11.1. MC Questions

(a) Consider the principal branch of the complex-valued logarithm, with domain $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Let z, z_1, z_2 be elements of Ω, and let n be a positive natural number. Which of the following equalities does not always hold?

A) $e^{\log(z_1)+\log(z_2)} = z_1 z_2$

B) $\log(e^{\log(z_1)+\log(z_2)}) = \log(z_1) + \log(z_2)$

C) $(\log(z))' = \frac{1}{z}$

D) $\log(z/n) = \log(z) - \log(n)$

Solution:

A) The equality $\log(z_1 z_2) = \log(z_1)+\log(z_2)$ holds modulo multiples of $2i\pi$. However, since the exponential map e^w is periodic with period $2i\pi$, any multiple of $2i\pi$ in the logarithm expression does not affect the result. Hence, the equality holds for all $z_1, z_2 \in \Omega$.

B) Since $e^{\log(z_1)+\log(z_2)} = z_1 z_2$, the left-hand side simplifies to $\log(z_1 z_2)$. However, $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ is true only modulo multiples of $2i\pi$. This means that $\log(z_1 z_2)$ and $\log(z_1) + \log(z_2)$ can differ by an integer multiple of $2i\pi$, and thus the equality is not always satisfied.

C) The domain $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is open and simply connected, allowing the complex logarithm $\log(z)$ to be well-defined and differentiable throughout Ω. Its derivative is $\frac{1}{z}$, as expected.

D) In general, $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ holds if $-\pi < \arg(z_1) + \arg(z_2) < \pi$. For z/n, since n is a positive real number, $\arg(n) = 0$, and thus $\arg(z) + \arg(1/n) = \arg(z)$ always satisfies the condition $-\pi < \arg(z) < \pi$. Therefore, the equality holds for all $z \in \Omega$ and positive integers n.

(b) Consider the following image, depicting five points $z_1, z_2, z_3, z_4, z_5 \in \mathbb{C}$ and a smooth path γ. What's the sum of the winding numbers $\sum_{i=1}^{5} W_\gamma(z_i)$?

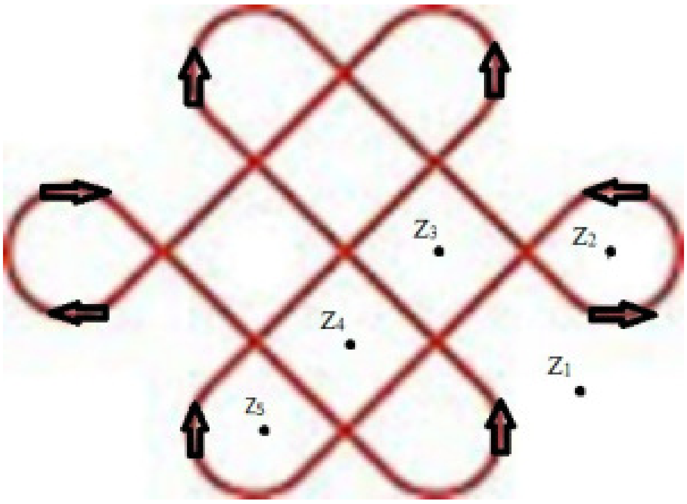
A) -1

B) 0

C) 1

D) 2

Solution: Consider a path from a point to another, say from z_1 to z_2. Starting from the winding number of z_1, if the curve crosses your path from left to right, the winding number increases, whereas if the curve crosses your path from right to left, the winding number decreases. Clearly $W_\gamma(z_1) = 0$. Going from z_1 to z_2, the curve



crosses our path from left to right. Therefore, $W_\gamma(z_2) = 1$. Going from z_1 to z_3, the curve crosses our path from right to left. Therefore, $W_\gamma(z_3) = -1$. Going from z_3 to z_4, the curve crosses our path from left to right. Therefore, $W_\gamma(z_4) = 0$. Going from z_4 to z_5, the curve crosses our path from right to left. Therefore, $W_\gamma(z_5) = -1$.

11.2. Logarithm Let U be an open and simply connected domain of $\mathbb{C}$, and $f : U \to \mathbb{C}$ a non-vanishing holomorphic function. Fix $z_0 \in U$ and denote with γ_z an arbitrary curve in U connecting z_0 to z.

(a) Show that the function

$$g(z) = \int_{\gamma_z} \frac{f'}{f}\, dw,$$

is well defined and holomorphic in U, and that $g'(z) = \frac{f'(z)}{f(z)}$ for all $z \in U$.

Solution: Since integrating an holomorphic function over a closed curve in a simply connected domain gives always zero, the integral defining g does not depend on the choice of γ_z. Fix $z \in U$ and $\gamma_z : [0,1] \to U$ connecting z_0 to z. Let $\tau \in \mathbb{C}$ with $|\tau|$ small enough so that the curve $\gamma_{z+\tau} : t \mapsto (\gamma_z(t) + t\tau)$ is contained in U. Obviously, $\gamma_{z+\tau}$ connects z_0 with $z + \tau$, and γ_z concatenated with the segment joining z to $z + \tau$ and $-\gamma_{z+\tau}$ is a closed curve. Hence

$$\begin{aligned}\frac{g(z+\tau) - g(z)}{\tau} &= \frac{1}{\tau}\Big(\int_{\gamma_{z+\tau}} \frac{f'}{f}\, dw - \int_{\gamma_z} \frac{f'}{f}\, dw\Big) = \frac{1}{\tau}\int_{\{z+t\tau : t\in[0,1]\}} \frac{f'}{f}\, dw \\ &= \frac{1}{\tau}\int_0^1 \frac{f'(z+t\tau)}{f(z+t\tau)}\tau\, dt = \int_0^1 \frac{f'(z+t\tau)}{f(z+t\tau)}\, dt,\end{aligned}$$

which by continuity of f'/f converges to $\int_0^1 f'(z)/f(z)\,dt = f'(z)/f(z)$ as $\tau \to 0$, proving that $g' = f'/f$.

(b) Compute the derivative of $\frac{\exp(g(z))}{f(z)}$.

Solution: By the previous point

$$\left(\frac{e^g}{f}\right)' = \frac{e^g g' f - e^g f'}{f^2} = e^g \frac{f'/f \cdot f - f'}{f^2} = 0.$$

(c) Deduce that there exists $\tilde{g}$ holomorphic in U such that $f = \exp(\tilde{g})$. Is this function unique?

Solution: From the previous point we get that e^g/f is equal to some constant $c \in \mathbb{C}$. Therefore, $cf = e^g$, so it suffices to take $c' \in \mathbb{C}$ so that $e^{c'} = c$ and set $\tilde{g} = g - c'$ to have $e^{\tilde{g}} = f$. Notice that the same works by adding to c' an integer multiple of $2\pi i$, so $\tilde{g}$ is not unique in general.

(d) Show that for every $n \in \mathbb{N}$ there exists an holomorphic function $h_n : U \to \mathbb{C}$ such that $(h_n)^n = f$.

Solution: Just take $h_n := \exp(\frac{1}{n}\tilde{g})$, where $\tilde{g}$ is as in the previous point.

11.3. Complex integral Evaluate

$$\int_{|z|=1} \frac{z^8 - 3iz}{4\pi z^9 + 5z^5 - 4z^3 - 2i}\,dz.$$

Hint: take advantage of Rouché's Theorem and the Homotopy Theorem.

Solution: Let $f(z) = 4\pi z^9$ and $g(z) = 5z^5 - 4z^3 - 2i$. Then, for $|z| = 1$,

$$|f(z)| = 4\pi > 11 \geq |5z^5 - 4z^3 - 2i| = |g(z)|.$$

By Rouché's Theorem, $f + g$ has all its nine zeros inside the unit circle. Therefore, the function that we want to integrate has all its poles in the interior of the unit circle. We can therefore apply the Homotopy Theorem for $\gamma_R = \{z \in \mathbb{C} : |z| = R\} \sim \gamma_1$, $R > 1$, obtaining

$$\int_{|z|=1} \frac{z^8 - 3iz}{4\pi z^9 + 5z^5 - 4z^3 - 2i}\,dz = \int_{|z|=R} \frac{z^8 - 3iz}{4\pi z^9 + 5z^5 - 4z^3 - 2i}\,dz$$

$$= \int_0^{2\pi} Rie^{it} \frac{R^8 e^{8it} - 3iRe^{it}}{4\pi R^9 e^{9it} + 5R^5 e^{5it} - 4R^3 e^{3it} - 2i}\,dt$$

$$= \int_0^{2\pi} ie^{it} \frac{e^{8it} - 3iR^{-7}e^{it}}{4\pi e^{9it} + 5R^{-4}e^{5it} - 4R^{-6}e^{3it} - 2i}\, dt.$$

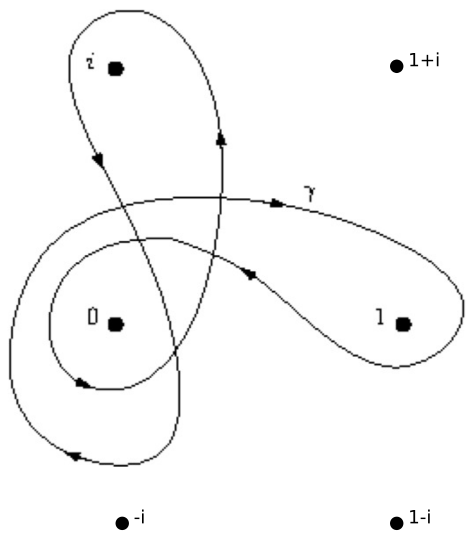
But, since

$$\lim_{R\to+\infty} ie^{it} \frac{e^{8it} - 3iR^{-7}e^{it}}{4\pi e^{9it} + 5R^{-4}e^{5it} - 4R^{-6}e^{3it} - 2i} = (4\pi)^{-1} ie^{it} e^{8it} ie^{-9it} = i(4\pi)^{-1},$$

uniformly in $t \in [0, 2\pi]$, we get by interchanging the limit $R \to +\infty$ and the integral that the answer is

$$2\pi i(4\pi)^{-1} = \frac{i}{2}.$$

11.4. Winding number Evaluate the integral $\int_\gamma g\, dz$ when $g(z) = \frac{z}{(z^2+1)(z^2-2z+2)}$ and γ is as follows:



Solution: we note that the poles of $g(z)$ are:

- $z = i$ and $z = -i$ (from $z^2 + 1 = 0$),
- $z = 1 + i$ and $z = 1 - i$ (from $z^2 - 2z + 2 = 0$).

The curve γ has the following winding conditions:

- Winds once counterclockwise around $z = i$,
- Does not wind around $z = -i$,
- Does not wind around $z = 1 + i$,

- Does not wind around $z = 1 - i$.

Thus, the integral depends only on the residue of $g(z)$ at $z = i$, as the curve does not enclose the other poles. To compute the residue of $g(z)$ at $z = i$:

$$\mathrm{Res}_{z=i} g = \lim_{z \to i} \frac{z}{(z+i)(z-1-i)(z-1+i)}.$$

Substituting $z = i$:

$$\mathrm{Res}_{z=i} g = \frac{i}{(i+i)(i-1-i)(i-1+i)} = \frac{i}{(2i)(-1)(2i-1)} = \frac{1}{2(1-2i)}.$$

Since only $z = i$ contributes to the integral:

$$\int_\gamma g \, dz = 2\pi i \cdot \mathrm{Res}_{z=i} g.$$

Substituting $\mathrm{Res}_{z=i} g = \frac{1}{2(1-2i)}$:

$$\int_\gamma g \, dz = 2\pi i \cdot \left(\frac{1}{2(1-2i)} \right) = \frac{\pi i}{1 - 2i}.$$

11.5. Fractional Residues Prove the following: if z_0 is a simple pole of a meromorphic function f and A_ε is an arc of the circle $\{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ of an angle $\alpha \in (0, 2\pi]$, then

$$\lim_{\varepsilon \to 0} \int_{A_\varepsilon} f \, dz = \alpha i \, \mathrm{res}_{z_0}(f).$$

Solution: Since f is meromorphic its poles are isolated, and hence there exists $\varepsilon_0 > 0$ such that z_0 is the unique pole inside $C_{\varepsilon_0} = \{z : |z - z_0| \leq \varepsilon_0\}$. Let $0 < \varepsilon < \varepsilon_0$. Since z_0 is simple, we can write $f(z) = \frac{a_{-1}}{z - z_0} + g(z)$ inside C_{ε_0} for some function g holomorphic, where $a_{-1} = \mathrm{res}_{z_0}(f)$. Then, by parametrizing A_ε as $t \mapsto z_0 + \varepsilon e^{it}$, $t \in [t_0, t_0 + \alpha]$, we get that

$$\int_{A_\varepsilon} f \, dz = \int_{A_\varepsilon} \frac{a_{-1}}{z - z_0} + g \, dz = \int_{t_0}^{t_0+\alpha} \frac{a_{-1}}{\varepsilon e^{it}} \varepsilon i e^{it} \, dt + \int_{C_\varepsilon} g \, dz = i\alpha a_{-1} + \int_{A_\epsilon} g \, dz.$$

Notocing now $\left| \int_{A_\varepsilon} g \, dz \right| \leq \alpha\varepsilon \max_{|z - z_0| \leq \varepsilon_0} |g(z)| = O(\varepsilon)$, we obtain that

$$\lim_{\varepsilon \to 0} \int_{A_\varepsilon} f \, dz = i\alpha a_{-1} + \lim_{\varepsilon \to 0} \int_{A_\varepsilon} g \, dz = i\alpha a_{-1},$$

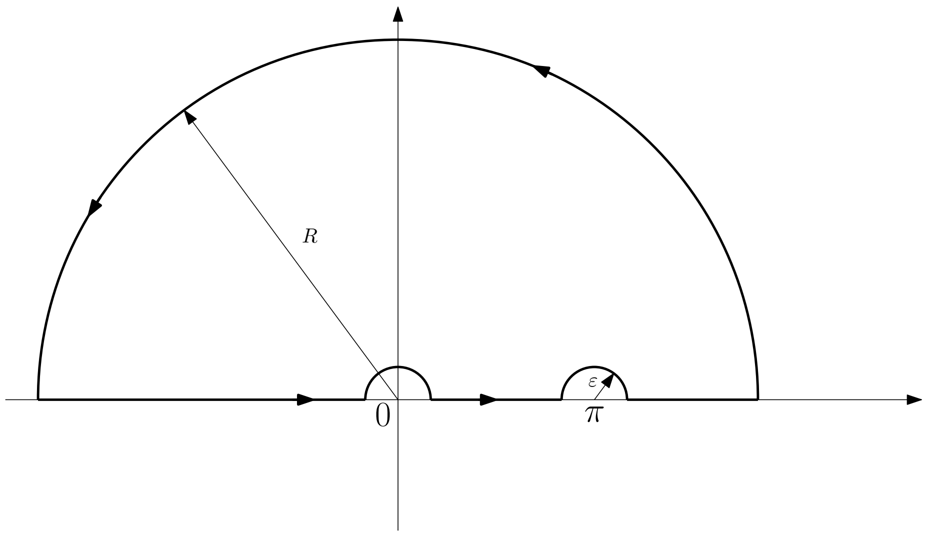
as wished.

11.6. Real integral Evaluate

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x(x-\pi)}\, dx.$$

Hint: take a suitable contour in $\mathbb{C}$ that avoids the zeros of the denominator. Take advantage of Exercise 11.5.

Solution: For $R > 2\pi$ and $\varepsilon \in (0,1)$, consider $\gamma_{\varepsilon,R}$ to be the boundary of the domain



$\{z : \varepsilon < |z| < 1, |z-\pi| > \pi, \arg(z) \in [0,\pi]\}$. Consider the function $f(z) = \frac{-ie^{iz}}{z(z-\pi)}$, and let $A_\varepsilon^1 = \{\varepsilon e^{it} : t \in [0,\pi]\}$, $A_\varepsilon^2 = \{\pi + \varepsilon e^{it} : t \in [0,\pi]\}$, and $C_R = \{Re^{it} : t \in [0,\pi]\}$. Then, since $\int_{\gamma_{\varepsilon,R}} f\, dz = 0$, we get that

$$\int_{-R}^{R} \frac{\sin(x)}{x(x-\pi)}\, dx = \lim_{\varepsilon\to 0}\Big(\int_{A_\varepsilon^1} f\, dz + \int_{A_\varepsilon^2} f\, dz\Big) - \int_{C_R} f\, dz.$$

By Exercise 11.5

$$\lim_{\varepsilon\to 0}\Big(\int_{A_\varepsilon^1} f\, dz + \int_{A_\varepsilon^2} f\, dz\Big) = \pi i(\operatorname{res}_0 f + \operatorname{res}_\pi f) = \pi i(-i/(0-\pi) - ie^{i\pi}/\pi) = -2.$$

On the other side

$$\Big|\int_{C_R} f\, dz\Big| \le \frac{\pi R}{R(R-\pi)} = O(R^{-1}).$$

Hence, we conclude that

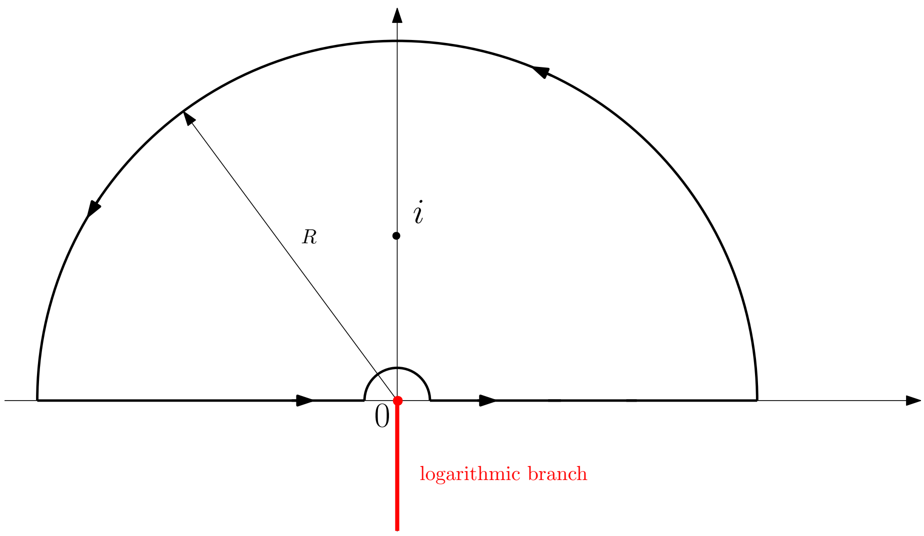
$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x(x-\pi)}\,dx = \lim_{R\to+\infty} \int_{-R}^{+R} \frac{\sin(x)}{x(x-\pi)}\,dx = -2 - \lim_{R\to+\infty} \int_{C_R} f\,dz = -2.$$

11.7. Real integral II Let $\alpha \in (0,1)$. Evaluate

$$\int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}\,dx,$$

choosing a suitable branch of the logarithm.

Solution: For $R > 1$ and $\varepsilon \in (0,1)$ let $\gamma_{\varepsilon,R}$ be the curve parametrizing the boundary of the domain $\Omega = \{z : \varepsilon < |z| < R, \Im(z) > 0\}$, like in picture. Let $f(z) = \frac{z^{2\alpha-1}}{1+z^2}$.



Since $z^{2\alpha-1} = e^{\log(z)(2\alpha-1)}$, it is convenient to chose the branch of the logarithm to be with argument between $-\pi/2$ and $3\pi/2$, so that the singularity cuts along the negative imaginary axis, and hence does not intersect $\bar{\Omega}$. By the residue Theorem

$$\int_{\gamma_{\varepsilon,R}} f\,dz = 2\pi i \operatorname{res}_i f = 2\pi i \frac{e^{\log(i)(2\alpha-1)}}{2i} = -\pi i e^{\alpha\pi i}.$$

As $\varepsilon \to 0$ and $R \to +\infty$ the integral of f along $[\varepsilon, R]$ converges to the desired integral. On the other side, the integral over $[-R, -\varepsilon]$ also converges to a multiple of the same value since

$$\int_{[-R,-\varepsilon]} \frac{e^{\log(z)(2\alpha-1)}}{1+z^2}\,dz = -e^{2\alpha\pi i} \int_{[\varepsilon,R]} \frac{e^{\log(w)(2\alpha-1)}}{1+w^2}\,dw,$$

by setting $w = -z$. In fact,

$$\int_{[-R,-\varepsilon]} \frac{e^{\log(z)(2\alpha-1)}}{1+z^2}\,dz = \int_{-R}^{-\varepsilon} \frac{e^{\log(t)(2\alpha-1)}}{1+t^2}\,dt = \int_{\varepsilon}^{R} \frac{e^{\log(-s)(2\alpha-1)}}{1+s^2}\,ds,$$

and since by our choice of the logarithmic branch $\log(-s) = \log(s) + i\pi$, we get

$$\begin{aligned}\int_{[-R,-\varepsilon]} \frac{e^{\log(z)(2\alpha-1)}}{1+z^2}\,dz &= \int_{\varepsilon}^{R} \frac{e^{\log(-s)(2\alpha-1)}}{1+s^2}\,ds = e^{i\pi(2\alpha-1)} \int_{\varepsilon}^{R} \frac{e^{\log(s)(2\alpha-1)}}{1+s^2}\,ds \\ &= -e^{2\alpha\pi i} \int_{[\varepsilon,R]} \frac{e^{\log(w)(2\alpha-1)}}{1+w^2}\,dw.\end{aligned}$$

as wished. Now, observing that

$$|z^{2\alpha-1}| = |z|^{2\alpha-1},$$

the integral over the arc of radius R (that we will call C_R) is of order $O(R^{2\alpha-2})$, and since $\alpha \in (0,1)$ it tends to zero as $R \to +\infty$. In fact:

$$\left|\int_{C_R} f\,dz\right| \le \int_{C_R} \left|\frac{z^{2\alpha-1}}{1+z^2}\right| dz = \int_{C_R} \frac{|z|^{2\alpha-1}}{|1+z^2|}\,dz \le \pi R \frac{R^{2\alpha-1}}{R^2-1} = O(R^{2\alpha-2}).$$

For the same reason, the integral over the arc of radius $\varepsilon > 0$ (that we will call c_ε)is of order $O(\varepsilon^{2\alpha})$:

$$\left|\int_{c_\varepsilon} f\,dz\right| \le \int_{c_\varepsilon} \left|\frac{z^{2\alpha-1}}{1+z^2}\right| dz = \int_{c_\varepsilon} \frac{|z|^{2\alpha-1}}{|1+z^2|}\,dz \le \pi\varepsilon \frac{\varepsilon^{2\alpha-1}}{1-\varepsilon^2} = O(R^{2\alpha}),$$

and also tends to zero as $\varepsilon \to 0$. We get that

$$(1 - e^{2\alpha\pi i}) \int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}\,dx = -\pi i e^{\alpha\pi i},$$

proving finally that

$$\int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}\,dx = \frac{\pi}{2\sin(\pi\alpha)}.$$