Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

4.1. MC Questions

(a) What's the value of $\int_0^{2\pi} e^{e^{it}+it}\,dt$?

A) 2π

B) 0

C) -2π

D) 4π

Solution: Express this as a line integral over the unit circle $z = e^{it}$:

$$\int_0^{2\pi} e^{e^{it}} e^{it}\,dt = -i\int_0^{2\pi} e^{e^{it}} ie^{it}\,dt = -i\int_{|z|=1} e^z\,dz = 0.$$

(b) Given $\gamma = \{z \in \mathbb{C} : |z| = 3\}$, what's the value of $\int_\gamma \frac{z^2+2z}{z^2-1}\,dz$?

A) $6\pi i$

B) -6π

C) $4\pi i$

D) $\frac{3}{2}\pi i$

Solution: We have that

$$\begin{aligned}\int_\gamma \frac{z^2+2z}{z^2-1}\,dz &= \frac{1}{2}\int_\gamma \frac{3z}{z-1} + \frac{-z}{z+1}\,dz \\ &= \frac{2\pi i}{2}(3-(-1)) = 4\pi i,\end{aligned}$$

by applying the Cauchy integral formula to both terms in the second integral.

4.2. An analytic identity Let γ be the counter-clockwise oriented circle of radius $r > 0$ and center $z_0 \in \mathbb{C}$, and let f be a function which is analytic in all of $\mathbb{C}$. Show that

$$\int_\gamma f(\bar{z})\,dz = 2\pi i r^2 f'(\bar{z}_0).$$

Solution: Let $f : \mathbb{C} \to \mathbb{C}$ analytic and $z_0 \in \mathbb{C}$ with $\overline{z_0} \in \mathbb{C}, r > 0$ such that the closed disc of radious r and center $\overline{z_0}$ is contained in $\mathbb{C}$. Then

$$\int_{|z-z_0|=r} f(\overline{z})dz = 2\pi i r^2 f'(\overline{z_0}).$$

Indeed, we have a nicely convergent series expansion $f(z) = \sum a_n(z-\overline{z_0})^n$ for $z \in \mathbb{C}$ with $|z-\overline{z_0}| < r+\varepsilon$ for some $\varepsilon > 0$ (ε exists because $\mathbb{C}$ is open and f is analytic in $\mathbb{C}$). Define

$$g(z) = \sum \overline{a_n}(z-z_0)^n, \text{ for } |z-z_0| < r+\varepsilon.$$

Then g is analytic where it's defined. We also have $f(\overline{z}) = \overline{g(z)}$ hence

$$\begin{aligned}
\int_{|z-z_0|=r} f(\overline{z})dz &= \int_{|z-z_0|=r} \overline{g(z)}dz \\
&= \overline{\int_{|z-z_0|=r} g(z)\overline{dz}} \\
&= \overline{\int_0^{2\pi} g(z_0+re^{it})\overline{ire^{it}}dt} \\
&= -r^2\overline{\int_0^{2\pi} \frac{g(z_0+re^{it})}{(re^{it})^2} ire^{it}dt} \\
&= -r^2\overline{\int_{|z-z_0|=r} \frac{g(z)}{(z-z_0)^2}dz}.
\end{aligned}$$

By Cauchy's formula (the general one involving derivatives), the last integral equals

$$2\pi i g'(z_0) = 2\pi i \overline{a_1} = 2\pi i \overline{f'(\overline{z_0})},$$

hence we get

$$\int_{|z-z_0|=r} f(\overline{z})dz = -r^2\overline{2\pi i\overline{f'(\overline{z_0})}} = 2\pi i r^2 f'(\overline{z_0}).$$

4.3. Let $r > 0$. Show the estimate

$$\left|\int_\gamma e^{iz^2}dz\right| \leq \frac{\pi(1-e^{-r^2})}{4r},$$

where γ is the curve with $\gamma(t) = re^{it}$, for $0 \leq t \leq \pi/4$

Solution: We begin by using a standard inequality for integrals of complex functions. For any integrable function $f : [a,b] \to \mathbb{C}$, the following inequality holds:

$$\left|\int_a^b f(z)\,dz\right| \leq \int_a^b |f(z)|\,|dz|.$$

Given that the curve is given by $\gamma(t) = re^{it}$, we parametrize the integral as:

$$z(t) = re^{it}, \quad \text{for}\, t \in \left[0, \frac{\pi}{4}\right].$$

Thus, we have $dz = ire^{it}dt$, and $|dz| = rdt$. Next, we compute the modulus of e^{iz^2}. First, calculate z^2 for $z = re^{it}$:

$$z^2 = r^2 e^{2it} = r^2(\cos(2t) + i\sin(2t)).$$

Thus, the real part of iz^2 is

$$\Re(iz^2) = \Re\left(ir^2(\cos(2t) + i\sin(2t))\right) = -r^2\sin(2t).$$

Therefore, we have

$$|e^{iz^2}| = e^{\Re(iz^2)} = e^{-r^2\sin(2t)}.$$

Now, we estimate the original integral:

$$\left|\int_\gamma e^{iz^2}dz\right| \leq \int_\gamma |e^{iz^2}||dz| = \int_0^{\frac{\pi}{4}} e^{-r^2\sin(2t)} r\,dt.$$

We substitute $u = 2t$, which implies $du = 2dt$. The limits of integration change accordingly: when $t = 0$, $u = 0$, and when $t = \frac{\pi}{4}$, $u = \frac{\pi}{2}$. Thus, the integral becomes:

$$\int_0^{\frac{\pi}{4}} e^{-r^2\sin(2t)} r\,dt = \frac{r}{2}\int_0^{\frac{\pi}{2}} e^{-r^2\sin(u)}\,du.$$

Since for $0 \leq u \leq \pi/2$, $\sin(u) \geq \frac{2u}{\pi}$ we have that

$$\left|\int_\gamma e^{iz^2}dz\right| \leq \frac{r}{2}\int_0^{\frac{\pi}{2}} e^{-r^2\sin(u)}\,du \leq \frac{r}{2}\int_0^{\frac{\pi}{2}} e^{-2ur^2/\pi}\,du = \frac{\pi(1-e^{-r^2})}{4r}.$$

4.4. $\star$ **Converse of Theorem 3.2, Chapter 1** Suppose that a function f is continuous on a plane domain Ω in $\mathbb{C}$ and that $\int_\gamma f(z)dz = 0$ for every closed piecewise smooth path on Ω. Show that f has a primitive in Ω.

Solution: Let $z_0 \in \Omega$ be a fixed point. For any point $z \in \Omega$, we define the function $F(z)$ as the integral of f along a smooth path from z_0 to z. More precisely, for a smooth path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$, we define:

$$F(z) = \int_\gamma f(w)\,dw.$$

To show that $F(z)$ is well-defined, suppose γ_1 and γ_2 are two smooth paths from z_0 to z in Ω. Consider the closed curve Γ formed by following γ_1 from z_0 to z, and then following γ_2 in reverse from z back to z_0. Since Γ is a closed piecewise smooth path in Ω, and by the given condition of the problem, we know that:

$$\int_\Gamma f(w)\,dw = 0.$$

Now, using the linearity of the integral, we can split the integral over Γ as:

$$\int_\Gamma f(w)\,dw = \int_{\gamma_1} f(w)\,dw - \int_{\gamma_2} f(w)\,dw = 0,$$

which implies that

$$\int_{\gamma_1} f(w)\,dw = \int_{\gamma_2} f(w)\,dw.$$

Thus, the value of $F(z)$ is independent of the choice of path, so $F(z)$ is well-defined.

Next, we need to show that F is holomorphic in Ω, and that $F'(z) = f(z)$ for all $z \in \Omega$. Fix an arbitrary point $z \in \Omega$, and consider a small open disk D centered at z, such that $D \subset \Omega$. For any point $z' \in D$, we want to compute $F(z')$. To do this, choose a smooth path from z_0 to z' that goes through z. Specifically, we can take the path to consist of two parts: one path from z_0 to z, and then another path from z to z' within D. We know that since f is continuous and satisfies the condition $\int_\gamma f(z)\,dz = 0$ for all closed curves γ in $D \subset \Omega$, its integral along rectangles whose sides are parallel to the coordinate axis is also zero. Hence by Theorem 2.1' from the notes,f admits a primitive in D. That is, there exists a holomorphic function $G : D \to \mathbb{C}$ such that $G'(z) = f(z)$ for all $z \in D$.

Thus, for any $z' \in D$, we can compute $F(z')$ as:

$$F(z') = F(z) + \int_z^{z'} f(w)\,dw = F(z) + G(z') - G(z),$$

where the second equality follows from the fact that $G'(w) = f(w)$ in D.

Differentiating both sides with respect to z', we get:

$$F'(z') = G'(z') = f(z').$$

Therefore, $F'(z) = f(z)$ for all $z \in \Omega$.

4.5. $\star$ Application of Liouville's theorem Let f and g be entire functions such that, for all $z \in \mathbb{C}$, $\Re(f(z)) \leq k\Re(g(z))$ for some real constant k, independent of z. Prove that there are constants a, b such that $f(z) = ag(z) + b$

Solution: Consider the function $h(z) = e^{f(z)-kg(z)}$, where $f(z)-kg(z)$ is the difference of two entire functions. Since the sum or difference of entire functions is entire, $f(z) - kg(z)$ is an entire function. The exponential of an entire function is also entire, so $h(z) = e^{f(z)-kg(z)}$ is entire.

Now, let's examine the modulus of $h(z)$. We have:

$$|h(z)| = \left|e^{f(z)-kg(z)}\right| = e^{\Re(f(z)-kg(z))}.$$

By the given condition, we know that $\Re(f(z)) \leq k\Re(g(z))$ for all $z \in \mathbb{C}$. Therefore,

$$\Re(f(z)) - k\Re(g(z)) \leq 0.$$

This implies that

$$|h(z)| = e^{\Re(f(z))-k\Re(g(z))} \leq 1$$

for all $z \in \mathbb{C}$. We have shown that $h(z) = e^{f(z)-kg(z)}$ is an entire function and that $|h(z)| \leq 1$ for all $z \in \mathbb{C}$. By Liouville's Theorem, any bounded entire function must be constant. Hence, $h(z) = C$, where C is a constant. Since $h(z) = C$, we have:

$$e^{f(z)-kg(z)} = C.$$

Therefore, there exists some constant $b \in \mathbb{C}$ such that

$$f(z) - kg(z) = b.$$

Thus, we conclude that:

$$f(z) = kg(z) + b.$$