x,y) = -h_y(x,y)$. As h is harmonic one has $u_x \equiv v_y$, furthermore $u_y \equiv -v_x$ by equality of the mixed derivatives. So $f\colon D \to \mathbb{C}$ is C^∞ and satisfies the CR equations; therefore it is an analytic function of $z = x + iy \in D$.

Now, by Theorem 2.1 in the notes, we know that the function f has a primitive F in D.

Let $(x,y) \mapsto U(x,y)$ be the real part of F. Then by the CR equations, this time applied to F, we have that, for all $z \in D$,

$$U_x(z) - iU_y(z) = F'(z) = f(z) = h_x(z) - ih_y(z).$$

Let $g(x,y) := U(x,y) - h(x,y)$. Then $g_x = g_y = 0$ and it follows by exercise 5(a) of Serie 1 that in fact in D, $U(x,y) = h(x,y) + C$ for some constant C. Hence h is the real part of the holomorphic function $F(z) + C$.

3.6. ⋆ Real integrals via complex integration

(a) (i) Show that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \lim_{R \to +\infty} \frac{1}{2i} \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx.$$

Solution: By writing

$$\frac{1}{2i} \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx = \frac{1}{2i} \int_{-R}^{R} \frac{\cos(x) - 1 + i\sin(x)}{x} dx$$

and taking the limit for $R \to \infty$ we get

$$\lim_{R\to\infty} \frac{1}{2i} \int_{-R}^{R} \frac{e^{ix}-1}{x} dx = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{\cos(x)-1}{x} dx + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx.$$

The first integral vanishes as we're integrating an odd function over an around 0 symmetric domain. By using the fact that $\frac{\sin(x)}{x}$ is instead even, we can write

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \int_{0}^{\infty} \frac{\sin(x)}{x} dx.$$

(ii) Let $R > 0$ be large and $\varepsilon > 0$ be small. Explain why

$$\int_{\gamma} \frac{e^{iz}-1}{z} dz = 0,$$

where γ is the "indented semicircle" curve described in the picture below.

Solution: We note that the function $z \mapsto \frac{e^{iz}-1}{z}$ is holomorphic in $\mathbb{C} \setminus \{0\}$. Since γ and the region it includes do not contain 0, we can apply Cauchy's theorem and get the wanted result.

(iii) Deduce the value of

$$\int_{0}^{\infty} \frac{\sin(x)}{x} dx.$$

Solution: Using the previous items, we have:

$$\int_{\varepsilon}^{R} \frac{e^{ix}-1}{x} dx + \int_{-R}^{-\varepsilon} \frac{e^{ix}-1}{x} dx = -\int_{\gamma_R^+} \frac{e^{iz}-1}{z} dz + \int_{\gamma_\varepsilon^+} \frac{e^{iz}-1}{z} dz.$$

Let us analyze the first integral. We parametrize γ_R as $\gamma_R \colon [0,\pi] \to \mathbb{C}$, $z \mapsto Re^{zi}$. Obtaining

$$\int_{\gamma_R^+} \frac{e^{iz}-1}{z} dz = i \int_{0}^{\pi} (e^{iR\cos(t)-R\sin(t)} - 1) dt = -\pi i + i \int_{0}^{\pi} e^{iR\cos(t)-R\sin(t)} dt.$$

Observe that

$$\left| \int_{0}^{\pi} e^{iR\cos(t)-R\sin(t)} dt \right| \leq \int_{0}^{\pi} e^{-R\sin(t)} dt,$$

as well as

$$\lim_{R\to\infty} \int_{0}^{\pi} e^{-R\sin(t)} dt = 0,$$

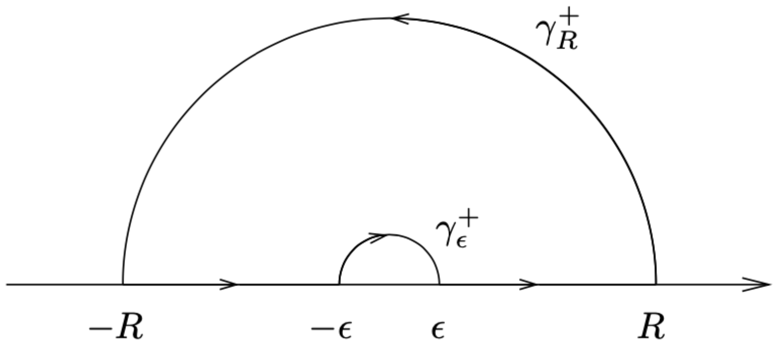
which concludes the analysis of the first integral. Analogously, for the second one, we get

$$\int_{\gamma_\varepsilon^+} \frac{e^{iz}-1}{z} dz = -\pi i + i \int_0^\pi e^{i\varepsilon\cos(t)} e^{-\varepsilon\sin(t)} dt.$$

Again, by uniform convergence on a bounded domain, we can write

$$\lim_{\varepsilon\to 0} \int_0^\pi e^{i\varepsilon\cos(t)-\varepsilon\sin(t)} dt = \int_0^\pi \lim_{\varepsilon\to 0} e^{i\varepsilon\cos(t)-\varepsilon\sin(t)} dt = \pi,$$

which yields the desired result.



(b) Let γ be the counter clockwise oriented unit circle and $n \in \mathbb{N}$. Compute

$$\int_\gamma z^{-1}(z-z^{-1})^n \, dz,$$

and deduce that

$$\int_0^{2\pi} \sin(t)^n \, dt = \begin{cases} \frac{\pi}{2^{n-1}} \binom{n}{n/2}, & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Solution: Using the binomial expansion for $(z-z^{-1})^n$, we have:

$$(z-z^{-1})^n = \sum_{k=0}^n \binom{n}{k} (-1)^k z^{n-2k}.$$

Thus, the integral becomes:

$$\int_\gamma z^{-1}(z-z^{-1})^n \, dz = \sum_{k=0}^n \binom{n}{k} (-1)^k \int_\gamma z^{n-2k-1} \, dz.$$

By Cauchy's integral theorem, $\int_\gamma z^m\,dz = 0$ unless $m = -1$. Therefore, the only non-zero contribution occurs when:

$$n - 2k - 1 = -1 \quad \Rightarrow \quad 2k = n \quad \Rightarrow \quad k = \frac{n}{2}.$$

Thus, for odd n, there is no integer k that satisfies this condition, and the integral is zero. For even n, we have $k = \frac{n}{2}$, so the integral becomes:

$$\int_\gamma z^{-1}(z - z^{-1})^n\,dz = \binom{n}{n/2}(-1)^{n/2}\int_\gamma z^{-1}\,dz.$$

Since $\int_\gamma z^{-1}\,dz = 2\pi i$, we conclude that for even n,

$$\int_\gamma z^{-1}(z - z^{-1})^n\,dz = \binom{n}{n/2}(-1)^{n/2}2\pi i.$$

Now, let's compute the integral in terms of trigonometric functions. Using the substitution $z = e^{it}$, we get:

$$z - z^{-1} = e^{it} - e^{-it} = 2i\sin(t).$$

Thus, the integral becomes:

$$\int_\gamma z^{-1}(z - z^{-1})^n\,dz = \int_0^{2\pi} ie^{it}e^{-it}(2i\sin(t))^n\,dt = i\int_0^{2\pi}(2i\sin(t))^n\,dt.$$

For odd n, since $\sin(t)^n$ is an odd function, the integral over $[0, 2\pi]$ vanishes, confirming that:

$$\int_0^{2\pi}\sin(t)^n\,dt = 0 \quad \text{for odd } n.$$

For even $n = 2m$, we have:

$$i\int_0^{2\pi}(2i)^n\sin(t)^n\,dt = (-1)^m 2^n\int_0^{2\pi}\sin(t)^{2m}\,dt.$$

Equating this with the earlier result $\binom{n}{n/2}(-1)^m 2\pi i$, we get:

$$\int_0^{2\pi}\sin(t)^{2m}\,dt = \frac{\pi}{2^{2m-1}}\binom{2m}{m}.$$

Thus, we deduce that:

$$\int_0^{2\pi}\sin(t)^n\,dt = \begin{cases} \frac{\pi}{2^{n-1}}\binom{n}{n/2}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.}\end{cases}$$