1.1. MC Questions

(a) Let $f(z) = z^2$. How does f map the upper half-plane (the set of complex numbers with positive imaginary parts)?

A) It maps the upper half-plane to the left half-plane.

B) It maps the upper half-plane to the lower half-plane.

C) It maps the upper half-plane to both the upper and lower half-planes.

D) It maps the upper half-plane to the entire complex plane except the real axis.

Solution: The correct answer is C. A complex number in the form $r(\cos\theta + i\sin\theta)$ is mapped to $r^2(\cos 2\theta + i\sin 2\theta)$, from which we deduce that $f(z)$ is in the uppper half-plane if $0 < \theta < \frac{\pi}{2}$, $f(z)$ lies on the real line if $\theta = \frac{\pi}{2}$ and $f(z)$ is in the lower half-plane if $\frac{\pi}{2} < \theta < \pi$.

(b) Let $f(z) = u + iv$ be a holomorphic function in the unit disc $D_1(0)$. Let $F_1(z) := \overline{f(\bar{z})}$ and $F_2(z) := f(\bar{z})$. Which one of the following statemnets is correct.

A) F_1 is holomorphic but F_2 is not.

B) F_2 is holomorphic but F_1 is not.

C) Both F_1 and F_2 are holomorphic.

D) Neither F_1 nor F_2 is holomorphic.

Solution: We know that F_2 can't be in general holomorphic as it's not holomorphic in the case in which f is the identity $f = \mathrm{Id}$. Next, to show that $F_1(z)$ is holomorphic, we need to show that it satisfies the Cauchy-Riemann equations, as differentiability follows from the fact that f was differentiable. Let $F_1(z) = U(x,y) + iV(x,y)$. Then $U(x,y) = u(x,-y) \quad \text{and} \quad V(x,y) = -v(x,-y)$. Computing the partial derivatives,

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x}(x,-y) \quad \text{and} \quad \frac{\partial V}{\partial y} = \frac{\partial v}{\partial y}(x,-y).$$

Using the fact that f satisfies by assumption the CR equations, we get

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x},$$

which is what we wanted to prove.

1.2. Complex Numbers Review

(a) Simplify the following expressions

- $\left(1+i\sqrt{3}\right)^{50}$
- $(1+i)^{2n}(1-i)^{2m}$ for every $m, n \in \mathbb{N}$.

Solution:

- We express the number in its polar form and compute directly:

$$\left(1+i\sqrt{3}\right)^{50} = 2(\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3}))^{50} = 2^{50}(\cos(\frac{\pi\cdot 50}{3})+i\sin(\frac{\pi\cdot 50}{3})) = 2^{50}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})$$

- Since $(1+i)^2 = 2i$ and $(1-i)^2 = -2i$ we have that

$$\begin{aligned}(1+i)^{2n}(1-i)^{2m} &= 2^{m+n}(i)^n(-i)^m = 2^{m+n}(i)^n\left(\frac{1}{i}\right)^m \\ &= 2^{m+n}(i)^{m-n} = \begin{cases} 2^{m+n}, & \text{if } m-n = 0 \bmod 4, \\ 2^{m+n}i, & \text{if } m-n = 1 \bmod 4, \\ -2^{m+n}, & \text{if } m-n = 2 \bmod 4, \\ -2^{m+n}i, & \text{if } m-n = 3 \bmod 4. \end{cases}\end{aligned}$$

(b) Express the complex number $z = -1 + i\sqrt{3}$ in polar form and compute all of its cubic roots.

Solution: We use the fact that given a number $z = r(\cos\theta + i\sin\theta)$, its n-th roots are given by $r^{\frac{1}{n}}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$, for $k \in \{0, \ldots, n-1\}$. It follows that the cubic roots are:

$$\begin{aligned} z_0 &= \sqrt[3]{2}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right), \\ z_1 &= \sqrt[3]{2}\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right), \\ z_2 &= \sqrt[3]{2}\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right). \end{aligned}$$

(c) Find all $z \in \mathbb{C}$ such that $z^2 + (3+4i)z + (5+6i) = 0$.

Solution: Applying the standard formula for the roots of a polynomial of degree 2 yields that the solutions of the equations are

$$z_{1,2} = \frac{-(3+4i) \pm 3\sqrt{3}i}{2},$$

which gives $z_1 = -\frac{3}{2} + (-2 + \frac{3\sqrt{3}}{2})i$ and $z_2 = -\frac{3}{2} + (-2 - \frac{3\sqrt{3}}{2})i$.

1.3. Power Series Investigate the absolute convergence and radius of convergence of the following power series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{3^n} z^n, \qquad \sum_{n=0}^{+\infty} \frac{n!}{(2n)!} z^n, \qquad \sum_{n=0}^{+\infty} n^2 z^n.$$

Solution: Let (a_n) be a sequence of complex numbers. We recall that if we set

$$R = \begin{cases} \frac{1}{\limsup_{n\to+\infty} |a_n|^{1/n}}, & \text{if } \limsup_{n\to+\infty} |a_n|^{1/n} > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

then the associated complex power serie $\sum_{n=0}^{+\infty} a_n z^n$ converges absolutely if $|z| < R$ and diverges if $|z| > R$. Since

$$\limsup_{n\to+\infty} \left| \frac{(-1)^n}{3^n} \right|^{1/n} = \lim_{n\to+\infty} \frac{1}{(3^n)^{1/n}} = \frac{1}{3},$$

we have that the first power serie of the exercise is absolutely convergent if $|z| < 3$. Since

$$\limsup_{n\to+\infty} \left| \frac{n!}{(2n)!} \right|^{1/n} = \lim_{n\to+\infty} \left(\frac{n!}{(2n)!} \right)^{1/n} \leq \lim_{n\to+\infty} \left(\frac{1}{n!} \right)^{1/n} = 0,$$

we have that the second power serie of the exercise is absolutely convergent if $|z| < +\infty$. Since

$$\limsup_{n\to+\infty} \left| n^2 \right|^{1/n} = \lim_{n\to+\infty} \left(n^2 \right)^{1/n} = 1,$$

we have that the third power serie of the exercise is absolutely convergent if $|z| < 1$.

1.4. Differentiability, Cauchy-Riemann and Holomorphicity Provide, with proof:

(a) some function $f\colon \mathbb{C} \to \mathbb{C}$ such that, for some $z_0 \in \mathbb{C}$, f satisfies the Cauchy-Riemann equations at z_0, but is *not* holomoprhic at z_0;

Solution: Define $f(x+iy) = \sqrt{|x| \cdot |y|}$. In this case, setting $f = u + iv$ we have that $u(z) = u(x+iy) = \sqrt{|x||y|}$ and $v \equiv 0$. By the very definition of partial derivative, we get that

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x\to 0} \frac{0}{x} = 0,$$

and similarly $\frac{\partial u}{\partial y}(0,0) = 0$. So the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are clearly satisfied in $z = 0$. However, f is not holomorphic at the origin, since choosing for instance the particular complex increment of differentiation $H = 1 + i$, we get that

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \frac{f(\tau H) - f(0)}{\tau H} = \lim_{\substack{\tau \to 0 \\ \tau > 0}} \frac{1}{\tau} \frac{\tau}{1+i} = \frac{1}{1+i} \neq 0,$$

On the other hand if we choose $H = 2 + i$ than a similar argument gives that the above limit is $\frac{\sqrt{2}}{2+i}$ Hence the limit of the differential quotient $(f(h) - f(0))/h$ depends on how $h \in \mathbb{C}$ approaches zero.

(b) functions $u, v \colon \mathbb{R}^2 \to \mathbb{R}$ such that, for some $p_0 = (x_0, y_0) \in \mathbb{R}^2$, the function $(u, v) \colon (x, y) \mapsto (u(x, y), v(x, y))$ is differentiable at p_0, but the function $u + iv \colon \mathbb{C} \to \mathbb{C}$ defined for $z = x + iy$ by $(u + iv)(z) = (u + iv)(x + iy) = u(x, y) + i \cdot v(x, y)$ is *not* holomorphic at $z_0 = x_0 + iy_0$.

Solution: Consider $u(x, y) = v(x, y) = x + y$. Then the function $(u, v) \colon (x, y) \mapsto (u(x, y), v(x, y))$ is differentiable at every point in $\mathbb{R}^2$. On the other hand, the equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ is clearly not satisfied.

1.5. Applications of CR equations Let $\Omega \subset \mathbb{C}$ be a domain, i.e an open connected subset of $\mathbb{C}$.

(a) Let $u : \Omega \to \mathbb{R}$ be a differentiable function such that $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$ for all $z \in \Omega$. Prove that u is constant on Ω

Solution: Fix $z \in \Omega$ and let $w \in \Omega$ so that the segment $\gamma(t) = (1-t)z + tw$, $t \in [0, 1]$, is contained if Ω. Define the function $g : [0, 1] \to \mathbb{R}$ by $g(t) := u(\gamma(t))$. Since u and γ are differentiable, we have that $g \in C^1(0, 1)$, with derivative

$$g'(t) = \nabla u(\gamma(t)) \cdot \gamma'(t),$$

where $\nabla u(\gamma(t)) = (\frac{\partial u}{\partial x}(\gamma(t)), \frac{\partial u}{\partial y}(\gamma(t)))$ and $\gamma'(t) = w - z$. By assumption, $\nabla u \equiv 0$ everywhere, hence $g' \equiv 0$, implying $u(w) = g(1) = g(0) = u(z)$. Suppose now $w \in \Omega$ is arbitrary. Since the domain Ω is open and connected there exists a finite sequence of points $w_0, w_1, \ldots, w_N$ in Ω so that $w_0 = z$, $w_N = w$ and for every $j = 0, \ldots, N-1$ the segment joining w_j to w_{j+1} is contained in Ω. Repeating the previous argument on each segment, we obtain that $u(z) = u(w_0) = u(w_1) = \cdots = u(w_{N-1}) = u(w_N) = u(w)$ for all $w \in \Omega$, showing that u is indeed a constant function.

(b) Let $f : \Omega \to \mathbb{C}$ be holomorphic and $f'(z) = 0$ for all $z \in \Omega$. Prove that f is constant in Ω.

Solution: Since f is holomorphic in Ω, its real and imaginary parts are is particular differentiable in the sense of real analysis by looking at Ω as a subset of $\mathbb{R}^2$. Moreover, since $0 = f'(z) = 2\frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right)$ we have that $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \equiv 0$ in Ω. By point (a) we get that u must be constant in Ω. The same holds for v by noticing that $f'(z) = 2i\frac{\partial v}{\partial z}$. Hence, $f = u + iv$ is constant in Ω.

(c) If $f = u+iv$ is holomorphic on Ω and if any of the functions u, v or $|f|$ is constant on Ω then f is constant.

Solution: If v is constant, we get from the Cauchy-Riemann equations that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$ implying by point (a) that u is also constant, and therefore f is constant. The same holds if u is constant by interchanging the roles of u and v. If $|f| = \sqrt{u^2 + v^2}$ is constant, then also $|f|^2$ must be constant. Hence, by applying the Cauchy-Riemann identities, we get that

$$0 = \nabla|f|^2 = \left(2v\frac{\partial v}{\partial x} + 2u\frac{\partial u}{\partial x}, 2v\frac{\partial v}{\partial y} + 2u\frac{\partial u}{\partial y}\right) = \left(2v\frac{\partial v}{\partial x} + 2u\frac{\partial v}{\partial y}, 2v\frac{\partial v}{\partial y} - 2u\frac{\partial v}{\partial x}\right)$$

implying that

$$v\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial y} = 0 = v\frac{\partial v}{\partial y} - u\frac{\partial v}{\partial x}. \tag{1}$$

Now, multiplying the left hand side by v and the right hand side by u we get that the expression simplifies in

$$(u^2 + v^2)\frac{\partial v}{\partial x} = 0.$$

We have now two possibilities: if $u^2 + v^2$ vanishes somewhere, by the assumption $|f| =$ constant we get that $u = v = 0$ everywhere in Ω. Otherwise, from the above expression we deduce that $\frac{\partial v}{\partial x} = 0$ in Ω. The same argument proves that $\frac{\partial v}{\partial y} = 0$ by multiplying the left hand side of (1) by u and the right hand side by $-v$. This proves by part (a) that v is constant and hence as above f is constant.

1.6. ⋆ Geometric transformations of the complex plane

(a) Describe the transformation $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ in terms of geometric operations on the complex plane. Find the image of the unit circle under this transformation.

Solution: The transformation $g(r(\cos\theta + i\sin\theta)) = r^{-1}(\cos\theta + i\sin\theta)$ performs an inversion with respect to the unit circle. This means that points inside the unit circle are mapped to points outside the unit circle, and vice versa: if $|z| < 1$, then $|g(z)| > 1$, and if $|z| > 1$, then $|g(z)| < 1$. The given transformation f can be seen as composition of g with the reflection across the real axis. In formulae, the argument θ of the complex number $z = re^{i\theta}$ is changed to $-\theta$. Thus, the transformation $f(z) = \frac{1}{z}$ can be viewed as a combination of an inversion in the unit circle and a reflection across the real axis.

Notice that the reflection across the real axis leaves the modulus of a complex number unchanged. It follows from this, together with the property of g above, that the image of the unit circle through f has to be a subset of the unit circle. Being f an involution (that is, $f = f^{-1}$) we get that in particular f is invertible, hence the image of the unit circle is the whole unit circle.

(b) Let $f\colon \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ be defined as $f(z) = \frac{z-i}{z+i}$. Show that this transformation maps the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to the unit disk $\{z \in \mathbb{C} : |w| < 1\}$.

Solution: For $w = f(z) = f(x+iy)$, the modulus of w is:

$$|w| = \left|\frac{z-i}{z+i}\right| = \frac{|z-i|}{|z+i|} = \frac{\sqrt{x^2 + (y-1)^2}}{\sqrt{x^2 + (y+1)^2}}.$$

For $z \in \mathbb{C}$ with $\mathrm{Im}(z) = y > 0$, we see immediately that the denominator is larger than the numerator, from which the claims follows.

1.7. $\star$ Let $u : \mathbb{C} \to \mathbb{R}$ be a real-valued function on $\mathbb{C}$.

(a) Show that there is at most one holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f(0)) = 0$.

Solution: Suppose there are two holomorphic functions f_1 and f_2 such that:

$$\mathrm{Re}(f_1(z)) = \mathrm{Re}(f_2(z)) = u(z) \quad \text{for all } z \in \mathbb{C}$$

and

$$\mathrm{Im}(f_1(0)) = \mathrm{Im}(f_2(0)) = 0.$$

Define the difference $h(z) = f_1(z) - f_2(z)$. Then:

$$\mathrm{Re}(h(z)) = \mathrm{Re}(f_1(z)) - \mathrm{Re}(f_2(z)) = 0 \quad \text{for all } z \in \mathbb{C}.$$

Since the sum of holomorphic functions is itself holomorphic, h is holomorphic and its real part is zero. But using Exercise 1.5 c, this gives that h is constant. This in return says that $\operatorname{Im} h$ is constant. Since $(\operatorname{Im} h)(0) = \operatorname{Im}(f_1)(0) - \operatorname{Im}(f_2)(0) = 0$ this constant is zero.This then implies that $h \equiv 0$, i.e. $f_1 = f_2$.

(b) Give an example of a C^∞ function u such that there is no f as in the previous item.

Solution: Consider the function $u(z) = u(x+iy) = x^2$. Suppose there is such a holomorphic function $f(z) = u(z) + iv(z) = x^2 + iv(x,y)$. Since f is holomorphic, it must satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

For $u(z) = x^2$, we compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 0.$$

From the Cauchy-Riemann equations, we then obtain:

$$\frac{\partial v}{\partial y} = 2x \quad \text{and} \quad \frac{\partial v}{\partial x} = 0.$$

From $\frac{\partial v}{\partial x} = 0$, we conclude that $v(x,y)$ is independent of x, so we can write $v(x,y) = g(y)$ for some function g depending only on y.

Next, from $\frac{\partial v}{\partial y} = 2x$, we get:

$$g'(y) = 2x.$$

However, this is a contradiction because the right-hand side $2x$ depends on x, whereas the left-hand side $g'(y)$ depends only on y. This shows that no such function $v(x,y)$ can exist.