Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

2.1. MC Questions

(a) Which of the following functions is NOT holomorphic?

A) $f(z) = z^6 + 5$

B) $f(z) = x^2 - y^2 + x + i(y + 2xy)$

C) $f(z) = (\cos(x) + i\sin(x))e^{-y}$

D) $f(z) = x - iy + 2$

Solution: One can check using Cauchy Riemann equations that each function is holomorphic except for D). Note that the function in B) can be rewritten as $z^2 + z$ where as the one in C) is e^{iz}, Hence A),B) and C) are clearly holomorphic everywhere. Finally the function is D) is $\bar{z} + 2$ which is antiholomorphic.

(b) Given that the derivative of a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is expressed as a power series

$$f'(z) = \sum_{n=0}^{\infty} b_n z^n,$$

which of the following is, for some value of the constant $C \in \mathbb{C}$, the correct expression for $f(z)$?

A) $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n+1} z^n + C$

B) $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n} z^{n+1} + C$

C) $f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} + C$

D) $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n+1} z^{n+1} + C$

Solution: The correct answer is D. Given $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, we know that $f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}$, which can be rewritten as $f'(z) = \sum_{m=0}^{+\infty} (m+1) a_{m+1} z^m$. We get that $b_n = (n+1) a_{n+1}$, inverting which gives the desired result.

2.2. Complex numbers and geometry I Denote with $A_y := \{iy : y \in \mathbb{R}\} \subset \mathbb{C}$ the y-axis in the complex plane. Describe geometrically the image of A_y under the exponential map $\{e^z : z \in A_y\}$. Repeat the same replacing A_y with the x-axis $A_x := \{x : x \in \mathbb{R}\} \subset \mathbb{C}$, the diagonal $D := \{a + ia : a \in \mathbb{R}\} \subset \mathbb{C}$, and the curve $\{\log(a) + ia : a > 0\} \subset \mathbb{C}$.

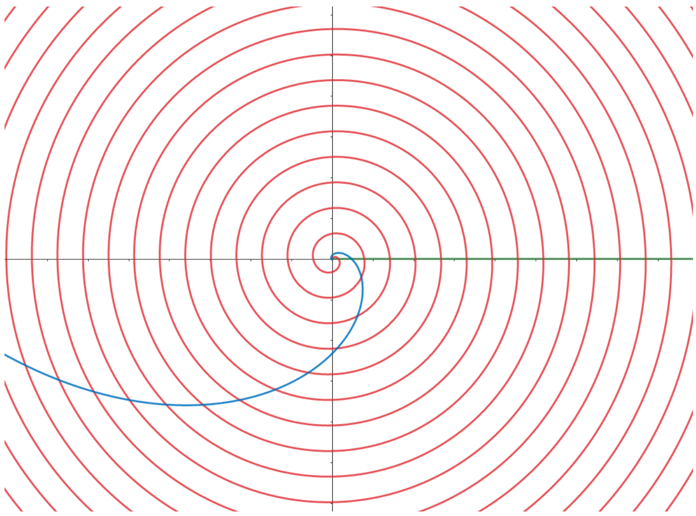
Solution: We recall that from the definiton of exponential function the following identity holds:

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)),$$

for every $z = x + iy \in \mathbb{C}$. Therefore, it follows that

$$\begin{aligned}
\exp(A_y) &= \{\cos(y) + i\sin(y) : y \in \mathbb{R}\},\\
\exp(A_x) &= \{e^x : x \in \mathbb{R}\}, \text{ (in green in the picture)}\\
\exp(D) &= \{e^a(\cos(a) + i\sin(a)) : a \in \mathbb{R}\}, \text{ (in blue in the picture)}\\
\exp(G) &= \{a(\cos(a) + i\sin(a)) : a > 0\}, \text{ (in red in the picture)}.
\end{aligned}$$

These sets represent geometrically in the complex plane: the unit circle, the open positive part of the x-axis, a logarithmic spiral, and an Archimedean spiral.



2.3. Integrating over a triangle Let Ω be an open subset of $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic, and that $f' : \Omega \to \mathbb{C}$ is continuous. Show taking advantage of the Green formula [1] that

$$\int_T f\,dz = 0,$$

[1] Let C be a positively oriented, piecewise-smooth simple curve in the plane, and let D be the region bounded by C. If $\vec{F} = (F^1, F^2) : \bar{D} \to \mathbb{R}^2$ is a vector field whose components have continuous partial derivatives, then Green's theorem states: $\int_C \vec{F} \cdot dr = \iint_D (\partial_x F^2 - \partial_y F^1)\,dx\,dy$.

where the integration is along an arbitrary triangle T contained in Ω.

Solution: Write $f = u + iv$, let $\gamma(t) = x(t) + iy(t) : [a, b] \to \mathbb{C}$ be a parametrization of T, and call Ω the interior of T, that is $\partial\Omega = T$. Then, by definition of complex line integration we have that

$$\int_T f\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t))\,dt$$
$$= \int_a^b (u(\gamma(t))x'(t) - v(\gamma(t))y'(t)))\,dt + i\int_a^b (u(\gamma(t))y'(t) + v(\gamma(t))x'(t))\,dt.$$

Set $\tilde{\gamma}(t) = (x(t), y(t))$ the identification of γ as a curve in $\mathbb{R}^2$. Then, defining the vector fields $\vec{F}(x, y) = (u(x, y), -v(x, y))$ and $\vec{G}(x, y) = (v(x, y), u(x, y))$ we get that the above integral is equal to

$$\int_{\tilde{\gamma}} \vec{F} \cdot dr + i\int_{\tilde{\gamma}} \vec{G} \cdot dr,$$

which by Green formula and the Cauchy-Riemann equations gives finally

$$\int_T f\,dz = \iint_\Omega (-\partial_x v - \partial_y u)\,dx\,dy + i\iint_\Omega (\partial_x u - \partial_y v)\,dx\,dy = 0.$$

2.4. * Line integral I Compute the following complex line integrals. Here $\Re(z)$ and $\Im(z)$ denote respectively the real and imaginary parts of z.

(a) $\int_\gamma (z^2 + z)\,dz$, when γ is the segment joining 1 to $1 + i$.

Solution: We parametrize γ as $\gamma(t) = 1 + it$, $t \in [0, 1]$. Then

$$\int_\gamma (z^2 + z)\,dz = \int_0^1 ((1 + it)^2 + (1 + it))(1 + it)'\,dt = i\int_0^1 (2 - t^2 + 3it)\,dt$$
$$= i(2 - 1/3 + i3/2) = -3/2 + i5/3.$$

(b) $\int_\gamma \bar{z}\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Solution: We parametrize the curve γ as $t \mapsto e^{it}$ for $t \in [0, 2\pi]$. Then

$$\int_\gamma \bar{z}\,dz = \int_0^{2\pi} (\cos(t) - i\sin(t))(-\sin(t) + i\cos(t))\,dt = 2i\pi.$$

(c) $\int_\gamma z^n\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $n \in \mathbb{Z}$.

Solution: Denote with $C_1 = \{z : |z| = 1\}$ the unit circle. If $n \neq -1$. Then

$$\int_{C_1} z^n \, dz = i \int_0^{2\pi} e^{int} e^{it} \, dt = i \int_0^{2\pi} e^{it(n+1)} \, dt = i \frac{e^{it(n+1)}}{n+1} \Big|_0^{2\pi} = 0.$$

For $n = -1$ we have

$$\int_{C_1} \frac{1}{z} \, dz = \int_0^{2\pi} e^{-it} i e^{it} \, dt = \int_0^{2\pi} i dt = 2\pi i$$

(d) $\int_\gamma z^n \, dz$, when γ is the circle $\{z \in \mathbb{C} : |z - 2| = 1\}$ and $n \in \mathbb{N}$.

Solution: Note that z^n is continuous and has a primitive in $\mathbb{C}$ which is equal to $\frac{z^{n+1}}{n+1}$. Since we are integrating over a closed curve the integral is zero.

Alternatively note that by linearity of the integral and previous part the integral of any polynomial over the unit circle $C_1(0)$ is zero. Let $g(z)$ be the polynomial $(z_0 + rz)^n$. Then

$$0 = \int_{C_1(0)} g(z) dz = \int_0^{2\pi} (z_0 + re^{it})^n i e^{it} dt$$

On the other hand

$$\int_{C_r(0)} z^n dz = \int_0^{2\pi} (z_0 + re^{it})^n r i e^{it} dt = r \int_{C_1(0)} g(z) dz.$$

Hence it is also equal to zero.

(e) $\int_\gamma \frac{dz}{(z-a)(z-b)}$ when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $a, b \in \mathbb{C}$ with $|a| < 1 < |b|$.

Solution: We have that

$$\begin{aligned}\int_C \frac{dz}{(z-a)(z-b)} &= \frac{1}{a-b} \int_C \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz \\ &= \frac{i}{a-b} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - a)} - \frac{e^{it}}{(e^{it} - b)} \, dt.\end{aligned}$$

We compute this integral in two steps: first, since $|a/e^{it}| < 1$, we have that

$$\int_0^{2\pi} \frac{e^{it}}{e^{it} - a} \, dt = \int_0^{2\pi} \frac{1}{1 - a/e^{it}} \, dt = \int_0^{2\pi} \sum_{n=0}^{+\infty} a^n e^{-itn} \, dt = 2\pi,$$

because the only term that is not zero when integrated is when $n = 0$. On the other side, since $|1/b| < 1$, we can rewrite

$$\int_0^{2\pi} \frac{e^{it}}{e^{it} - b} \, dt = \frac{1}{b} \int_0^{2\pi} \frac{e^{it}}{e^{it}/b - 1} \, dt = -\frac{1}{b} \int_0^{2\pi} e^{it} \sum_{n=0}^{+\infty} e^{int} b^{-n} \, dt = 0,$$

because term by term the integral is zero. Hence,

$$\int_C \frac{1}{(z-b)(z-a)} \, dz = \frac{2i\pi}{a-b},$$

as wished. In both cases, we took advantage of Fubini's theorem to interchange the integration with the sum. We recall the statement: let $(f_n)_{n\geq 0}$ be a sequence of functions such that

- $\int \sum_n |f_n| \, dx < +\infty$
- $\sum_n \int |f_n| \, dx < +\infty$

then $\sum_n \int f_n \, dx = \int \sum_n f \, dx$.

2.5. Line Integral II Is it true that for any $f : \mathbb{C} \mapsto \mathbb{C}$

$$\Re \int_\gamma f(z) \, dz = \int_\gamma \Re(f(z) \, dz$$

If so prove it, if not give a counterexample.

Solution: The statement is **false**. We provide a counterexample with the function $f(z) = z$ and the contour γ as the unit circle parameterized by $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. We have $f(z) = z = e^{it}$ along the contour, and $dz = ie^{it} \, dt$. Therefore, the LHS integral becomes:

$$\int_\gamma f(z) \, dz = \int_0^{2\pi} e^{it} \cdot ie^{it} \, dt = i \int_0^{2\pi} e^{2it} \, dt = 0.$$

Thus,

$$\Re \left(\int_\gamma f(z) \, dz \right) = \Re(0) = 0.$$

Along the unit circle, $f(z) = e^{it} = \cos(t) + i \sin(t)$, so $\Re(f(z)) = \cos(t)$. Thus, the RHS integral becomes:

$$\int_\gamma \Re(f(z)) \, dz = \int_0^{2\pi} \cos(t) \cdot ie^{it} \, dt.$$

Expanding $e^{it} = \cos(t) + i\sin(t)$, we get:

$$i\int_0^{2\pi} \cos(t)\left(\cos(t) + i\sin(t)\right)\, dt = i\int_0^{2\pi} \left(\cos^2(t) + i\cos(t)\sin(t)\right)\, dt.$$

This simplifies to:

$$i\left(\int_0^{2\pi} \cos^2(t)\, dt + i\int_0^{2\pi} \cos(t)\sin(t)\, dt\right).$$

We know that:

$$\int_0^{2\pi} \cos^2(t)\, dt = \pi \quad \text{and} \quad \int_0^{2\pi} \cos(t)\sin(t)\, dt = 0.$$

Therefore, the integral is:

$$i \cdot \pi.$$

2.6. * Differentiability

(a) Prove (without using the Cauchy-Riemann equation) that the functions

$$f(z) = \Re(z), \quad g(z) = \Im(z)$$

are not differentiable at any point.

Solution: We have:

$$\lim_{h\to 0, h\in\mathbb{R}} \frac{f(z+h) - f(z)}{h} = 1.$$

On the other hand:

$$\lim_{h\to 0, h\in i\mathbb{R}} \frac{f(z+h) - f(z)}{h} = 0.$$

Therefore, the general limit does not exist. A similar argument holds for $g(z)$.

(b) Let $a, b \in \mathbb{C}$. Find all points in $\mathbb{C}$ where $af(z) + bg(z)$ is differentiable.

Solution: We can rewrite the expression as:

$$af(z) + bg(z) = \frac{a(z + \overline{z})}{2} + \frac{b(z - \overline{z})}{2i}.$$

Simplifying this, we get:

$$af(z) + bg(z) = z\left(\frac{a}{2} - i\frac{b}{2}\right) + \overline{z}\left(\frac{a}{2} + i\frac{b}{2}\right).$$

The function is differentiable if and only if $\overline{z}\left(\frac{a}{2} + i\frac{b}{2}\right)$ is differentiable. But since $\bar{z}$ is not differentiable unless $z = 0$, we conclude that for $z \neq 0$ the function is differentiable if and only if $a = -ib$.