7.1. MC Questions

(a) Consider the function $f\colon z \mapsto \frac{e^z}{e^z-1}$, defined for all $z \in \mathbb{C}$ such that $e^z \neq 1$. Which of the following statements holds?

A) The function f is holomorphic on $\mathbb{C} \setminus \{0\}$.

B) The function f has poles and each pole is simple.

C) The function f has both poles and removable singularities.

D) The function f has finitely many singularities.

Solution: To determine the points at which f is not holomorphic, observe that f is undefined when $e^z - 1 = 0$. This equation is satisfied when $z = 2\pi i n$ for some integer $n \in \mathbb{Z}$. Around these points, we can determine the nature of the singularity using the limit:

$$\lim_{z\to 2\pi i n} \frac{z - 2\pi i n}{e^z - 1}.$$

By applying L'Hopital's rule to evaluate this limit, we find that each singularity at $z = 2\pi i n$ is indeed a simple pole. Thus, f has only simple poles.

(b) Which of the following equalities is **false**?

A) $\operatorname{res}_{2i}\left(\frac{1}{z^2+4}\right) = \frac{1}{4i}$

B) $\operatorname{res}_{0}\left(\frac{\sin(z)}{z^2}\right) = 1$

C) $\operatorname{res}_{0}\left(\frac{\cos(z)}{z^2}\right) = 0$

D) $\operatorname{res}_{1}\left(\frac{1}{z^5-1}\right) = \frac{1}{5!}$

Solution: Since $(1/(z^2+4))^{-1} = (z^2+4) = (z-2i)(z+2i)$ has a zero of order 1 in $2i$, we get that $1/(z^2+4)$ has a pole of order 1 at the same point. The residue is given by

$$\operatorname{res}_{2i} \frac{1}{z^2+4} = \lim_{z\to 2i} (z-2i)\frac{1}{z^2+4} = \frac{1}{4i}.$$

For the second function, taking advantage of the Taylor expansion of $\sin(z)$ at 0 we have that

$$\frac{\sin(z)}{z^2} = z^{-2}\sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z^{-1} + \sum_{k=1}^{+\infty} \frac{(-1)^k z^{2k-1}}{(2k+1)!},$$

showing at the same time that the pole at zero is of order 1 and the residue is

$$\operatorname{res}_0 \frac{\sin(z)}{z^2} = 1.$$

We argue similarly for the third function:

$$\frac{\cos(z)}{z^2} = z^{-2} \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{(2k)!} = z^{-2} + \sum_{k=1}^{+\infty} \frac{(-1)^k z^{2k-2}}{(2k)!},$$

and hence the pole is of order 2, and

$$\operatorname{res}_0 \frac{\cos(z)}{z^2} = 0.$$

Finally, $(1/(z^5-1))^{-1} = z^5 - 1$ has a zero of order 1 in 1, and therefore the pole of $1/(z^5-1)$ is also of order 1 by definition. The residue is

$$\operatorname{res}_1 \frac{1}{z^5-1} = \lim_{z\to 1} \frac{(z-1)}{z^5-1} = \lim_{z\to 1} \frac{1}{5z^4} = \frac{1}{5},$$

where we took advantage of Bernoulli-l'Hôpital's rule to compute the limit.

7.2. Schwarz reflection principle Let Ω be open, **connected**, and symmetric with respect to the x-axis (i.e. $z \mapsto \bar{z}$ preserves Ω), and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $L := \{z \in \Omega : \Im(z) = 0\}$. Note that L is non-empty. Prove that $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega$ if and only if f is real valued on L.

Hint: consider g to be the restriction of f to the upper half plane intersected with Ω. 'Reflect' g by imposing $g^*(z) := \overline{g(\bar{z})}$. Argue taking advantage of Morera's Theorem.

Solution: One direction is elementary: since $z \in L$ implies $\bar{z} = z$, the relation $\overline{f(z)} = f(\bar{z})$ gives on L that $\overline{f(z)} = f(z)$, and hence $2\Im(f(z)) = 0$, showing that f has real image on L. To prove the other direction suppose f real valued on L. Define the function

$$h(z) := \begin{cases} f(z), & \text{if } z \in \Omega, \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \text{if } z \in \Omega, \Im(z) < 0. \end{cases}$$

We claim that h is continuous. In fact, by construction of f we only have to check continuity approaching L, that is

$$\lim_{\substack{\Im(z)\to 0^+ \\ z\in\Omega}} h(z) = \lim_{\substack{\Im(z)\to 0^- \\ z\in\Omega}} h(z) \Leftrightarrow \lim_{\substack{\Im(z)\to 0^+ \\ z\in\Omega}} f(z) = \lim_{\substack{\Im(z)\to 0^- \\ z\in\Omega}} \overline{f(\bar{z})}.$$

This holds because by assumption $\overline{f(\bar{z})} = f(z)$ for $z \in L$ and the conjugation $w \mapsto \bar{w}$ is continuous. We prove now that h is holomorphic: by Morera's Theorem we have to check that $\int_T h\, dz = 0$ for every triangle $T \subset \Omega$. We split this into three cases:

– Type 1: $T \subset (\Omega \cap \{z : \Im(z) \geq 0\})$. In this case $h = f$ on T and it suffices to apply Goursat's Theorem for holomorphic functions.

– Type 2: $T \subset (\Omega \cap \{z : \Im(z) < 0\})$. In this case we can compute

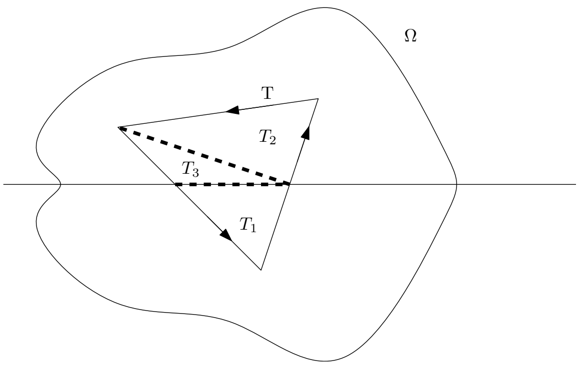
$$\int_T h(z)\, dz = \int_T \overline{f(\bar{z})}\, dz = \overline{\int_T f(\bar{z})\, \overline{dz}} = \overline{\int_{\bar{T}} f(z)\, dz} = \bar{0} = 0,$$

since $\bar{T}$ is a triangle in the upper halfplane, and therefore we can apply the result for Type 1.

– Generic type: T is a generic triangle in Ω. In this case one can check that there exist (at most) 3 oriented triangles T_1, T_2, and T_3, all of them of type 1 or 2 such that

$$\int_T h\, dz = \int_{T_1} h\, dz + \int_{T_2} h\, dz + \int_{T_3} h\, dz.$$

Hence, by applying the previous cases, we deduce finally $\int_T h\, dz = 0$.



To conclude, observe that we constructed h holomorphic in Ω that agrees with f when $\Im(z) \geq 0$. By the uniqueness of analytic extensions, we deduce that $f = h$, and hence $f(\bar{z}) = h(\bar{z}) = \overline{h(z)} = \overline{f(z)}$ as wished.

7.3. Dense image Show that the image of a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is *dense* in $\mathbb{C}$, that is: for every $z \in \mathbb{C}$ and $\varepsilon > 0$, there exists $w \in \mathbb{C}$ such that $|z - f(w)| < \varepsilon$.

Remark: In fact there is a theorem, called the little Picard Theorem, which asserts that $f(\mathbb{C})$ misses at most one single point of $\mathbb{C}$!

Solution: By contradiction suppose that there exists $z^* \in \mathbb{C}$ and $\varepsilon^* > 0$ such that $f(w) \notin \{z \in \mathbb{C} : |z - z^*| < \varepsilon^*\}$ for all $w \in \mathbb{C}$. Define the function

$$g(w) := \frac{1}{f(w) - z^*}.$$

By assumption $|f(w) - z^*| \geq \varepsilon^*$ and hence g is a well defined holomorphic function since the denominator is never zero. On the other side

$$|g(w)| = \frac{1}{|f(w) - z^*|} \leq \frac{1}{\varepsilon^*}, \quad \forall w \in \mathbb{C},$$

contradicting Liouville's Theorem (every holomorphic function on $\mathbb{C}$ is either constant or unbounded). Hence, for all $\varepsilon > 0$ and $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that $|f(w) - z| < \varepsilon$, proving the density of the image of f in $\mathbb{C}$.

7.4. Let $D = \{z \mid |z| < 1\}$ be the unit disk, and let $\overline{D}$ be its closure. Give an example of a continuous function $f : \overline{D} \to \mathbb{C}$ that is holomorphic on D, but does not have a holomorphic continuation on any domain in $\mathbb{C}$ containing $\overline{D}$.

Solution. We can define such a function f as a power series with radius of convergence 1 that converges everywhere on the boundary of D. For example, consider the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

This series converges for $|z| \leq 1$, making f continuous on $\overline{D}$ and holomorphic on D. Suppose there is a function F which is holomorphic on an open domain $\Omega \supset \overline{D}$. By compactness of $\overline{D}$ and the fact that $\mathbb{C} \setminus \Omega$ is closed, it follows that Ω includes some bigger disc $D_r = \{z \in \mathbb{C} \mid |z| < r\}$, $r > 1$ containing $\overline{D}$ and agreeing with f in D. Then for any $z \in D_r$, F has a power series expansion

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

This power series expansion is clearly also valid for $z \in D$. By the uniqueness of the coefficients in the power series expansions of holomoprhic functions, we have that $a_0 = 0$ and $a_n = 1/n^2$ for $n \geq 1$.

But this will mean f has a radius of convergence bigger than 1 which contradicts the fact that the power series defining f has a radius of convergence exactly equal to 1

7.5. Complex integrals Compute the following complex integrals taking advantage of the Residue Theorem.

(a)

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)} \, dz.$$

Solution: The poles of $f(z) = e^z/(z^2(z-1))$ are at 0 and 1, with multiplicity 2 and 1 respectively. We compute

$$\text{res}_0(f) = \lim_{z\to 0}(z^2 f(z))' = \lim_{z\to 0}\left(\frac{e^z}{(z-1)}\right)' = -2,$$

and

$$\text{res}_1(f) = \lim_{z\to 1}(z-1)f(z) = e.$$

By the Residue Theorem, since 0 and 1 are in the interior of the disc of radius 2 centered at the origin, we can compute

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)}\, dz = 2\pi i(e-2).$$

(b)

$$\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\, dz.$$

Solution: The poles of $f(z) = 1/(z^2(z^2-4))$ are at 0, $\sqrt{2}$ and $-\sqrt{2}$ with multiplicity 2, 1, 1 respectively. However, since only 0 belongs to the interior of the circumference of radius 1 centered at the origin, we get that

$$\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\, dz = 2\pi i\,\text{res}_0(f) = 2\pi i \lim_{z\to 0}(z^2 f(z))' = 2\pi i \lim_{z\to 0}\left(\frac{1}{(z^2-4)e^z}\right)' = \frac{\pi i}{2}.$$

(c)

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz.$$

Hint: Note that the function $\frac{1}{z\sin(1/z)}$ has infinitely many singularities accumulating at 0. Hence you cannot use the residue theorem directly. To go around this problem first prove

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz = \int_{|w|=2} \frac{1}{w\sin(w)}\, dw.$$

Solution: By parametrizing the contour as $t \mapsto e^{it}/2$ we get that

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz = \int_0^{2\pi} \frac{ie^{it}/2}{e^{it}/2\sin(2e^{-it})}\, dt = \int_{|w|=2} \frac{1}{w\sin(w)}\, dw$$

where we recognised $t \mapsto 2e^{-it}$ as the circle of radius 2 oriented in the *clockwise* direction (hence the change of sign). The only pole contained in $|w| \le 2$ is $z = 0$, and it is of order 2. We get

$$\int_{|w|=2} \frac{1}{w\sin(w)}\, dw = 2\pi i\,\text{res}_0\left(\frac{1}{w\sin(w)}\right) = 2\pi i \lim_{w\to 0}\left(\frac{w^2}{w\sin(w)}\right)' = 0.$$

(d)
$$\int_{|z|=5} \frac{1}{(z-i)(z+2)(z-4)}\,dz.$$

Solution: The poles of $f(z) = \frac{1}{(z-i)(z+2)(z-4)}$ are all of order 1, and equal to i, -2, and 4. The associated residues are

$$\begin{aligned}
\operatorname{res}_i f &= \frac{1}{(2+i)(i-4)}, \\
\operatorname{res}_{-2} f &= \frac{1}{6(2+i)}, \\
\operatorname{res}_4 f &= \frac{1}{6(4-i)}.
\end{aligned}$$

Since they are all in the interior of $\{|z|=5\}$, we have that

$$\int_{|z|=5} \frac{1}{(z-i)(z+2)(z-4)}\,dz = 2\pi i\left(\frac{1}{(2+i)(i-4)} + \frac{1}{6(2+i)} + \frac{1}{6(4-i)}\right),$$

7.6. The Gamma function Let $Z_- := \{0,-1,-2,\dots\}$ the set of all non-positive integers, and define for all $\tau \in \mathbb{R}$ the set $U_\tau := \{z \in \mathbb{C} : \Re(z) > \tau, z \notin Z_-\}$, and $U := \mathbb{C} \setminus Z_-$.

(a) Show that the function defined by the complex improper Riemann integral

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1}\,dt$$

is well defined for all $z \in U_1$. (Here $t^{z-1} = \exp((z-1)\log(t))$).

Solution: First of all, fix $z \in U_1$. Then, $\Re(z-1) > 0$ by definition of U_1, and therefore there exists $a > 0$ large enough (depending on $\Re(z)$) such that $\Re(z-1)\log(t) < t/2$ for all $t > a$ (this follows from the elementary observation $\lim_{s\to+\infty} \log(s)/s = 0$). Now for every $n > a$ one has that

$$\begin{aligned}
\left|\int_a^n e^{-t} t^{z-1}\,dt\right| &= \left|\int_a^n e^{-t} e^{(\Re(z-1)+i\Im(z-1))\log(t)}\,dt\right| \\
&= \left|\int_a^n e^{-t} e^{i\Im(z-1)\log(t)} e^{\Re(z-1)\log(t)}\,dt\right| \\
&\le \int_a^n e^{-t} |e^{i\Im(z-1)\log(t)}||e^{\Re(z-1)\log(t)}|\,dt = \int_a^n e^{-t} e^{\Re(z-1)\log(t)}\,dt \\
&\le \int_a^n e^{-t} e^{t/2}\,dt = [-2e^{-t/2}]_{t=a}^{t=n} = -2e^{-a/2} + 2e^{-n/2}.
\end{aligned}$$

On the other side, notice that on the interval $[0, a]$ the function $t \mapsto e^{-t+\Re(z-1)\ln(t)}$ is continuous, and therefore the integral $\int_0^a |e^{-t}t^{z-1}|\,dt =: \alpha$ is well defined. Hence, we conclude that the improper integral defining Γ converges absolutely:

$$\lim_{n\to+\infty}\int_0^n |e^{-t}t^{z-1}|\,dt = \int_0^a |e^{-t}t^{z-1}|\,dt + \lim_{n\to+\infty}\int_a^n |e^{-t}t^{z-1}|\,dt \leq \alpha - 2e^{-a/2} < +\infty$$

proving that $\Gamma(z)$ is well defined for all $z \in U_1$.

(b) Prove that Γ is holomorphic in U_1.

Hint: First show that the functions of the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ given by truncating the integral at height n ($\Gamma_n(z) = \int_0^n e^{-t}t^{z-1}\,dt$) are holomorphic. Then, show that $\Gamma_n \to \Gamma$ uniformly in all compact subsets of U_1.

Solution: Define the sequence

$$(\Gamma_n(z))_{n\in\mathbb{N}} = \int_0^n e^{-t}t^{z-1}\,dt.$$

We first prove that $z \mapsto \Gamma_n(z)$ is continuous: let $\varepsilon > 0$ and fix $w \in U_1$ and $n \in \mathbb{N}$. Since $z \mapsto t^{z-1}$ is continuous in U_1, there exists $\delta > 0$ such that for every $v \in \mathbb{C}$ such that $|w - v| < \delta$ one has that $|t^{w-1} - t^{v-1}| < \varepsilon/(1 - e^{-n})$ and $v \in U_1$. In this case we can perform the following estimate:

$$|\Gamma_n(w) - \Gamma_n(v)| \leq \int_0^n e^{-t}|t^{w-1} - t^{v-1}|\,dt < (1 - e^{-n})\varepsilon/(1 - e^{-n}) = \varepsilon,$$

proving the continuity of Γ_n in $w \in U_1$ arbitrary, and therefore in all U_1. By Morera's Theorem, we prove Γ_n holomorphic in U_1 by checking that $\int_T \Gamma_n(z)\,dz = 0$ for all triangle $T \subset U_1$. Now, observe that for such a given triangle

$$\int_T \Gamma_n(z)dz = \int_T\int_0^n e^{-t}t^{z-1}\,dt\,dz = \int_0^n\int_T e^{-t}t^{z-1}\,dz\,dt = \int_0^n 0\,dz = 0,$$

since $z \mapsto t^{z-1}$ is holomorphic for all $t > 0$, and we can interchange the integration because both T and $[0, n]$ are compact, and $(t, z) \mapsto e^{-t}t^{z-1}$ is continuous and hence uniformly bounded in $[0, n] \times T$. This shows that $(\Gamma_n)_{n\in\mathbb{N}}$ define a sequence of holomorphic functions on U_1. By taking advantage of Theorem 5.2 of last lecture, to show that Γ is holomorphic in U_1 is suffices to prove that $\Gamma_n \to \Gamma$ uniformly on every compact subset of U_1. Let $K \subset U_1$ be compact, and let $b = \max\{\Re(z-1) : z \in K\} > 0$. Let $N = N(b) > 0$ big enough so that $t/2 \geq b\log(t)$ for all $t > N$. Then, for all $z \in K$ and $n \geq N$ one has that

$$|\Gamma(z) - \Gamma_n(z)| \leq \int_n^{+\infty} e^{-t}e^{\Re(z-1)\log(t)}\,dt \leq \int_n^{+\infty} e^{-t}e^{b\log(t)}\,dt \leq 2e^{-2n},$$

which converges to zero uniformly in K.

(c) Show that $\Gamma(z+1) = z\Gamma(z)$ for all $z \in U_1$.

Solution: This follows by integration by parts:

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^z \, dt = [-e^{-t} t^z]_0^{+\infty} + \int_0^{+\infty} e^{-t} z t^{z-1} \, dt = z\Gamma(z).$$

(d) Deduce that Γ allows a unique holomorphic extension to U_0.

Solution: Define the function $\tilde{\Gamma}$ on U_0 by setting

$$\tilde{\Gamma}(z) = \frac{\Gamma(z+1)}{z}, \quad z \in U_0.$$

Since $z \in U_0$ implies $z + 1 \in U_1$ and $z \neq 0$, we deduce that $\tilde{\Gamma}$ is a well defined holomorphic function. On the other side, by the previous point $\tilde{\Gamma}$ coincides with Γ on U_1, showing that it is the unique analytic continuation of Γ from U_1 to U_0.

(e) Deduce that Γ allows a unique holomorphic extension to U.

Solution: We construct the extension on U inductively on $m \in \mathbb{N}_0$ over $U_{-m/2}$ preserving the property $\Gamma(z+1) = z\Gamma(z)$. The case $m = 0$ has been proved in the previous point. Supposing now Γ extended in $U_{-m/2}$, then

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad z \in U_{-(m+1)/2}.$$

defines again an analytic extension, agreeing with the previous one on the set $U_{-m/2}$. The property $\Gamma(z+1) = z\Gamma(z)$ is ensured by the very definition, an the uniqueness by the properties of analytic functions.