9.1. MC Questions

(a) Suppose $f(z) = z^3 + 3z + 2$ and $g(z) = z^3 + 2$. What's the number of zeros of $f(z) + g(z)$ inside $|z| = 2$?

A) 0
C) 1
B) 6
D) 3

Solution: Choose:

$$u(z) = 2z^3, \quad v(z) = 3z + 4,$$

and consider the contour $|z| = 2$.

On $|z| = 2$:

- For $u(z) = 2z^3$, we calculate:

$$|u(z)| = |2z^3| = 2|z|^3 = 2(2^3) = 16.$$

- For $v(z) = 3z + 4$, we use the triangle inequality:

$$|v(z)| \leq |3z| + |4| = 3|z| + 4 = 3(2) + 4 = 10.$$

Thus, Rouche's theorem can be directly applied. The function $u(z) = 2z^3$ has exactly 3 zeros (counting multiplicities) inside $|z| = 2$. Since $u(z)$ and $u(z) + v(z) = h(z)$ have the same number of zeros inside $|z| = 2$, the number of zeros of $h(z)$ is also **3**.

(b) Let $f(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function on $\mathbb{C}$. Which of the following conditions is both necessary **and** sufficient for $f(z)$ to be a rational function?

A) $f(z)$ has no singularities on $\mathbb{C}$.

B) $f(z)$ is holomorphic everywhere except for a finite number of poles.

C) $f(z)$ has at most a pole at infinity and at most finitely many poles in $\mathbb{C}$.

D) $f(z)$ has finitely many singularities.

Solution:

- A) is incorrect because having **no singularities** implies that $f(z)$ is an *entire function*, not a rational function. Entire functions are not necessarily rational (e.g., e^z is entire but not rational).

- B) is *necessary* but not *sufficient*. For example $\frac{e^z}{z}$ has finitely many poles but is not a rational function.
- C) is the correct answer. The conditions in C) imply that we can extend the function f to a function $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is meromorphic on $\hat{\mathbb{C}}$. Using exercise 9.3(c) we conclude that $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function, and hence so is f. It is clear that of f is a rational function then it has the stated properties.
- D) is incorrect. For example $e^{1/z}$ has a singularity only at $z = 0$ (which is an essential singularity) and it is not a rational function.

9.2. Laurent Series A *Laurent series* centered at $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n\in\mathbb{Z}} a_n(z - z_0)^n = \dots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

where $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$. We define ρ_0, $\rho_I \in [0, +\infty]$ the *outer* and *inner* radius of convergence as

$$\rho_0 := \left(\limsup_{n\to+\infty} |a_n|^{1/n}\right)^{-1}, \qquad \rho_I := \limsup_{n\to+\infty} |a_{-n}|^{1/n}.$$

If $\rho_I < \rho_0$, we define the *annulus of convergence* as

$$\mathcal{A}(z_0, \rho_I, \rho_0) := \{z \in \mathbb{C} : \rho_I < |z - z_0| < \rho_0\},$$

with the convention $\mathcal{A}(z_0, \rho_I, +\infty) = \{z \in \mathbb{C} : \rho_I < |z - z_0|\}$, so that in particular $\mathcal{A}(z_0, 0, +\infty) = \mathbb{C} \setminus \{z_0\}$.

(a) Show that if $\rho_0 > 0$, then the series

$$f_0(z) := \sum_{n=0}^{+\infty} a_n(z - z_0)^n, \quad z \in \mathcal{D}_0(z_0, \rho_0) := \{z \in \mathbb{C} : |z - z_0| < \rho_0\},$$

converges absolutely and uniformly on compact sets. Show that if $\rho_I < +\infty$, then the series

$$f_I(z) := \sum_{n=1}^{+\infty} a_{-n}(z - z_0)^{-n}, \quad z \in \mathcal{D}_I(z_0, \rho_I) := \{z \in \mathbb{C} : \rho_I < |z - z_0|\},$$

converges absolutely and uniformly on compact sets.

Solution: In the case $\rho_0 > 0$, notice that f_0 and ρ_0 coincide with a Taylor expansion in z_0 and the radius of convergence of its associated power series. We know that the series defining f_0 converges absolutely and uniformly on compact subsets of $\mathcal{D}_0(z_0, \rho_0)$

(by Theorem 2.5 in the Lecture Notes). For the case $\rho_I < +\infty$, consider first the power series

$$g_I(\zeta) = \sum_{n=1}^{+\infty} a_{-n}\zeta^n.$$

Then, by the same argument as in the previous case, we know that g_I converges absolutely and uniformly on compact subset in $D(0, 1/\rho_I)$, the ball centered at 0 and of radius $(\limsup_{n\to+\infty}|a_{-n}|^{1/n})^{-1} = 1/\rho_I$. Consider the change of variable $\zeta = (z-z_0)^{-1}$. Now, the map $F(z) = (z-z_0)^{-1}$ sends $\mathcal{D}_I(z_0, \rho_I)$ to $D(0, 1/\rho_I) \setminus \{0\}$ continuously, and therefore it sends compact subsets of $\mathcal{D}_I(z_0, \rho_I)$ to compact subsets of $D(0, 1/\rho_I) \setminus \{0\}$. From the relation $f_I = g_I \circ F$ we deduce that f_I also converges uniformly on compact subsets in $\mathcal{D}_I(z_0, \rho_I)$ as wished.

(b) Show that a Laurent series is divergent for any z satisfying $|z - z_0| > \rho_0$ or $|z - z_0| < \rho_I$.

Solution: The argument is similar to point (a): if $|z - z_0| > \rho_0$ the series $f_0(z)$ diverges, again by Theorem 2.5 in the Lecture Notes. The same hold for $g_I(\zeta)$ when $|\zeta| = |z - z_0|^{-1} > 1/\rho_I$, and hence for $f_I(z)$ when $|z - z_0| < \rho_I$. Since $f = f_0 + f_I$ we conclude that $f(z)$ diverges if $|z - z_0| < \rho_I$ or $|z - z_0| > \rho_0$ as wished.

(c) Deduce that the full Laurent series

$$f(z) := \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$$

defines an analytic function in $\mathcal{A}(z_0, \rho_I, \rho_0)$, and its coefficients are related to f by the formula

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz,$$

for any $n \in \mathbb{Z}$ and $r \in (\rho_I, \rho_0)$.

Solution: Since $f = f_I + f_0$ and f_I is analytic in $\mathcal{D}_I(z_0, \rho_I)$ and f_0 is analytic in $\mathcal{D}_0(z_0, \rho_0)$ by point (a), we deduce that f is analytic in $\mathcal{A}(z_0, \rho_I, \rho_0) = \mathcal{D}_I(z_0, \rho_0) \cap \mathcal{D}_0(z_0, \rho_I)$. Let $r \in (\rho_I, \rho_0)$ and $\varepsilon > 0$ small enough so that $K = \overline{\mathcal{A}(z_0, r - \varepsilon, r + \varepsilon)} \subset \mathcal{A}(z_0, \rho_I, \rho_0)$. Since f converges absolutely an uniformly on the compact set K, we have that

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \sum_{k\in\mathbb{Z}} a_k \frac{1}{2\pi i}\int_{|z-z_0|=r} (z-z_0)^{k-(n+1)}\, dz = a_n,$$

where we exchanged sum and integration by Fubini thanks to the uniform convergence of the series defining f in the compact set K.

9.3. Meromorphic functions Recall the definition of $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

(a) Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Show that f has at most countably many poles.

Solution: Since by definition the poles of a meromorphic function cannot have limit points, any compact subset of $\mathbb{C}$ contains at most finitely many poles. Since every open set Ω in $\mathbb{C}$ is a union of countably many compact sets (for instance, $\mathbb{C} = \bigcup_{n=1}^{+\infty} \{z \in \mathbb{C} : |z| \leq n\}$), it follows that the set of poles of f is at most countable.

(b) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be meromorphic on $\hat{\mathbb{C}}$. Show that f has at most finitely many poles.

Solution: There exists $R > 0$ such that f is holomorphic for every $|z| > R$. Hence, the poles of f are contained in the compact set $\{|z| \leq R\}$ with the possible exception of ∞. Again, since by definition there is no accumulation point, the number of poles must be finite.

(c) Deduce that if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic on $\hat{\mathbb{C}}$, than it is a rational function.

Solution: By point (b) we know that the zeros of f in $\mathbb{C}$ are finite, and we can therefore denote them $\{z_1, \ldots, z_N\}$ with respective order $\{n_1, \ldots, n_N\}$. For each $k \in \{1, \ldots, N\}$ we can express f in a neighbourhood of z_k as

$$f(z) = \sum_{n=1}^{n_k} \frac{a_{-n}^k}{(z - z_k)^n} + \sum_{n=0}^{+\infty} a_n^k (z - z_k)^n = f_k(z) + g_k(z),$$

for coefficients $(a_n^k)_{n \geq -n_k}$, where f_k is the principal part of f at z_k, and g_k is holomorphic in a neighbourhood of z_k. Similarly,

$$f(1/z) = f_\infty(z) + g_\infty(z),$$

where g_∞ is holomorphic in a neighbourhood of the origin, and f_∞ is the principal part of $f(1/z)$ at zero. Define now $C(z) = f(z) - f_\infty(1/z) - \sum_{k=1}^N f_k(z)$. Notice that since we removed the principal parts of f at each z_k in the definition of $C(z)$, we deduce that $\{z_1, \ldots, z_N\}$ are removable singularities of $C(z)$. The same holds for the possible pole at ∞ since $C(1/z)$ is bounded in a neighbourhood of zero, and therefore $C(z)$ is bounded in $\mathbb{C}$. Hence, by Liouville's Theorem, $C(z) \equiv c \in \mathbb{C}$ is constant, and therefore $f(z) = c + f_\infty(1/z) + \sum_{k=1}^N f_k(z)$ is rational, as claimed.

9.4. Generalization of the Argument Principle

(a) Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$ and $r > 0$ such that $\bar{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset \Omega$. Suppose that $f : \Omega \to \mathbb{C}$ is homolorphic and that $f(z) \neq 0$ on the circle

$\partial D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. Show that for any holomorphic function $\varphi : \Omega \to \mathbb{C}$ we have that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'}{f} \varphi \, dz = \sum_{w \in D(z_0,r): f(w)=0} (\operatorname{ord}_w f) \varphi(w).$$

Solution: Let w be a zero of f of order n. Then, there exists g holomorphic and non-vanishing such that $f(z) = (z - w)^n g(z)$. From

$$\frac{f'(z)}{f(z)} \varphi(z) = \frac{n}{z - w} g(z) \varphi(z) + \frac{g'(z)}{g(z)} \varphi(z)$$

we deduce that if $\varphi(w) = 0$, then w is not a zero of $f'\varphi/f$, and hence $\operatorname{ord}_w(f'\varphi/f) = 0 = (\operatorname{ord}_w f)\varphi(w)$. On the other side, if $\varphi(w) \neq 0$, then w is pole of order one of $f'\varphi/f$ with residue

$$\operatorname{res}_w(f'\varphi/f) = \lim_{z \to w} (ng(z)\varphi(z) + (z - w)g'(z)\varphi(z)/g(z)) = ng(w)\varphi(w) = (\operatorname{ord}_w f)\varphi(w).$$

We apply the Residue Theorem to conclude:

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'}{f} \varphi \, dz = \sum_{w \text{ pole in } |z-z_0|<r} \operatorname{res}_w \left(\frac{f'\varphi}{f} \right) = \sum_{w \in D(z_0,r): f(w)=0} (\operatorname{ord}_w f) \varphi(w).$$

(b) Compute

$$\int_{|z|=2} \frac{z e^{z^3+1}}{z^2 + 1} dz$$

Solution: The integral can be rewritten as:

$$\int_{|z|=2} \frac{z e^{z^3+1}}{z^2 + 1} \, dz = \int_{|z|=2} \frac{f'(z)}{f(z)} \cdot \varphi(z) \, dz,$$

where:

$$f(z) = z^2 + 1, \quad f'(z) = 2z, \quad \text{and} \quad \varphi(z) = \frac{e^{z^3+1}}{2}.$$

The zeros of $f(z) = z^2 + 1$ are $w = i$ and $w = -i$, each with $\operatorname{ord}_w f = 1$.

We compute:

$$\varphi(z) = \frac{e^{z^3+1}}{2}.$$

At $w = i$:

$$\varphi(i) = \frac{e^{i^3+1}}{2} = \frac{e^{-i+1}}{2}.$$

At $w = -i$:

$$\varphi(-i) = \frac{e^{(-i)^3+1}}{2} = \frac{e^{i+1}}{2}.$$

The formula gives:

$$\int_{|z|=2} \frac{ze^{z^3+1}}{z^2+1}\, dz = 2\pi i \cdot \sum_{w=i,-i} (\operatorname{ord}_w f)\varphi(w).$$

Substitute $\varphi(w)$ and $\operatorname{ord}_w f = 1$:

$$\int_{|z|=2} \frac{ze^{z^3+1}}{z^2+1}\, dz = 2\pi i \cdot (\varphi(i) + \varphi(-i)).$$

Substitute the values of $\varphi(i)$ and $\varphi(-i)$:

$$\varphi(i) + \varphi(-i) = \frac{e^{-i+1}}{2} + \frac{e^{i+1}}{2}.$$

Simplify:

$$\varphi(i) + \varphi(-i) = \frac{1}{2}\left(e^{-i+1} + e^{i+1}\right).$$

Using Euler's formula:

$$e^{-i+1} = e \cdot (\cos(1) - i\sin(1)), \quad e^{i+1} = e \cdot (\cos(1) + i\sin(1)).$$

Add:

$$e^{-i+1} + e^{i+1} = 2e\cos(1).$$

Thus:

$$\varphi(i) + \varphi(-i) = \frac{1}{2} \cdot 2e\cos(1) = e\cos(1).$$

The integral is:

$$\int_{|z|=2} \frac{ze^{z^3+1}}{z^2+1}\, dz = 2\pi i \cdot e\cos(1).$$

9.5. Application of Rouché Theorem Let $f(z)$ be a holomorphic function inside the unit disk $|z| < 1$, with the Taylor series expansion:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Suppose $f(z)$ is continuous on the closed unit disk and that it has exactly m zeros (counted with multiplicity) inside $|z| < 1$. Prove that:

$$\min_{|z|=1} |f(z)| \leq |c_0| + |c_1| + \cdots + |c_m|.$$

Solution: Assume, towards a contradiction, that

$$\min_{|z|=1} |f(z)| > |c_0| + |c_1| + \cdots + |c_m|.$$

By continuity of $f(z)$, this inequality also holds on circles of radius $1 - \epsilon$ for all sufficiently small $\epsilon > 0$, i.e.,

$$\min_{|z|=1-\epsilon} |f(z)| > |c_0| + |c_1| + \cdots + |c_m|.$$

Consider the polynomial $p(z)$ of degree m, consisting of the first $m + 1$ terms of the Taylor series of $f(z)$:

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m.$$

By assumption, $p(z)$ is not identically zero. If $p(z)$ were identically zero, then $f(z)$ would have at least $m+1$ zeros inside the open unit disk, contradicting the assumption that $f(z)$ has exactly m zeros.

Since $p(z)$ is a polynomial of degree m, it has at most m zeros. For $\epsilon > 0$ small enough, the open disk of radius $1 - \epsilon$ contains all the zeros of $p(z)$ that are within the open unit disk $|z| < 1$.

By continuity and the assumption that $\min_{|z|=1} |f(z)| > |c_0| + |c_1| + \cdots + |c_m|$, it follows that

$$|f(z)| > |c_0| + |c_1| + \cdots + |c_m| \geq |p(z)|,$$

on $|z| = 1 - \epsilon$. Therefore, $|f(z)| > |p(z)|$ on the circle $|z| = 1 - \epsilon$.

We apply Rouché's theorem: if $|f(z)| > |p(z)|$ on $|z| = 1 - \epsilon$, then $f(z)$ and $f(z) - p(z)$ have the same number of zeros (counted with multiplicity) inside the disk $|z| < 1 - \epsilon$.

Since $f(z) - p(z)$ involves higher-order terms starting from $c_{m+1}z^{m+1}$, this implies that $f(z) - p(z)$ has at least $m + 1$ zeros. Thus, $f(z)$ must also have at least $m + 1$ zeros in $|z| < 1 - \epsilon$. However, this contradicts the assumption that $f(z)$ has exactly m zeros in $|z| < 1$.

The assumption that $\min_{|z|=1} |f(z)| > |c_0| + |c_1| + \cdots + |c_m|$ leads to a contradiction. Therefore, we conclude that

$$\min_{|z|=1} |f(z)| \leq |c_0| + |c_1| + \cdots + |c_m|.$$