let $f : \Omega \to \mathbb{C}$

Primitives A **primitive** for f on Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ $\forall z \in \Omega$.

Existence of a primitive gives

Thm 3.2 let $f : \Omega \to \mathbb{C}$ be a continous function on open set $\Omega \subset \mathbb{C}$. If f has a primitive F in Ω and γ is a curve which begins at z_1 and ends at z_2 (ie $\gamma : [a, b] \to \mathbb{C}$, $\gamma(a) = z_1$, $\gamma(b) = z_2$)) then

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1)$$

An immediate corollary is

Cor 3.3 If γ is a closed curve in an open set Ω, f cont. and has a primitive in Ω then

$$\int_\gamma f(z)\,dz = 0.$$

Proof of Thm 3.2: let $F = U(x, y) + iV(x, y)$ $\gamma : [a, b] \to \mathbb{C}$. First assume γ is smooth

Define a function

$$G : [a, b] \to \mathbb{C}$$
$$t \longmapsto F(\gamma(t)) = F(x(t), y(t))$$

where we wrote $\gamma(t) = x(t) + iy(t)$

We need to check the compatibility of real derivative of G and complex derivative of F.

G is a continous function

$$G'(t) = [U(x(t), y(t)) + iV(x(t), y(t))]'$$

$$= [U_x(x(t), y(t)) \cdot x'(t) + U_y(x(t), y(t)) \cdot y'(t)]$$

$$+ i[V_x(x, y) \cdot x'(t) + V_y(x, y) \cdot y'(t)]$$

Using chain rule of vector calculus.

$$= (U_x(x, y)x'(t) - V_x(x, y)y'(t))$$

using CR eqns for F

$\boxed{U_x = V_y}$
$\boxed{U_y = -V_x}$

$$+ i[V_x(x, y)x'(t) + U_x(x, y)y'(t)]$$

$$= [U_x(x, y) + iV_x(x, y)]x'(t) +$$

$$[-V_x(x, y) + iU_x(x, y)]y'(t)$$

$$= \underbrace{[U_x(x, y) + iV_x(x, y)]}_{F'(x(t), y(t))}\underbrace{[x'(t) + iy'(t)]}_{\gamma'(t)}$$

$$F'(\gamma(t)) \cdot \gamma'(t) \overset{F' = f}{=} f(\gamma(t))\gamma'(t)$$

Hence $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)$$

$$= \int_a^b G'(t)\,dt = G(b) - G(a)$$

fund. thm analysis

$$= F(\gamma(b)) - F(\gamma(a))$$

$$= F(z_2) - F(z_1)$$

Another corollary of Thm 3.2 is the following

Cor 3.4 If $f : \Omega \to \mathbb{C}$ is holomorphic where Ω is open connected. If $f' = 0$ then f is constant

Proof: We want to show that for any 2 points $z, w \in \Omega$, $f(z) = f(w)$

Ω is open connected, then there is a (polygonal) path $\gamma : [0,1] \longrightarrow \Omega$ connecting z, w
$$\gamma(0) = z, \quad \gamma(1) = w .$$

Since f is holomorphic, f is clearly a primitive of f'

Hence
$$\int_\gamma f' \, dz = f(\gamma(1)) - f(\gamma(0))$$
$$= f(z) - f(w)$$

But since $f' = 0$ the integral on the left is zero. Hence $f(z) = f(w)$.

Important example

Let $f : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$

$z \longrightarrow 1/z$.

Then $f(z)$ has no primitive on $\mathbb{C}\setminus\{0\}$.

To see this

let γ be the circle at center 0 of radius 1. $\gamma : [0, 2\pi] \to \mathbb{C}$, $t \mapsto e^{it}$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(e^{it}) . ie^{it}\,dt$$

$$= \int_0^{2\pi} \frac{1}{e^{it}} . ie^{it}\,dt = 2\pi i \neq 0.$$

Ex : What is $\int_\gamma z^2\,dz$? where $\gamma : [0, 1] \longrightarrow \mathbb{C}$

$t \to t + \pi i t^2$

Since $F(z) = \frac{z^3}{3}$ is a primitive of z^2

$$\int_\gamma z^2\,dz = F(\gamma(1)) - F(\gamma(0)) = \frac{(1+\pi i)^3}{3}.$$

or $\int_0^1 (t + \pi i t^2)^2 (1 + 2\pi i t)\,dt = \text{- - - - -}$

which is much longer

II - Cauchy's Theorem and its applications

Cauchy's theorem is at the heart of complex analysis.
It "roughly" says that if f is holomorphic in an open set Ω and $\gamma \subset \Omega$ is a closed curve whose "interior" is contained in Ω then

$$\int_\gamma f(z)\,dz = 0.$$

Cauchy's thm as we will see has many applications, e.g. Liouville's thm, which in return gives a proof of fund. thm of algebra.

The interior of a path is not easy to define for a general curve. We will work around this difficulty by first proving Cauchy's thm for curves whose interior is easy to define, namely for triangles and rectangles (Goursat's thm).
We then use Goursat's thm to show that a holomorphic function on an open disc has a primitive in that disc.
This then will give as a corollary Cauchy's thm on a disc.

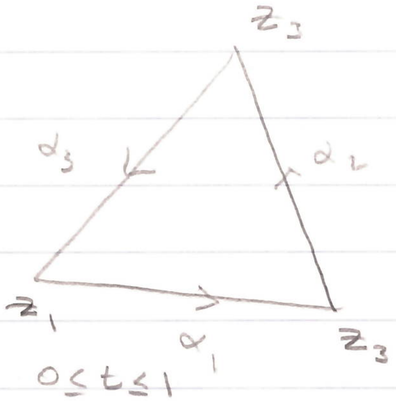
Thm (Goursat) (Thm 1.1 in chapter 2)

Let $\Omega \subset \mathbb{C}$ be open, $T \subset \Omega$ a triangle whose interior is also contained in Ω. Let $f: \Omega \to \mathbb{C}$ be holomorphic then

$$\int_T f(z)\,dz = 0$$

Proof: Note that a triangle is a closed curve which is the union of 3 line segments. If T has 3 corners at z_1, z_2, z_3, then we will write

$$T = \langle z_1, z_2, z_3 \rangle$$
$$:= \alpha_1 + \alpha_2 + \alpha_3$$



with

$$\alpha_1(t) = z_1 + (t-0)(z_2 - z_1) \qquad 0 \le t \le 1$$
$$\alpha_2(t) = z_2 + (t-1)(z_3 - z_2) \qquad 1 \le t \le 2$$
$$\alpha_3(t) = z_3 + (t-2)(z_1 - z_3), \qquad 2 \le t \le 3$$

$$\Delta := \{ z \in \mathbb{C} \mid z = t_1 z_1 + t_2 z_2 + t_3 z_3, \quad 0 \le t_1, t_2, t_3, \quad t_1 + t_2 + t_3 = 1 \}$$

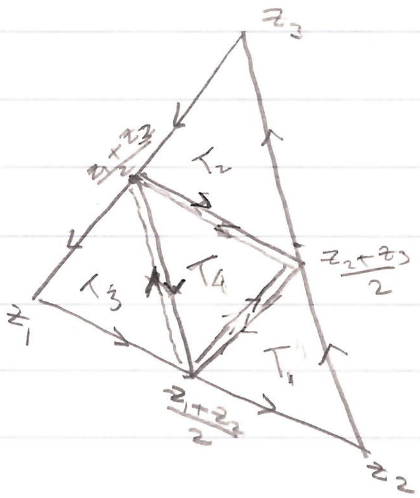
is the smallest convex set containing z_1, z_2, z_3.

Image $T \subset \Delta$, infact Image $T = \partial \Delta$.

We will inductively construct a sequence of triangular paths

We Define

$T^{(n)} = \langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \rangle$ as follows. let



(1) $T^{(0)} = \langle z_1^{(0)}, z_2^{(0)}, z_3^{(0)} \rangle$

$= \alpha_1^{(0)} + \alpha_2^{(0)} + \alpha_3^{(0)}$

when $T^{(n)}$ is defined as

$\langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \rangle$

(2) $T^{(n+1)}$ is one of the 4 triangular paths

$$T_1^{(n)} := \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, z_2^{(n)}, \frac{z_2^{(n)} + z_3^{(n)}}{2} \right\rangle$$

$$T_2^{(n)} := \left\langle \frac{z_2^{(n)} + z_3^{(n)}}{2}, z_3^{(n)}, \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle$$

$$T_3^{(n)} := \left\langle \frac{z_1^{(n)} + z_3^{(n)}}{2}, z_1^{(n)}, \frac{z_1^{(n)} + z_2^{(n)}}{2} \right\rangle$$

$$T_4^{(n)} := \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, \frac{z_2^{(n)} + z_3^{(n)}}{2}, \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle$$

(ie we are at each step partitioning Δ using lines parallel to the sides and passing through their midpoints)

The triangular paths $T_i^{(n)}$ $1 \le i \le 4$ are all entirely contained in Δ, and

$$\int_{T^{(n)}} f\,dz = \int_{T_1^{(n)}} + \int_{T_2^{(n)}} + \int_{T_3^{(n)}} + \int_{T_4^{(n)}}$$

Hence

$$\left|\int_{T^{(n)}} f\,dz\right| \le 4 \max_{1\le i\le 4} \left|\int_{T_i^{(n)}} f\,dz\right|$$

We choose $T^{(n+1)}$ one of $T_i^{(n)}$ $1\le i\le 4$ so that

$$\left|\int_{T^{(n)}} f\,dz\right| \le 4 \left|\int_{T^{(n+1)}} f\,dz\right|$$

It then follows that

$$\left|\int_{T} f\,dz\right| \le 4^n \left|\int_{T^{(n)}} f\,dz\right|$$

and the closed filled triangles $\Delta^{(n)}$ are nested compact sets

$$\Delta = \Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \ldots$$

Moreover if $\operatorname{diam} \Delta_0 = \sup_{z,w\in\Delta} |z-w|$

and d_n and p_n are diameter and perimeter of $\Delta^{(n)}$ respectively, then

$$d_n = \frac{d_0}{2^n} = \frac{d}{2^n}, \quad p_n = \frac{p_0}{2^n} = \frac{p}{2^n}$$

Hence $\operatorname{diam} \Delta^{(n)} \to 0$ as $n \to \infty$.

We now need the following Proposition about nested compact sets.

Prop (Compact nested sets) (Prop 1.4 of ch. 1)

If $\Delta^{(1)} \supset \Delta^{(2)} \ldots \supset \Delta^{(n)} \supset \ldots$ is a sequence of non-emty compact sets in $\mathbb{C}$ with the property that $\operatorname{diam} \Delta^{(n)} \to 0$ as $n \to \infty$.

Then $\exists$ a unique point $z_0 \in \mathbb{C}$ s.t $z_0 \subset \Delta^{(n)}$ $\forall n$.

Pf of Prop:

Choose z_n any point in $\Delta^{(n)}$.

For $n \geq m \geq 1$, integers $|z_n - z_m| \leq \operatorname{diam} \Delta^{(m)}$

Since $\operatorname{diam} \Delta^{(n)} \to 0$ this says that (z_n) is a Cauchy sequence hence converges to a limit z_0.
Then for $m \geq 1$, note $z_n \in \Delta^{(m)}$ $\forall n \geq m$
since $z_n \in \Delta^{(n)} \subset \Delta^{(m)}$
$\Delta^{(m)}$ is compact, hence $\lim z_n = z_0$ is also in $\Delta^{(m)}$ $\forall m \geq$
And z_0 is unique since if z_0, z_0' are both in $\Delta^{(n)}$ $\forall n$ and $z_0 \neq z_0'$ Then $|z_0 - z_0'| > 0$ which contradicts $\operatorname{diam} \Delta^{n} \to 0$

$\boxtimes$

Going back to our integrals: we have

$$I := \left| \int_T f(z)\,dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)\,dz \right|$$

f is holomorphic at $z_0 \in \Delta^{(n)} \subset \Omega$
Hence

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z)$$

with $$\lim_{z \to z_0} \frac{|E(z)|}{|z - z_0|} = 0.$$

It is clear that $E(z)$ is continuous at z_0 and $\lim_{z \to z_0} E(z) = 0$.

Let $\varepsilon > 0$, we will show that $I = \left| \int_T f(z)\,dz \right|$
$\leq \varepsilon\, d\, p$

which will show that $\int_T f\,dz = 0$.

To this end we use

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} f(z_0) + f'(z_0)(z-z_0)\,dz + \int_{T^{(n)}} E(z)\,dz$$

Using cor. 3.3
The first integral on the right is zero, since
$g(z) = f(z_0) + f'(z_0)(z-z_0)$ has a primitive
(Take $G(z) = f(z_0)z + f'(z_0)\frac{(z-z_0)^2}{2}$)

Hence we are reduced to the estimate

$$\left|\int_T f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} E(z)\,dz\right|$$

let $I_n := \left|\int_{T^{(n)}} E(z)\,dz\right|$

For a given $\varepsilon > 0$, choose an open disc
$D_\delta(z_0) \subset \Omega$ s.t

$$|E(z)| \leq \varepsilon\,|z-z_0| \quad \forall\, z \in D_\delta(z_0)$$

Because $d_n \to 0$ as $n \to \infty$ $\exists$ an index N s.t $d_n < \delta$, $\forall n \geq N$. $z_0 \in \Delta^{(n)}$, $\forall z \in \Delta^{(n)}$ $|z - z_0| \leq d_n < \delta$ $\forall n \geq N$.

Hence $\Delta^{(n)} \subset D_\delta(z_0)$ $\forall n \geq N$. and we get

$$|I| \leq 4^n |I_n| \leq 4^n \left| \int_{T^{(n)}} E(z)\,dz \right| \leq 4^n \int_{T^{(n)}} |E(z)||dz|$$

$$\leq 4^n \varepsilon \int_{T^{(n)}} |z - z_0||dz|$$

$$\leq 4^n \varepsilon\, d_n P_n = 4^n \varepsilon \frac{d}{2^n} \frac{p}{2^n} = \varepsilon d p$$

Hence $I = 0$ since ε was arbitrary

As a corollary we get

Cor ((or 1.2 chapter 2) if f is holomorphic in an open set Ω that contains $\mathcal{R}$ a rectangle and its interior, then

$$\int_R f(z)\,dz = 0, \quad \text{where } R = \partial \mathcal{R}$$

(This is corollary 1.2, in chapter 2, p 36 of Stein & Shakarchi (S&S))