### 5.1. MC Questions

**(a)** A subset $\mathcal{A}$ of a domain $\Omega \subset \mathbb{C}$ is called *discrete* in $\Omega$ if it has no limit point in $\Omega$. For how many of the following pairs $(\Omega, \mathcal{A})$ is it not true that $\mathcal{A}$ is discrete in $\Omega$?

(i) $\Omega = \mathbb{C}$. Define $q_n = \sum_{k=1}^{n} \frac{1}{k}$ and $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{q_n e^{ix} : x = \frac{k2\pi}{n} \text{ for some } k \in \mathbb{N}\}$.

(ii) $\Omega = \mathbb{C}$. Define $q_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{q_n e^{ix} : x = \frac{k2\pi}{n} \text{ for some } k \in \mathbb{N}\}$.

(iii) $\Omega = \mathbb{C}$, $\mathcal{A} = \{\frac{1}{n} : n \in \mathbb{N}^+\}$.

(iv) $\Omega = \mathbb{C}\backslash\{0\}$, $\mathcal{A} = \{\frac{1}{n} : n \in \mathbb{N}^+\}$.

A) 0

B) 1

C) 2

D) 3

**Solution:** In options (iii) and (iv), 0 is clearly the only accumulation point for the set $\mathcal{A}$ in $\mathbb{C}$. Only (iii) should be counted in the answer, as in (iv) we have that $0 \notin \Omega$. We now claim that in option (i), the set is discrete in $\Omega$. Indeed, any bounded subset of $\mathbb{C}$ contains at most finitely many points from $\mathcal{A}$, as $\lim_{n\to\infty} q_n = \infty$. The same can't be said about option (ii): given that $\lim_{n\to\infty} q_n = \frac{\pi^2}{6}$ we have for instance that $\frac{\pi^2}{6}$ is an accumulation point for $\mathcal{A}$. All in all, two out of the four sets are discrete and two are not.

**(b)** Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on an open set $\Omega$ . For $w \in \mathbb{C}$ we define the set

$$E_w := \{z \in \Omega : f(z) = w.\}$$

Which of the following is true?

A) $E_w$ is a discrete set in $\Omega$ only for $w = 0$.

B) $E_w$ is a discrete set in $\Omega$ only for $w \neq 0$.

C) $E_w$ is a discrete set in $\Omega$ for every $w \in \mathbb{C}$.

D) If $\Omega$ is connected then $E_w$ is a discrete set in $\Omega$ for every $w \in \mathbb{C}$.

**Solution:** We start by proving that D) holds. Let $\Omega \subseteq \mathbb{C}$ be open and connected, and fix $w \in \Omega$. Define $g : \Omega \to \mathbb{C}$ as $g(z) = f(z) - w$. Then $E_w = \{z \in \Omega : f(z) = w\} = \{z \in \Omega : f(z) - w = 0\} = \{z \in \Omega : g(z) = 0\}$. Since $g$ is also non-constant and holomorphic, we know that its set of zeros has to be discrete (all zeros are isolated), and consequently $E_w$ is also discrete. On the other hand, option C) is false since if $\Omega$ is not connected then we can easily define a locally constant but non-constant holomorphic function $f$, which, being locally constant, clearly doesn't satisfy the fact that $E_w$ is a discrete set in $\Omega$ for every $w \in \mathbb{C}$. Since the value taken by $f$ in the part of the domain on which it is locally constant is arbitrary, A) and B) are also proven to be false.

### 5.2. Order of zeros

**(a)** Find the zeros of the function $z \mapsto \sin(z^2)$ and determine their order.

**Solution:** Taking advantage of the definition of complex sine, we have that $\sin(z_0^2) = 0$ if $z_0 = 0$ or $z_0 = \pm\sqrt{k\pi}$ or $z_0 = \pm i\sqrt{k\pi}$ for some $k \geq 1$. Consider now $(\sin(z^2))' = 2z\cos(z^2)$. Among the possible values for $z_0$ we just mentioned, we have that $(\sin(z_0^2))' = 0 \iff z_0 = 0$, so all the other zeroes have order 1. Computing $(\sin(z_0^2))''|_{z_0=0} \neq 0$ allows us to conclude that 0 has order 2.

**(b)** Let $p(z) := 1 + a_1 z + \cdots + a_n z^n$ be a polynomial and $f(z) := e^z - p(z)$. Clearly $z_0 = 0$ is a zero of the function $f(z)$. Compute $\mathrm{ord}_{z_0} f$, the order of the zero of $f$ at $z_0$, as a function of the coefficients of $p(z)$.

**Solution:** The Taylor series expansion of $e^z$ around $z = 0$ is:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

The function we are investigating is:

$$f(z) = e^z - p(z).$$

We subtract the polynomial $p(z)$ from the Taylor series of $e^z$:

$$f(z) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - \left(1 + a_1 z + a_2 z^2 + \cdots + a_n z^n\right).$$

Simplifying the expression, we get:

$$f(z) = (z - a_1 z) + \left(\frac{z^2}{2!} - a_2 z^2\right) + \cdots + \left(\frac{z^n}{n!} - a_n z^n\right) + \sum_{k=n+1}^{\infty} \frac{z^k}{k!}.$$

The order of the zero at $z_0 = 0$ is determined by the first non-zero coefficient in the above expansion. That is, we seek the smallest $k$ such that

$$\frac{1}{k!} - a_k \neq 0.$$

Thus, the order of the zero at $z_0 = 0$ is:

$$\min\left\{k : 1 \leq k \leq n \text{ and } a_k \neq \frac{1}{k!}\right\}.$$

If $a_k = \frac{1}{k!}$ for all $k \leq n$, then the order of the zero is $n+1$.

**5.3.** ⋆ **The complex logarithm** Let

$$U = \mathbb{C} \setminus \{z \in \mathbb{C} : \Im(z) = 0, \Re(z) \leq 0\}$$

be the open set obtained by removing the negative real axis from the complex plane $\mathbb{C}$. The complex logarithm is defined in $U$ as

$$\log(z) := \log(|z|) + i\arg(z), \quad z = |z|e^{i\arg(z)},$$

where $\arg(z) \in ]-\pi, \pi[$. Show that for every $z \in U$

$$\log(z) = \int_\gamma \frac{1}{w}\, dw,$$

where $\gamma$ is the segment connecting 1 to $z$.

*Hint: integrate over a well chosen closed curve containing $\gamma$ and passing through $|z|$.*

**Solution:** Fix $z \in U$ and let $\theta = \arg(z) \in ]-\pi, \pi[$ so that $z = |z|e^{i\theta}$. Then, consider the closed curve $\sigma$ defined as the concatenation of: $\gamma_1(t) = (1-t) + t|z|$, $t \in [0,1]$ (the segment joining 1 with $|z|$), then $\gamma_2(t) = |z|e^{it}$ for $t \in [0, \theta]$ (the arc centred at the origin connecting $|z|$ to $z$ ), and finally $\gamma^-$, (the segment joining $z$ to 1). Since $w \mapsto 1/w$ is holomorphic in $U$, by Cauchy Theorem we have by integrating over $\sigma$ that

$$\begin{aligned}\int_\gamma \frac{1}{w}\, dw = \int_{\gamma_1} \frac{1}{w}\, dw + \int_{\gamma_2} \frac{1}{w}\, dw = \int_0^1 \frac{|z|-1}{(1+t)+t|z|}\, dt + \int_0^\theta \frac{i|z|e^{it}}{|z|e^{it}}\, dt \\ = \log((1+t)+t|z|)|_{t=0}^{t=1} + i\theta = \log(|z|) + i\arg(z)\end{aligned}$$

which is exactly the definition of the complex logarithm.

**5.4. A complex ODE** Take advantage of the power series expansion around zero to find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f'(z) = zf(z)$ and $f(0) = 1$.

**Solution:** We know that $f(z)$ can be expressed as a power series around $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Taking the derivative term-by-term, we obtain:

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Substituting $f(z)$ and $f'(z)$ into the differential equation $f'(z) = zf(z)$, we get:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = z \sum_{n=0}^{\infty} a_n z^n.$$

The right-hand side is

$$z \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{m=1}^{\infty} a_{m-1} z^m \quad \text{(letting } m = n+1\text{)}.$$

Thus, the equation becomes:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{m=1}^{\infty} a_{m-1} z^m.$$

To align the powers of $z$ on both sides, perform a change of index on the left-hand side. Let $m = n - 1$, which implies $n = m + 1$. Then:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} z^m.$$

Now, the equation is:

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} z^m = \sum_{m=1}^{\infty} a_{m-1} z^m.$$

Separating the $m = 0$ term on the left-hand side:

$$a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} z^m = 0 + \sum_{m=1}^{\infty} a_{m-1} z^m.$$

By equating the coefficients of the two power series, we get $a_1 = 0$ and obtain the following recurrence relation for $m \geq 1$:

$$(m+1) a_{m+1} = a_{m-1} \quad \Rightarrow \quad a_{m+1} = \frac{a_{m-1}}{m+1}.$$

Using the initial condition $f(0) = 1$, we have:

$$a_0 = 1.$$

From $a_1 = 0$, we proceed to find the subsequent coefficients using the recurrence relation.

Let's compute a few coefficients to identify a pattern:

- For $m = 1$:

$$a_2 = \frac{a_0}{2} = \frac{1}{2}.$$

- For $m = 2$:

$$a_3 = \frac{a_1}{3} = 0.$$

- For $m = 3$:

$$a_4 = \frac{a_2}{4} = \frac{1}{2 \cdot 4} = \frac{1}{8}.$$

- For $m = 4$:

$$a_5 = \frac{a_3}{5} = 0.$$

Continuing this process, we observe that all coefficients $a_n$ for odd $n \geq 1$ are zero. For even $n = 2k$, the coefficients are given by:

$$a_{2k} = \frac{1}{2^k k!}.$$

Substituting the coefficients back into the power series, we obtain:

$$f(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^k k!}.$$

Recognizing the power series expansion of the exponential function, we can rewrite this as:

$$f(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{2}\right)^k}{k!} = e^{\frac{z^2}{2}}.$$

**5.5. Riemann continuation Theorem** Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic. Show that the following are equivalent:

1. There exists $g : \mathbb{C} \to \mathbb{C}$ holomorphic, such that $g(z) = f(z)$ for all $z \neq 0$.
2. There exists $g : \mathbb{C} \to \mathbb{C}$ continuous, such that $g(z) = f(z)$ for all $z \neq 0$.
3. There exists $\varepsilon > 0$ such that $f$ is bounded in $\dot{B}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\} \setminus \{0\}$.
4. $\lim_{z\to 0} z f(z) = 0$.

*Hint: to prove* 4. $\Rightarrow$ 1. *define* $h(z) = zf(z)$ *when* $z \neq 0$ *and* $h(0) = 0$. *Analyse the relation between* $f(z)$, $h(z)$ *and* $k(z) := zh(z)$.

**Solution:** Notice that the implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. are elementary. We are left to show 4. $\Rightarrow$ 1. Introduce the function

$$h(z) := \begin{cases} zf(z), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

and set $k(z) = zh(z)$. By assumption 4. $h$ and $k$ are holomorphic in $\mathbb{C} \setminus \{0\}$ and continuous in the whole complex plane $\mathbb{C}$. Since $k(z) = k(0) + zh(z)$ we deduce that $k$ is complex differentiable in zero and hence holomorphic in $\mathbb{C}$. By Taylor representation of holomorphic functions, $k(z) = a_0 + a_1 z + a_2 z^2 + \dots$ for coefficients $a_0, a_1, \dots \in \mathbb{C}$. Since $k(0) = 0$ and $k'(0) = h(0) = 0$ we deduce that $k(z) = a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots = z^2(a_2 + a_3 z + a_4 z^2 + \dots)$. Now, recalling that $k(z) = z^2 f(z)$ for $z \neq 0$ we deduce that $g(z) := a_2 + a_3 z + a_4 z^2 + \dots$ is indeed an holomorphic extension of $f$ in $\mathbb{C}$.

**5.6.** $\star$ Let $D \subset \mathbb{C}$ be the unit disk at the origin. Find all functions $f(z)$ which are holomorphic on $D$ and which satisfy

$$f\left(\frac{1}{n}\right) = n^2 f\left(\frac{1}{n}\right)^3, \quad n = 2, 3, 4, \dots$$

**Solution:** We rewrite this as

$$f\left(\frac{1}{n}\right)\left(f\left(\frac{1}{n}\right) - \frac{1}{n}\right)\left(f\left(\frac{1}{n}\right) + \frac{1}{n}\right) = 0.$$

At each $n$, one of the following holds:

$$f\left(\frac{1}{n}\right) = 0, \quad f\left(\frac{1}{n}\right) = \frac{1}{n}, \quad \text{or} \quad f\left(\frac{1}{n}\right) = -\frac{1}{n}.$$

At least one of these three equations must hold for infinitely many $n$. Therefore, either

$$f(z) = 0, \quad f(z) = z, \quad \text{or} \quad f(z) = -z.$$