8.1. MC Questions

(a) Consider the real integral $I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$. This can be computed using Cauchy's Residue Theorem. Which of the following is false?

A) Let $\gamma(R)$ be a closed semicircle of radius $R > 1$ (centered at the origin) in the lower half of the complex plane, which is traced counterclockwise. Then $I = \lim_{R\to\infty} \oint_{\gamma(R)} \frac{1}{1+z^2}\, dz$.

B) Let $\gamma(R)$ be a closed semicircle of radius $R > 1$(centered at the origin) in the lower half of the complex plane, traced clockwise. Then $I = \lim_{R\to\infty} \oint_{\gamma(R)} \frac{1}{1+z^2}\, dz$.

C) $I = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, z = i\right)$

D) $I = -2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, z = -i\right)$

Solution: A) is false as it results in $\int_{\infty}^{-\infty} \frac{1}{1+x^2}\, dx = -I$, given the orientation of the semicircle. B) is correct C) and D) are also true, as

$$\begin{aligned}
\operatorname{Res}_{z=-i}\left(\frac{1}{1+z^2}\right) &= \lim_{z\to -i}(z+i)\frac{1}{1+z^2} \\
&= \lim_{z\to -i}(z+i)\frac{1}{(z+i)(z-i)} \\
&= \lim_{z\to -i}\frac{1}{z-i} \\
&= \frac{1}{-i-i} \\
&= \frac{1}{-2i} = -\frac{1}{2i},
\end{aligned}$$

and similarly

$$\begin{aligned}
\operatorname{Res}_{z=i}\left(\frac{1}{1+z^2}\right) &= \lim_{z\to i}(z-i)\frac{1}{1+z^2} \\
&= \lim_{z\to i}(z-i)\frac{1}{(z+i)(z-i)} \\
&= \lim_{z\to i}\frac{1}{z+i} \\
&= \frac{1}{i+i} \\
&= \frac{1}{2i}.
\end{aligned}$$

(b) Let $f(z) = \frac{e^z}{(z-1)^3}$. What is the order of the pole of $f(z)$ at $z = 1$?

A) 0 (no pole)

B) 1

C) 2

D) 3

Solution: The order of the pole is the highest power of $(z-1)$ in the denominator that cannot be canceled by any term in the numerator. In this case the factor e^z is analytic (holomorphic) at $z=1$, so it does not affect the order of the pole. Hence, the term $(z-1)^3$ in the denominator determines the order of the pole. Since there is no cancellation, the pole at $z=1$ is of order 3.

8.2. Poles at infinity Let $f:\mathbb{C}\to\mathbb{C}$ be holomorphic. We say that f *has a pole at infinity of order* $N\in\mathbb{N}$ if the function $g(z):=f(1/z)$ has a pole of order N at the origin in the usual sense. Prove that if $f:\mathbb{C}\to\mathbb{C}$ has a pole of order $N\in\mathbb{N}$ at infinity, then it has to be a polynomial of degree $N\in\mathbb{N}$.

Solution: Since f is holomorphic, the expansion

$$f(z)=\sum_{k=0}^{+\infty}a_kz^k,$$

converges in any ball centered in 0. If f has a pole at infinity of order N, by definition for $z\neq 0$

$$g(z)=f(1/z)=\sum_{k=0}^{+\infty}a_kz^{-k}$$

has a pole of order N at zero, which means

$$z^Ng(z)=\sum_{k=0}^{+\infty}a_kz^{N-k}$$

is holomorphic in a neighbourhood of 0. This implies that $a_k=0$ for every $k>N$ and $a_N\neq 0$, proving that

$$f(z)=\sum_{k=0}^{N}a_kz^k,$$

that is, f is a polynomial of degree N as claimed.

8.3. Meromorphic functions For $z\in\mathbb{C}$ such that $\sin(z)\neq 0$ define the map

$$\text{cotan}(z)=\frac{\cos(z)}{\sin(z)}.$$

(a) Show that cotan is meromorphic in $\mathbb{C}$, determine its poles and their residues.

Solution: Notice that $\sin(z) = 0$ if and only if $z = k\pi$ for some $k \in \mathbb{Z}$, and therefore cotan is holomorphic in the open domain $\mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}$. Since $\{k\pi : k \in \mathbb{Z}\}$ has no accumulation points in $\mathbb{C}$, in order to prove that cotan is meromorphic we are left to show that its singularities are in fact poles. By definition $z = k\pi$ is a pole of cotan if it is a zero of $1/\operatorname{cotan} = \tan$, which is the case since $\cos(k\pi) = (-1)^k$. To compute the residues we notice that all poles have order one since the zeros of tan have order one:

$$\tan(z)' \,|_{z=k\pi} = \left.\frac{1}{\cos^2(z)}\right|_{z=k\pi} = 1 \neq 0.$$

Therefore,

$$\operatorname{res}_{k\pi} \operatorname{cotan} = \lim_{z\to k\pi} (z - k\pi)\frac{\cos(z)}{\sin(z)} = (-1)^k \lim_{z\to k\pi} \frac{(z - k\pi)}{\sin(z)} = (-1)^{2k} = 1,$$

since

$$\lim_{z\to k\pi} \frac{\sin(z)}{z - k\pi} = \lim_{z\to k\pi} \frac{\cos(k\pi)(z - k\pi) + O(|z - k\pi|^2)}{(z - k\pi)} = (-1)^k,$$

by expanding $\sin(z)$ around $k\pi$ at the first order.

(b) Let $w \in \mathbb{C} \setminus \mathbb{Z}$ and define

$$f(z) = \frac{\pi \operatorname{cotan}(\pi z)}{(z + w)^2}.$$

Show that f is meromorphic in $\mathbb{C}$, determine its poles and their residues.

Solution: Since $z \mapsto \operatorname{cotan}(\pi z)$ and $z \mapsto 1/(z + w)^2$ are meromorphic, f is also meromorphic by being the multiplication of the two. Thanks to the previous point, the set of poles of f are $\mathbb{Z} \cup \{-w\}$. The residues at $k \in \mathbb{Z}$ are given by

$$\begin{aligned}\operatorname{res}_k f &= \frac{1}{(k + w)^2} \lim_{z\to k} \frac{\pi(z - k)\cos(\pi z)}{\sin(\pi z)} \\ &= \frac{(-1)^k}{(k + w)^2} \lim_{z\to k} \frac{\pi(z - k)}{\pi\cos(\pi z)(z - k) + O(|z - k|^2)} = \frac{1}{(k + w)^2}.\end{aligned}$$

To compute the order of $-w$ observe that $\operatorname{cotan}(\pi z)$ is equal to zero if and only if $z = k + 1/2$, $k \in \mathbb{Z}$. Hence, if $-w = k + 1/2$, then the pole has order 1 and

$$\begin{aligned}\operatorname{res}_{-w} f &= \lim_{z\to -w} (z + w)f(z) = \lim_{z\to -w} \frac{\pi\cos(\pi z)}{\sin(\pi z)(z + w)} \\ &= \lim_{z\to -w} \frac{\pi(-\pi\sin(-\pi w)(z + w) + O(|z + w|^2))}{\sin(-\pi w)(z + w)} = -\pi^2 = -\frac{\pi^2}{\sin(\pi w)^2}.\end{aligned}$$

If $-w \neq k + 1/2$, then the pole has order 2, and

$$\operatorname{res}_{-w} f = \lim_{z \to -w} \Big((z+w)^2 f(z) \Big)' = \lim_{z \to -w} (\pi \operatorname{cotan}(\pi z))' = -\frac{\pi^2}{\sin^2(\pi w)^2}.$$

(c) Compute for every integer $n \geq 1$ such that $|w| < n$ the line integral

$$\int_{\gamma_n} f \, dz,$$

where γ_n is the circle or radius $n + 1/2$ centered at the origin and positively oriented.

Solution: Observe that γ_n does not intersect with any of the poles of f and contains the pole $-w$. We can therefore apply the Residue Theorem obtaining

$$\int_{\gamma_n} f \, dz = 2\pi i \Big(\operatorname{res}_{-w} f + \sum_{k=-n}^{n} \operatorname{res}_k f \Big) = 2\pi i \Big(-\frac{\pi^2}{\sin^2(\pi w)} + \sum_{k=-n}^{n} \frac{1}{(w+k)^2} \Big).$$

(d) Deduce that

$$\lim_{n \to +\infty} \sum_{k=-n}^{n} \frac{1}{(w+k)^2} = \frac{\pi^2}{\sin(\pi w)^2}.$$

Solution: From the previous point, since

$$\sum_{k=-k}^{k} \frac{1}{(w+k)^2} = \frac{1}{2\pi i} \int_{\gamma_n} f \, dz + \frac{\pi^2}{\sin(\pi w)^2},$$

it suffices to prove that the integral on γ_n vanishes as $n \to +\infty$. Observe that

$$|\operatorname{cotan}(\pi z)| = \left| i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{1 + e^{2i\pi z}}{e^{2i\pi z} - 1} \right| \leq \frac{1 + |e^{2i\pi z}|}{||e^{2\pi i z}| - 1|} = \frac{1 + e^{-2\pi \Im(z)}}{|e^{-2\pi \Im(z)} - 1|}.$$

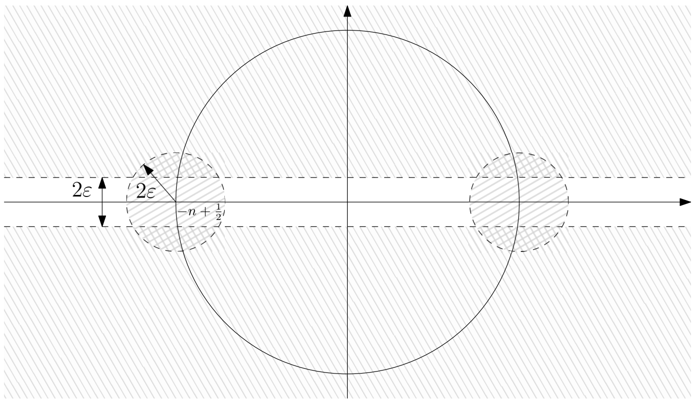
taking advantage of the reverse triangle inequality $|w - u| \geq ||w| - |u||$. Hence, for every $\varepsilon > 0$ the function $\operatorname{cotan}(\pi z)$ is uniformly bounded in the half plane $\{\Im(z) > \varepsilon\}$ by $C(\varepsilon) = 2/(1 - e^{-2\pi\varepsilon}) > 0$. The same holds true in the half plane $\{\Im(z) < -\varepsilon\}$ since $\operatorname{cotan}(-\pi z) = -\operatorname{cotan}(\pi z)$. Let now $n \in \mathbb{Z}$ and consider a point in a 2ε-neighbourhood of $n\pi + 1/2$, i.e. $u = n + 1/2 + \tau$, for $\tau \in \mathbb{C}$, $|\tau| < 2\varepsilon$. Then, taking advantage of the classical trigonometric identities we can compute

$$\begin{aligned} \operatorname{cotan}(\pi u) = \frac{\cos(u)}{\sin(u)} &= \frac{\cos(\pi(n+1/2))\cos(\pi\tau) - \sin(\pi(n+1/2))\sin(\pi\tau)}{\sin(\pi(n+1/2))\cos(\pi\tau) + \cos(\pi(n+1/2))\sin(\pi\tau)} \\ &= -\frac{\sin(\pi\tau)}{\cos(\pi\tau)} = -\tan(\pi\tau), \end{aligned}$$

whose norm is controlled uniformly in n by some constant $C' = C'(\varepsilon) > 0$ provided $\varepsilon < 1/2$. Hence, fixing $\varepsilon < 1/2$ and covering every circle γ_n with two half planes and two balls centered in the intersection of the real axis we can estimate

$$\left|\int_{\gamma_n} f\,dx\right| \leq \int_{\gamma_n} |f|\,dz \leq \text{length}(\gamma_n)\frac{\pi \max\{C, C'\}}{(n+1/2-|w|)^2}$$
$$= \frac{2\pi^2(n+1/2)\max\{C,C'\}}{(n+1/2-|w|)^2} \to 0,$$

as $n \to +\infty$, as wished.



8.4. $\star$ **Real integrals** Compute the following real integrals taking advantage of the Residue Theorem[1].

(a)

$$\int_0^{2\pi} \frac{1}{1+\sin^2(t)}\,dt$$

Solution: We start by noting that on the unit circle we can express $\sin t$ in terms of z:

$$\sin t = \frac{z - z^{-1}}{2i}$$

Thus,

$$\sin^2 t = \left(\frac{z-z^{-1}}{2i}\right)^2 = \frac{(z-z^{-1})^2}{-4}$$

[1]Recall: $\{z_1,\dots,z_N\} \subset \Omega$ poles and $f : \Omega \setminus \{z_1,\dots,z_N\} \to \mathbb{C}$ holomorphic. Then if $\{z_1,\dots,z_N\}$ are inside a simple closed curve γ in Ω, then $\int_\gamma f\,dz = 2\pi i \sum_{j=1}^N \text{res}_{z_j}(f)$.

Simplify the numerator:

$$(z - z^{-1})^2 = z^2 - 2 + z^{-2}$$

So,

$$\sin^2 t = \frac{z^2 + z^{-2} - 2}{-4}$$

Therefore,

$$1 + \sin^2 t = 1 - \frac{z^2 + z^{-2} - 2}{4} = \frac{6 - z^2 - z^{-2}}{4}$$

Using $z = e^{it}$, we have $dt = \frac{dz}{iz}$. Substituting into I, we get:

$$I = \oint_C \frac{4}{6 - z^2 - z^{-2}} \cdot \frac{dz}{iz}$$

Simplify:

$$I = \frac{1}{i} \oint_C \frac{4z}{6z^2 - z^4 - 1}\, dz$$

Set $D(z) = -(z^4 - 6z^2 + 1)$. Factor $D(z)$ by finding the roots of $z^4 - 6z^2 + 1 = 0$:

$$z^2 = 3 \pm 2\sqrt{2} \implies z = \pm\sqrt{3 + 2\sqrt{2}}, \quad z = \pm\sqrt{3 - 2\sqrt{2}}$$

Simplify the radicals:

$$\sqrt{3 + 2\sqrt{2}} = \sqrt{(\sqrt{2} + 1)^2} = \sqrt{2} + 1$$

$$\sqrt{3 - 2\sqrt{2}} = \sqrt{(\sqrt{2} - 1)^2} = \sqrt{2} - 1$$

Thus,

$$D(z) = -\left(z - (\sqrt{2} + 1)\right)\left(z + (\sqrt{2} + 1)\right)\left(z - (\sqrt{2} - 1)\right)\left(z + (\sqrt{2} - 1)\right)$$

Write the integrand as:

$$\frac{z}{(z^2 - a)(z^2 - b)} = \frac{Cz}{z^2 - a} + \frac{Dz}{z^2 - b}$$

where $a = (\sqrt{2} + 1)^2$ and $b = (\sqrt{2} - 1)^2$.

Solving for C and D yields:

$$C = \frac{1}{4\sqrt{2}}, \quad D = -\frac{1}{4\sqrt{2}}$$

So,

$$I = \frac{i}{\sqrt{2}} \oint_C \left(\frac{z}{z^2 - a} - \frac{z}{z^2 - b} \right) dz$$

Only the poles inside the unit circle ($|z| < 1$) contribute to the integral, which are $z = \pm(\sqrt{2} - 1)$. The residues at these poles are:

$$\text{Res}_{z=\pm(\sqrt{2}-1)} \left(\frac{z}{z^2 - b} \right) = \frac{1}{2}$$

The total residue inside the contour is 1.

Using the residue theorem:

$$I = \frac{i}{\sqrt{2}} \cdot (-2\pi i) \cdot (1) = \sqrt{2}\pi$$

(b)

$$\int_0^\infty \frac{\cos(x)}{x^2 + 1} \, dx$$

Solution: Consider the function:

$$f(z) = \frac{e^{iz}}{z^2 + 1}.$$

The function $f(z)$ has simple poles at $z = \pm i$. We consider the contour consisting of the real axis from $-R$ to R, and a semicircle in the upper half-plane of radius R.

By the residue theorem, the value of the contour integral is:

$$\int_{\text{contour}} f(z) \, dz = 2\pi i \cdot \text{Res}(f, i),$$

where the residue of $f(z)$ at $z = i$ is:

$$\text{Res}(f, i) = \lim_{z \to i} (z - i) \frac{e^{iz}}{z^2 + 1} = \frac{e^{-1}}{2i}.$$

Thus, the contour integral evaluates to:

$$\int_{\text{contour}} f(z)\,dz = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}.$$

The integral along the real axis can be expressed as:

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1}\,dx.$$

Since $\sin(x)$ is an odd function and $\frac{\sin(x)}{x^2+1}$ is odd as well, the integral of the sine term vanishes:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1}\,dx = 0.$$

This leaves:

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \pi e^{-1}.$$

Since $\cos(x)$ is an even function, the integral over $[0,\infty)$ is half of the integral over $(-\infty,\infty)$:

$$\int_{0}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx.$$

Thus:

$$\int_{0}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \frac{\pi e^{-1}}{2}.$$

8.5. Quotient of holomorphic functions Let f, g be two non-constant holomorphic functions on $\mathbb{C}$. Show that if $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$, then there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$.

Solution: Let $h(z) = \frac{f(z)}{g(z)}$. Since g is not constant, it has isolated zeros, and hence h has isolated singularities. By assumption $|h(z)| \le 1$ for all z such that $g(z) \neq 0$. In particular, h is bounded in a neighbourhood of the zeros of g, and therefore we extend h to an entire function on the whole complex plane taking advantage of the Riemann continuation Theorem (cf Exercise 5.5). By continuity, the extension h is also uniformly bounded by 1, and therefore by Liouville's Theorem it has to be equal to some constant $c \in \mathbb{C}$. This proves that for all $z \in \mathbb{C}$ such that $g(z) \neq 0$ one has that $f(z) = cg(z)$. If $g(z) = 0$ the assumption $|f(z)| \leq |g(z)| = 0$ concludes the argument: $f(z) = 0 = cg(z)$.

8.6. $\star$ Let $P(z)$ be a complex polynomial of degree n and $R > 0$ so large that $P(z)$ does not vanish in $\{z : |z| \geq R\}$. Let γ be the path with $\gamma(t) = Re^{it}$, with $0 \leq t \leq 2\pi$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{P'(z)}{P(z)} dz = n.$$

Solution: Let $P(z) = \prod_k (z - a_k)^{m_k}$ with $a_k \in \mathbb{C}$ and $\sum_k m_k = n$. Then using the product rule for differentiation it is easy to see that

$$\frac{P'(z)}{P(z)} = \sum_k \frac{m_k}{z - a_k}.$$

Since all the zeroes, a_k, of $P(z)$ lie in the region $\{z : |z| < R\}$ we have

$$\frac{1}{2\pi i} \int_\gamma \frac{P'(z)}{P(z)} dz = \frac{1}{2\pi i} \sum_k m_k \int_\gamma \frac{1}{z - a_k} dz = \frac{1}{2\pi i} \sum_k m_k (2\pi i) = n$$