6.1. MC Questions

(a) Consider the sequence of functions $f_n(z) = \frac{z^n}{n+1}$ on $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Which of the following is true?

A) The sequence $\{f_n(z)\}$ converges uniformly on D.

B) The sequence $\{f_n(z)\}$ converges locally uniformly on D, but not uniformly.

C) The sequence $\{f_n(z)\}$ converges pointwise but not uniformly on D.

D) The sequence $\{f_n(z)\}$ does not converge on D.

Solution: We directly check for uniform convergence. To test this, we consider the absolute value of $f_n(z)$:

$$|f_n(z)| = \left|\frac{z^n}{n+1}\right| = \frac{|z|^n}{n+1}.$$

For $z \in D$, observe that $|f_n(z)| \leq \frac{1}{n+1}$ because $|z^n| \leq 1$ for all $z \in D$. Since $\frac{1}{n+1} \to 0$ uniformly (indeed, independently from $z \in D$) as $n \to \infty$, we can conclude that A) is the correct answer.

(b) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n\colon \mathbb{C} \to \mathbb{C}$. Which of the following statements is true?

A) If $\sum_{n\in\mathbb{N}} f_n$ converges uniformly on $\mathbb{C}$ to some f and for each $n \in \mathbb{N}$ there exists some $M_n \in \mathbb{R}^+$ such that $\sup_{z\in\mathbb{C}} |f_n(z)| \geq M_n$, then $\sum_{n\in\mathbb{N}} M_n$ converges.

B) If $\sum_{n\in\mathbb{N}} f_n$ converges locally uniformly on $\mathbb{C}$ to some f and for each $n \in \mathbb{N}$ there exists some $M_n \in \mathbb{R}^+$ such that $\sup_{z\in\mathbb{C}} |f_n(z)| \geq M_n$, then $\sup_{n\in\mathbb{N}} M_n$ is finite, i.e. the sequence $(M_n)_{n\in\mathbb{N}}$ is bounded.

C) If for all $n \in \mathbb{N}$ we have that $\sup_{z\in\mathbb{C}} |f_n(z)| = +\infty$, then $\sum_{n\in\mathbb{N}} f_n$ can not converge uniformly to any $f\colon \mathbb{C} \to \mathbb{C}$.

D) If for all $n \in \mathbb{N}$ we have that $\sup_{z\in\mathbb{C}} |f_n(z)| = +\infty$, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly to some $f\colon \mathbb{C} \to \mathbb{C}$.

Solution: A) is false. Consider the counterexample where $\{f_n(z)\}_n$ given by the sequence of constant functions $\{1, -1, \frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{3}, \dots, \frac{1}{n}, \frac{-1}{n}, \dots\}$. The series $\sum f_n(z)$ converges uniformly since

$$\sum_{n}^{N} f_n(z) = \begin{cases} 0 & N \text{ even}, \\ \frac{2}{N+1} & N \text{ odd}. \end{cases}$$

but for each n,

$$\sup_{z\in\mathbb{C}} |f_n(z)| \geq \frac{1}{n+1} \quad \text{so} \quad M_n = \frac{1}{n+1}.$$

However, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

B) is also false. Consider the sequence $f_n(z) = \frac{z^n}{n!}$. The sequence converges locally uniformly to e^z on $\mathbb{C}$, but

$$\sup_{z\in\mathbb{C}} |f_n(z)| = \infty,$$

so we can take $M_n = n$, and the sequence (M_n) is unbounded. Therefore, the statement is false.

C) is true. Assume, towards a contradiction, that $\sum_{n\in\mathbb{N}} f_n(z)$ converges uniformly to some function $f : \mathbb{C} \to \mathbb{C}$. By the definition of uniform convergence, this means that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $z \in \mathbb{C}$, we have

$$\left| \sum_{k=n}^{\infty} f_k(z) \right| < \epsilon.$$

In particular, this implies that $|f_n(z)| = |\sum_{k=n}^{\infty} f_k(z) - \sum_{k=n+1}^{\infty} f_k(z)| < 2\epsilon$ for all $n \geq N$ and for all $z \in \mathbb{C}$. However, by assumption, for each $n \in \mathbb{N}$, we have that

$$\sup_{z\in\mathbb{C}} |f_n(z)| = +\infty.$$

In particular, for any $n \geq N$, this contradicts the earlier conclusion that $|f_n(z)| < 2\epsilon$ for all $z \in \mathbb{C}$.

The fact that D) is false follows from the fact that C) is true.

6.2. Show that the following functions exist and are holomorphic on the indicated open sets; furthermore, give a similar expression for their derivatives:

(a) $f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$ on $D_1(0)$

Solution: One can give a simple proof using Weierstrass M-test. More precisely, let $0 < R < 1$. For any z with $|z| \leq R$ we have that $|\frac{z^n}{1-z^n}| \leq \frac{R^n}{1-R}$. Since $\frac{1}{1-R}\sum_{n=1}^{\infty} R^n$ converges, $f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$ converges on every compact subset of $D_1(0)$ using the Weierstrass M-test. Hence $f_1(z)$ is holomorphic.

Or one can use the definition of uniform convergence directly. Namely first observe that for each $n \geq 1$, the term

$$f_{1,n}(z) = \frac{z^n}{1 - z^n}$$

is holomorphic on the open unit disk $D_1(0)$. This is because z^n is a holomorphic function, and the denominator $1 - z^n$ does not vanish for $z \in D_1(0)$, since $|z| < 1$ implies $|z^n| < 1$. Hence, each term of the series is holomorphic in the open set $D_1(0)$. Since any finite partial sum of holomorphic functions is holomorphic, the partial sums

$$S_N(z) = \sum_{n=1}^{N} \frac{z^n}{1 - z^n}$$

are holomorphic for all N in $D_1(0)$. To prove that the series converges uniformly on compact subsets of $D_1(0)$, consider the tail of the series:

$$T_N(z) = \sum_{n=N+1}^{\infty} \frac{z^n}{1 - z^n}.$$

We want to show that $T_N(z) \to 0$ uniformly as $N \to \infty$. Let K be a compact subset of $D_1(0)$. Since K is compact, there exists $r < 1$ such that $|z| \leq r$ for all $z \in K$. For $|z| \leq r$, we have

$$\left| \frac{z^n}{1 - z^n} \right| \leq \frac{|z|^n}{1 - |z|^n} \leq \frac{r^n}{1 - r^n}.$$

Therefore, the tail sum satisfies

$$|T_N(z)| \leq \sum_{n=N+1}^{\infty} \frac{r^n}{1 - r^n} = \frac{r^{N+1}}{(1 - r)(1 - r^n)}.$$

Since this bound tends to zero as $N \to \infty$, the tail $T_N(z)$ goes to zero uniformly on K. Hence, the series converges uniformly on compact subsets of $D_1(0)$. Since the series converges uniformly on compact subsets of $D_1(0)$ and each term is holomorphic, we can differentiate the series term by term. That is, the derivative of the sum is the sum of the derivatives:

$$f_1^{(k)}(z) = \sum_{n=1}^{\infty} \frac{d^k}{dz^k} \left(\frac{z^n}{1 - z^n} \right).$$

(b) $f_2(z) = \int_0^1 (1 - tz)^4 e^{tz} \, dt$ on $\mathbb{C}$

Solution: Note that the function $F(z,t) = (1-tz)^4 e^{tz}$ is continuous on $\mathbb{C} \times [0,1]$ and holomorphic in z for each fixed $t \in [0,1]$. Now the result is a simple application of Theorem 5.4 from the notes.

Once can also prove this without using the Theorem 5.4 as follows. The function $f_2(z)$ is defined as an integral over the bounded domain $[0,1]$ of the integrand

$$g(z,t) = (1-tz)^4 e^{tz}.$$

Since $g(z,t)$ is continuous with respect to z for each fixed t, and the integrand is uniformly bounded on $[0,1]$ for any compact subset of $\mathbb{C}$ (as t ranges only from 0 to 1), the function $f_2(z)$ is continuous. Specifically, for each $t \in [0,1]$,

$$|(1-tz)^4 e^{tz}| \leq C \quad \text{for } z \in K,$$

where K is any compact set and C is a constant depending on K. Thus, $f_2(z)$ is continuous on $\mathbb{C}$. Next, we need to show that $f_2(z)$ is holomorphic. The integrand $g(z,t) = (1-tz)^4 e^{tz}$ is holomorphic in z for each fixed $t \in [0,1]$. Now, we need to show that $f_2(z)$ is holomorphic by integrating over a triangular region. Let T be a triangular region in $\mathbb{C}$. We wish to show that

$$\int_T f_2(z)\, dz = 0.$$

Using Fubini's theorem, we can interchange the order of integration:

$$\int_T f_2(z)\, dz = \int_0^1 \left(\int_T (1-tz)^4 e^{tz}\, dz \right) dt.$$

Since the integrand $(1-tz)^4 e^{tz}$ is holomorphic in z, by Cauchy's theorem, the integral over the triangular region T is zero:

$$\int_T (1-tz)^4 e^{tz}\, dz = 0 \quad \text{for all } t \in [0,1].$$

Thus, by Fubini's theorem, we conclude that

$$\int_T f_2(z)\, dz = 0,$$

which shows that $f_2(z)$ is holomorphic on $\mathbb{C}$. To compute the n-th derivative of $f_2(z)$, we apply Cauchy's differentiation formula. The n-th derivative of $f_2(z)$ is given by

$$f_2^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f_2(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta,$$

where γ is a small contour around z.

Using Fubini's theorem again, we can exchange the order of integration:

$$f_2^{(n)}(z) = \int_0^1 \left(\frac{n!}{2\pi i} \int_\gamma \frac{(1-t\zeta)^4 e^{t\zeta}}{(\zeta - z)^{n+1}} \, d\zeta \right) dt.$$

By Cauchy's differentiation formula, the inner contour integral gives the n-th derivative of the integrand with respect to ζ:

$$f_2^{(n)}(z) = \int_0^1 \frac{d^n}{dz^n} \left((1-tz)^4 e^{tz} \right) dt.$$

(c) $f_3(z) = \sum_{n=0}^{\infty} n^2 \exp(2i\pi n^3 z)$ on $H = \{z \in \mathbb{C} \mid \Im(z) > 0\}$

Solution: Let $\delta > 0$ and $H_\delta := \{z \in \mathbb{C} \mid \Im(z) \geq \delta\}$. Then for any $z \in H_\delta$, $|n^2 \exp(2i\pi n^3 z)| \leq n^2 \exp(-2\pi n^3 \delta)$.

Since $\sum_{n=0}^{\infty} n^2 \exp(-2\pi n^3 \delta) < \infty$, once again by Weierstrass M-test, $f_3(z)$ converges uniformly on the compact set H_δ. Since any compact subset of H is contained in H_δ for some $\delta > 0$, the result follows.

Alternatively let us write each term in the series as

$$f_{3,n}(z) = n^2 \exp(2i\pi n^3 z),$$

where $n \geq 0$. For each fixed n, the function $f_{3,n}(z)$ is holomorphic on all of $\mathbb{C}$, in particular each term is holomorphic on the given domain $\{z \in \mathbb{C} \mid \Im(z) > 0\}$.

Since any finite sum of holomorphic functions is holomorphic, we conclude that the partial sums

$$S_N(z) = \sum_{n=0}^{N} n^2 \exp(2i\pi n^3 z)$$

are holomorphic on $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. Next, we need to show that the series converges uniformly on compact subsets of $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. Let K be a compact subset of $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. Since $\Im(z) > 0$ for all $z \in K$, there exists a constant $\delta > 0$ such that $\Im(z) \geq \delta > 0$ for all $z \in K$. For $z \in K$, the exponential term in the series satisfies

$$|\exp(2i\pi n^3 z)| = \exp(-2\pi n^3 \Im(z)) \leq \exp(-2\pi n^3 \delta).$$

Therefore, for large n, the terms decay exponentially. This gives the following bound on the tail of the series:

$$\left| \sum_{n=N+1}^{\infty} n^2 \exp(2i\pi n^3 z) \right| \leq \sum_{n=N+1}^{\infty} n^2 \exp(-2\pi n^3 \delta).$$

Since this is a rapidly decreasing series for large n, the tail tends to zero uniformly as $N \to \infty$. Hence, the series converges uniformly on compact subsets of $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. Since the series converges uniformly on compact subsets, we can differentiate term by term. The derivative of the sum is the sum of the derivatives of each term. The k-th derivative of the n-th term is:

$$f_{3,n}^{(k)}(z) = \frac{d^k}{dz^k}\left(n^2 \exp(2i\pi n^3 z)\right).$$

Using the chain rule, the k-th derivative of the exponential function is:

$$f_{3,n}^{(k)}(z) = (2i\pi n^3)^k n^2 \exp(2i\pi n^3 z) = (2i\pi)^k n^{3k+2} \exp(2i\pi n^3 z).$$

Thus, the k-th derivative of $f_3(z)$ is:

$$f_3^{(k)}(z) = \sum_{n=0}^{\infty} (2i\pi)^k n^{3k+2} \exp(2i\pi n^3 z).$$

6.3.

(a) Prove that the sequence $f_n(z) = z^n$, $n \geq 1$ converges locally uniformly but not uniformly on $\{z : |z| < 1\}$.

Solution. Since $z^n \to 0$ as $n \to \infty$ for every $|z| < 1$, $f_n \to 0$ pointwise. Convergence is not uniform since $\sup_{|z|<1} |f_n(z) - 0| = 1$. Locally uniform convergence is equivalent to uniform convergence on compact subsets. Let K be a compact subset of the open unit disk. Define $r = \max_{z \in K} |z|$. Then since $r < 1$,

$$\max_{z \in K} |f_n(z) - 0| = \max_{z \in K} |z^n| = r^n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $f_n \to 0$ uniformly on K and hence converges locally uniformly.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary (not necessarily continuous) function and define for $n \in \mathbb{N}$

$$f_n(z) = \begin{cases} f(z), & \text{if } |z| \leq n, \\ 0, & \text{if } |z| > n. \end{cases}$$

Show that the sequence (f_n) converges pointwise and locally uniformly to f, and that it converges uniformly to f if and only if $\lim_{|z| \to \infty} f(z) = 0$.

Solution. Let $K \subset \mathbb{C}$ be a compact subset and define $r = \max_{z \in K} |z|$. Then for all $z \in K$, the sequence $f_n(z)$ becomes stationary and equal to $f(z)$ for $n \geq r$. Thus,

(f_n) converges uniformly on compact subsets of $\mathbb{C}$ and hence locally uniformly on $\mathbb{C}$. Moreover, (f_n) converges uniformly on $\mathbb{C}$ if and only if

$$\lim_{n\to\infty} \sup_{z\in\mathbb{C}} |f_n(z) - f(z)| = \lim_{n\to\infty} \sup_{|z|>n} |f(z)| = 0,$$

which is equivalent to $\lim_{|z|\to\infty} f(z) = 0$.

6.4. Let f be a holomorphic function on $D = \{z : |z| < 1\}$ with $f(0) = 0$. Prove that the series $\phi(z) = \sum_{n=1}^{\infty} f(z^n)$ converges locally uniformly on D.

Solution. Let $0 < R < 1$. We first prove that ϕ converges uniformly on $\{|z| \leq R\}$. For $|z| \leq R$, we take the path γ as the straight line segment joining 0 and z. Then

$$|f(z)| = \left| \int_\gamma f'(w)\, dw + f(0) \right| \leq M|z|,$$

where $M = \max_{|w|\leq R} |f'(w)|$.

$$|\phi(z)| \leq \sum_{n=1}^{\infty} |f(z^n)| \leq M \sum_{n=1}^{\infty} |z^n| \leq M \sum_{n=1}^{\infty} R^n,$$

which converges uniformly on $\{|z| \leq R\}$ by the Weierstrass criterion. Let $K \subset D$ be a compact subset. Then there exists $R < 1$ such that $K \subset \{|z| \leq R\}$. Therefore, ϕ converges uniformly on each compact subset K, and hence converges locally uniformly on D.

6.5. Weierstrass M-test Let $f_n\colon A \to \mathbb{C}$ be a sequence of functions and M_n be a sequence of real numbers such that

$$|f_n(z)| \leq M_n, \ \forall n \geq 1, \ \forall z \in A \ \text{ and } \sum_{n=1}^{\infty} M_n \text{ converges.}$$

Prove that $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on A.

Solution: For each fixed $z \in A$, we have the inequality

$$|f_n(z)| \leq M_n.$$

Since the series $\sum_{n=1}^{\infty} M_n$ converges, by the comparison test, it follows that the series

$$\sum_{n=1}^{\infty} |f_n(z)|$$

also converges. Therefore, the series $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely for each $z \in A$. Next, we show that the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on A. To do this, we use the Cauchy criterion for uniform convergence. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, there exists an integer $N \geq 1$ such that for all $p, q \geq N$,

$$\sum_{n=p}^{q} M_n < \epsilon.$$

Now, for all $z \in A$ and for all $p, q \geq N$, we have

$$\left|\sum_{n=p}^{q} f_n(z)\right| \leq \sum_{n=p}^{q} |f_n(z)| \leq \sum_{n=p}^{q} M_n.$$

Thus, for all $z \in A$, we get

$$\left|\sum_{n=p}^{q} f_n(z)\right| < \epsilon.$$

This shows that the sequence of partial sums of $\sum_{n=1}^{\infty} f_n(z)$ satisfies the Cauchy criterion uniformly on A. Therefore, the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on A.