10.1. Multiple Choice

(a) Let f be a holomorphic function on the closed unit disk $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Consider the relationship between the location of the minimum modulus of $|f|$ and whether f has zeroes inside D. Which of the following statements is **true**?

A) If f has zeroes inside D, then the minimum modulus of $|f|$ is attained on the boundary ∂D.

B) If f has no zeroes inside D, then the minimum modulus of $|f|$ is attained on the boundary ∂D.

C) If the minimum modulus of $|f|$ is attained on the boundary ∂D, then f has zeroes inside D.

D) If the minimum modulus of $|f|$ is attained on the boundary ∂D, then f has no zeroes inside D.

Solution:

- **A is false:** If f has zeroes inside D, the minimum modulus $|f(z)|$ is zero, and this zero must occur in the interior of D, and not necessarily on ∂D.
- **B is true:** If f has no zeroes inside D, then $1/f$ is holomorphic on D. By the Maximum Modulus Principle, the maximum modulus of $|1/f|$ (and hence the minimum modulus of $|f|$) occurs on the boundary ∂D.
- **C is false:** Take the constant function $f \equiv 1$.
- **D is false:** Clearly $f \equiv 0$ is a counterexample.

(b) Which of the following sets is not simply connected?

A) $\{z = x + iy \in \mathbb{C} \mid 0 < y < x^2 \text{ or } x = 0\}$

B) $\mathbb{C} \setminus \{re^{i\theta} : r > 0, \theta = \pi/4\}$

C) $\{z = x + iy \in \mathbb{C} \mid |x| < 1, |y| < 1\}$

D) $\{z \in \mathbb{C} \mid |z| > 1\}$

Solution:

A) This set includes the region bounded by the parabola $y = x^2$ above the x-axis, as well as the imaginary axis ($x = 0$). The inclusion of $x = 0$ connects disjoint parts of the parabola-bounded region. The set is therefore connected, and there are no holes or obstructions that prevent loops from contracting to a point.

B) This is just the cut plane $\mathbb{C}^-$ rotated by $3\pi/4$ degrees, hence simply connected.

C) This is the unit square which is convex hence simply connected.

D) This is not simply connected. Since for the closed curve $\gamma : [0, 2\pi] \to \mathbb{C}$ with $\gamma(t) = 2e^{it}$,
$$\int_\gamma \frac{1}{z} = 2\pi i \neq 0.$$

10.2. Laurent Series II Let $0 \leq s_1 < r_1 < r_2 < s_2$, and set $U = \mathcal{A}(0, s_1, s_2)$ and $V = \mathcal{A}(0, r_1, r_2)$ (like in Exercise 9.1). Denote with γ_1 and γ_2 the circles of radius r_1 and r_2, respectively, positively oriented. Let $f : U \to \mathbb{C}$ be a general holomorphic function.

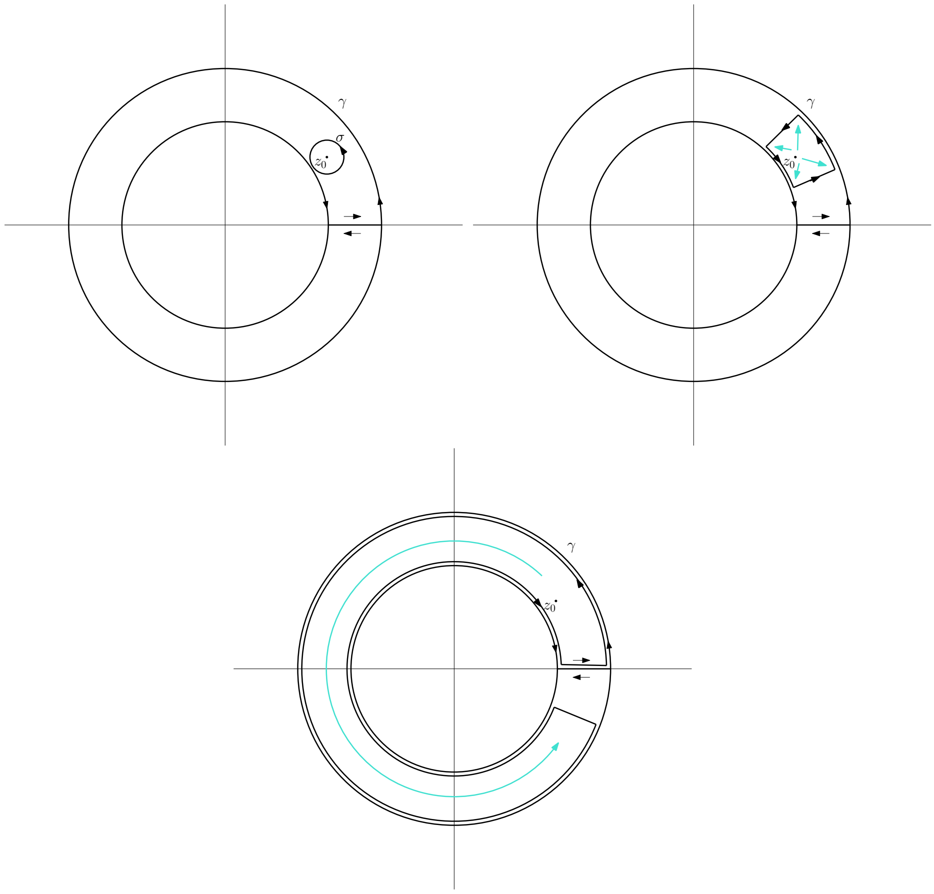
(a) Show that the functions
$$g_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw, \quad \text{for } |z| > r_1,$$
and
$$g_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw, \quad \text{for } |z| < r_2,$$
are well defined and holomorphic.

Solution: We prove this for g_1 via Morera's Theorem. The proof for g_2 is similar. First, notice that g_1 is continuous in $W = \{z : |z| > r_1\}$: fix $z_1 \in W$ distant $d > 0$ from γ_1. Then, for every $z_2 \in W$ distant $\delta > 0$ from z_1 ($\delta < d/2$) we get
$$\begin{aligned}|g_1(z_1) - g_1(z_2)| &= \left|\frac{1}{2\pi i} \int_{\gamma_1} f(w) \Big(\frac{1}{w - z_1} - \frac{1}{w - z_2}\Big)\, dz\right| \\ &\leq 2r_1 \max_{w \in \gamma_1} |f(w)| d^{-2} \delta.\end{aligned}$$
for any $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ small enough. Now given $\varepsilon > 0$ choose $0 < \delta = \delta(\varepsilon) < \min(d/2, \varepsilon d^2/2r_1 \max_{w \in \gamma_1} |f(w)|)$ to conclude the argument. Let now $T \subset W$ a generic triangle in W. Since $z \mapsto 1/(w - z)$ is holomorphic (and hence continuous) in W, by Fubini we check that
$$\int_T g_1(z)\, dz = \frac{1}{2\pi i} \int_T \int_{\gamma_1} \frac{f(w)}{w - z}\, dw\, dz = \frac{1}{2\pi i} \int_{\gamma_1} f(w) \underbrace{\int_T \frac{1}{w - z}\, dz}_{=0 \text{ by Goursat}}\, dw = 0.$$

Hence, g_1 is holomorphic in W by Morera's theorem.

(b) Let γ be the closed curve obtained by going along γ_2 starting at r_2, then along the segment joining r_2 to r_1, then along $-\gamma_1$, and finally back via the segment joining r_1 to r_2. Let $z_0 \in V$ and $r > 0$ small enough such that $\sigma = \{z \in \mathbb{C} : |z - z_0| = r\}$ is in V. Explain why σ and γ are homotopic in U.

Solution: By 'inflating' σ, one can show that it is homotopic to a little sector of annulus. Then, by deforming this sector continuously in the interior of γ it is clear that by overlapping its two flat ends, one obtains the curve γ with the correct orientation. See figure below.



(c) Show that $f = g_2 - g_1$ in V.

Solution: By independence of Cauchy formula under homotopies, we get that

$$\begin{aligned} f(z) &= \int_\sigma \frac{f(w)}{w-z}\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw \\ &= \int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \int_{\gamma_1} \frac{f(w)}{w-z}\,dw + \int_{r_1}^{r_2} \frac{f(w)}{w-z}\,dw - \int_{r_1}^{r_2} \frac{f(w)}{w-z}\,dw = g_2(z) - g_1(z). \end{aligned}$$

(d) Deduce that f can be represented as a Laurent serie, meaning: there exists a sequence $(a_n)_{n\in\mathbb{Z}}$ such that the series $\sum_{n\geq 1} a_n z^n$ and $\sum_{n\geq 1} a_{-n} z^{-n}$ are absolutely convergent in V, and satisfy

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \qquad \text{in } V.$$

Solution: By the previous point, it suffices to show that g_1 and g_2 can be represented as a Laurent series. Since g_2 is holomorphic in $\{|z| < r_2\}$ it admits a Taylor expansion (which is in particular a Laurent series) in the disk and $g_2(z) = \sum_{n\geq 0} a_n z^n$. For g_1 we can write

$$\begin{aligned} g_1(z) &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw = -\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z}\frac{1}{1-w/z}\,dw \\ &= -\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z}\sum_{k\geq 0}\left(\frac{w}{z}\right)^k dw \\ &= \sum_{n\leq -1}\left(-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w^{n+1}}\,dw\right) z^n = \sum_{n\leq -1} a_n z^n, \end{aligned}$$

as wished, where we took advantage of Fubini's Theorem to interchange sum and integration.

10.3. Complex vs Real Is it true that if $u, v : \mathbb{C} \to \mathbb{R}$ are smooth and open maps, then $f = u + iv$ is open? Answer from the perspective of the Open Mapping Theorem.

Solution: No, in general this is false: just consider $u(x,y) = v(x,y) = x$ for instance. Both functions are open since they are projections on the real axis, but the images of $f = u + iv$ are never open because the real axis is not open in $\mathbb{C}$. We deduce that the Open Mapping Theorem is a property of holomoprhic functions which is ensured by the extra condition of Cauchy-Riemann equations.

10.4. Maps preserving orthogonality Let $\Omega \in \mathbb{R}^2$ open, and $f : \Omega \to \mathbb{R}^2$ smooth. Show that if f is orientation preserving [1] and sends curves intersecting orthogonally to curves intersecting orthogonally, then f is holomorphic (by identifying $\mathbb{R}^2$ with $\mathbb{C}$).

Solution: By the Cauchy-Riemann equations, it is sufficient to prove that the Jacobian matrix of $f = u + iv$ is pointwise equal to

$$Df(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

[1] That is the determinant of its Jacobian is positive.

for some functions a, b. Now, since f sends curves that intersects orthogonally to curves that intersects orthogonally, we get in particular that

$$Df(x,y)\cdot(1,0)^t \perp Df(x,y)\cdot(0,1)^t,$$

that is $(A,C)\perp(B,D)$, implying that $(-C,A)$ is collinear to (B,D), meaning that there exists $\kappa\in\mathbb{R}$ such that $-\kappa C=B$ and $\kappa A=D$. Also, since f preserves the orientation, $0<\det(Df(x,y))=\kappa A^2+\kappa C^2$, implying that $\kappa>0$. We are left to prove that $\kappa=1$. Let now $(x,y)\in\mathbb{R}^2\setminus\{0\}$, then from

$$Df(x,y)\cdot(x,y)^t \perp Df(x,y)\cdot(-y,x)^t \quad\Leftrightarrow\quad (\kappa^2-1)(A^2+C^2)xy=0,$$

implying $\kappa=1$, as wished.

10.5. Let A be a square centered at the origin. Denote by s **one** arbitrarily fixed side of A. Let $f:A\to\mathbb{C}$ be holomorphic on the interior of A and continuous on the boundary of A, such that $f(z)=0$ for all $z\in s$. Prove that $f=0$ on A.

Solution: Let $f_1(z)=f(z)$, $f_2(z)=f(iz)$, $f_3(z)=f(-z)$, and $f_4(z)=f(-iz)$. Then each side of A corresponds to one of the functions f_1,f_2,f_3, or f_4, and each of these functions vanishes on one of the sides of A.

Define $g(z):=f_1(z)f_2(z)f_3(z)f_4(z)$. By construction, $g(z)$ vanishes on the entire boundary of A. Since g is holomorphic on A, by the Maximum Modulus Principle, $g\equiv 0$ on A.

Thus, for all $z\in A$, at least one of the functions $f_1(z),f_2(z),f_3(z),f_4(z)$ must be zero. Equivalently, $f(z)$ has a zero at one of the four points $\{\pm z,\pm iz\}\subset A$.

Now, consider a sequence $\{z_k\}$ in the interior of A, such that $z_k\to 0$. For each z_k, there exists a corresponding zero $w_k\in\{\pm z_k,\pm iz_k\}$ of f. Since $w_k\to 0$ as $z_k\to 0$, we obtain a sequence of zeros of f converging to 0.

By the Identity Theorem for holomorphic functions, $f\equiv 0$ on A.

10.6. Multiple Ways Let f be a holomorphic function on the closed unit disk $\overline{D}=\{z\in\mathbb{C}:|z|\le 1\}$. This exercise asks you to prove that

$$\max_{|z|=1}\left|f(z)-\frac{e^z}{z}\right|\ge 1$$

in **two** out of the three following ways:

(a) prove the claim using the Maximum Principle;

(b) prove the claim using Rouché's Theorem;

(c) prove the claim using Cauchy's Integral Formula.

Solution Using the Maximum Principle: Define the function $g(z) = zf(z) - e^z$, which is holomorphic on $\overline{D}$. Evaluating g at $z = 0$, we have:

$$g(0) = 0 \cdot f(0) - e^0 = -1.$$

By the Maximum Modulus Principle, the maximum of $|g(z)|$ on $\overline{D}$ is attained on the boundary $\partial D = \{z \in \mathbb{C} : |z| = 1\}$. Therefore, there exists $z_0 \in \partial D$ such that:

$$|g(z_0)| = \max_{|z| \le 1} |g(z)| \ge |g(0)| = 1.$$

Since $|z_0| = 1$ for $z_0 \in \partial D$, we have:

$$|g(z_0)| = |z_0 f(z_0) - e^{z_0}| = \left| z_0 \left(f(z_0) - \frac{e^{z_0}}{z_0} \right) \right| = \left| f(z_0) - \frac{e^{z_0}}{z_0} \right|.$$

Thus, it follows that:

$$\max_{|z|=1} \left| f(z) - \frac{e^z}{z} \right| \ge 1.$$

Solution Using Cauchy's Integral Formula: Define the function $g(z) = zf(z) - e^z$, which is holomorphic on $\overline{D}$. Evaluating g at $z = 0$, we have:

$$g(0) = 0 \cdot f(0) - e^0 = -1.$$

Applying Cauchy's integral formula for g at $z = 0$, we get:

$$g(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{z} \, dz.$$

Taking the modulus on both sides:

$$|g(0)| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{z} \, dz \right| \le \frac{1}{2\pi} \int_{|z|=1} \left| \frac{g(z)}{z} \right| |dz|.$$

Since $|z| = 1$ on the contour $|z| = 1$, we have $|1/z| = 1$. Therefore:

$$1 = |g(0)| \le \frac{1}{2\pi} \int_{|z|=1} |g(z)| \, |dz|.$$

This implies that the average value of $|g(z)|$ on the unit circle is at least 1. Consequently, there must exist some point z_0 on the unit circle $|z| = 1$ where $|g(z_0)| \geq 1$. Since $g(z) = zf(z) - e^z$ and $|z_0| = 1$, we have:

$$|g(z_0)| = |z_0 f(z_0) - e^{z_0}| = \left| z_0 \left(f(z_0) - \frac{e^{z_0}}{z_0} \right) \right| = \left| f(z_0) - \frac{e^{z_0}}{z_0} \right|.$$

Therefore:

$$\max_{|z|=1} \left| f(z) - \frac{e^z}{z} \right| \geq 1.$$

Solution Using Rouché's Theorem: Assume, for the sake of contradiction, that:

$$\left| f(z) - \frac{e^z}{z} \right| < 1 \quad \text{for all } |z| = 1.$$

Multiplying both sides by z, for $|z| = 1$, we get:

$$|zf(z) - e^z| < 1 \quad \text{for all } |z| = 1.$$

Define $h(z) = zf(z) - e^z$. On the unit circle $|z| = 1$, we have $|h(z)| < 1$. Consider the function $h(z) + 1 = zf(z) - e^z + 1$. By Rouché's theorem, since $|h(z)| < |1|$ on $|z| = 1$, $h(z) + 1$ and 1 have the same number of zeros inside the unit disk D.

However, $h(z) + 1 = zf(z) - e^z + 1$ has a zero at $z = 0$ because:

$$h(0) + 1 = 0 \cdot f(0) - e^0 + 1 = -1 + 1 = 0.$$

This implies that $h(z) + 1$ has at least one zero inside D, while the constant function 1 has no zeros. This contradiction arises from our assumption. Therefore, there must exist some point z_0 on the unit circle $|z| = 1$ where:

$$\left| f(z_0) - \frac{e^{z_0}}{z_0} \right| \geq 1.$$

Thus, we conclude that:

$$\max_{|z|=1} \left| f(z) - \frac{e^z}{z} \right| \geq 1.$$