§ Conformal Mappings (Chapter 8 in the book).

Motivating question. ① Given 2 open sets $U, V \subset \mathbb{C}$, does there exists a holomorphic bijection between them?

ie $f : U \to V$ $\quad f \in \mathcal{H}(U)$
and bijective

We'll see that inverse map

$f^{-1} : V \to U$ is automatically also holomorphic.

② Given an open sset $\Omega \subset \mathbb{C}$ what conditions guarantee that there is a holomorphic bijection from Ω to $\mathbb{D}$?
where
$\mathbb{D}$ = unit disc.

Why $\mathbb{D}$? $\mathbb{D}$ has a very nice geometric structure and most properties of holom functions we developed for $\mathbb{D}$ first. If there is a holom bijection between Ω and $\mathbb{D}$.

we can hope to transfer questions about holom functions on Ω to holom functions on $\mathbb{D}$.

Plan:

(1) We'll start by examples of such maps and show for example that there is a holom bijection between $\mathbb{D}$ and the upper half plane $\mathbb{H}$.

We can then compose simple maps to get more examples of holomorphic bijections.

(2) We'll then prove Schwarz lemma which says any $f: \mathbb{D} \to \mathbb{D}$ s.t $f(0)=0$ must satisfy

(1) $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$ (f is contracting)

(2) If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$ then f is a rotation

(3) $|f'(0)| \leq 1$ and if equality holds then f is a rotation

(3) Schwarz lemma will then give us all holom bijections of $\mathbb{D}$ to itself.

(4) Then we'll get to Riemann-mapping thm which says if $\Omega \neq \mathbb{C}$ or $\emptyset$ and simply

connected then there is a holom bijection between Ω and $\mathbb{D}$.

More precisely, for any $z_0 \in \Omega$ $\exists$ a <u>unique</u> $f : \Omega \to \mathbb{D}$ s.t

$f(z_0) = 0$ and $f'(z_0) > 0$

<u>Rk.</u> Riemann mapping thm says there are only 3 kinds of simply connected domains in $\mathbb{C}$ (up to holom bijections) ϕ, $\mathbb{C}$, $\mathbb{D}$

Note there can be no holom bijection between $f : \mathbb{C} \to \mathbb{D}$ since then f will be bounded and entire hence by Liouville's f is constant

Note Ω connected is also necessary since $\mathbb{D}$ is connected, same is true for simply connected since if $f : U \to V$ holom biject, U simply connected then so is V.

Defn Let U, V be 2 open sets in $\mathbb{C}$. An injective holomorphic map $f : U \to V$ is called a conformal map from U to V.

If f is bijective the we say that f is a conformal equivalence or a biholomorphism or a holomorphic isomorphism and U and V are conformally equivalent.

If $U = V$ a conformal equivalence is an automorphism. (Note there is a small difference in the defn of conformal compared to the book. In the book f is taken to be bijective).

Proposition 1.1 (8.1.1) If $f : U \to V$ conformal (ie holomorphic and injective) then $\forall z \in U$ $f'(z) \neq 0$. The inverse of f, which is defined on the image of f is holomorphic ie $f : U \to \text{Image}(f) \subset V$ is a conformal equivalence and $f^{-1} : \text{Image}(f) \to U$ is also a conformal equivalence.

Pf. Suppose f is injective and holom. but on the contrary $\exists z_0 \in U$ s.t $f'(z_0) = 0$. We w.t show that f cannot be injective. Let $h(z) = f(z) - f(z_0)$.

Then $h(z_0) = 0$ and $h'(z_0) = 0$

If $k := \mathrm{ord}_{z_0} \big(\underbrace{f(z) - f(z_0)}_{h(z)}\big)$ then by our assumption $k \geq 2$ $(h(z_0) = 0,\ h'(z_0) = 2)$

If $k = \infty$ then $f(z) - f(z_0) \equiv 0$ hence f is constant and cannot be injective.

So can assume $k < \infty$. Then $\exists r > 0$ s.t $\forall z \in D_r(z_0)$, we have

$$f(z) - f(z_0) = \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

$$+ G(z)(z - z_0)^{k+1}$$

$$f(z) - f(z_0) = a(z - z_0)^k + G(z)(z - z_0)^{k+1}$$

where $\frac{f^k(z_0)}{k!} = a \neq 0$, $z \in D_r(z_0)$

Note that since zeroes of f' are isolated we can also choose $D_r(z_0)$ s.t $f'(z) \neq 0$ for $z \in D_r^*(z_0)$.

We will use Rouche's thm to show that for a $w \in \mathbb{C}$, $g(z) = f(z) - f(z_0) - w$ has the same number of zeroes as $a(z-z_0)^k - w$ in some disc around z_0.

If $a(z-z_0)^k = w$ has k solns in that disc, we'll have $g(z) = f(z) - f(z_0) - w$ has k zeroes for z sufficiently close to z_0.

Say $z_1, \ldots z_k$ are the zeroes of g.

If $w \neq 0$ then these zeroes are not equal to z_0 (Since if $z_k = z_0$, then $0 = g(z_k) = g(z_0) = f(z_0) - f(z_0) + w = w \neq 0$.)

Since $f'(z) \neq 0$ for $z \in D_r^*(z_0)$

we have that $g'(z) = f'(z) \neq 0$ for $z \in D_r^*(z_0)$

Hence each zero has order 1 and they are distinct. But that means $\exists$ k distinct $z_1, z_2, \ldots z_k$ s.t $f(z_i) = f(z_0) + w$

ie f is not injective.

To show that in some nbhd of z_0, $g(z) = f(z) - f(z_0) - w$ has k zeroes we write for $z \in D_r(z_0)$

$$f(z) - f(z_0) - w = a(z - z_0)^k + G(z)(z - z_0)^{k+1} - w$$

$$= \left(a(z - z_0)^k - w\right) + \left(G(z)(z - z_0)^{k+1}\right)$$

We apply Rouche's thm as follows

Let $C = \sup_{|z - z_0| = \frac{r}{2}} |G(z)|$, C exists since G is continuous

Pick $0 < s < r/2$ and assume $|w| < |a| \left(\frac{s}{2}\right)^k$. Then on the circle $|z - z_0| = s$,

we have $|a(z - z_0)^k - w| \geq |a| s^k - |a| \left(\frac{s}{2}\right)^k \geq |a| \left(\frac{s}{2}\right)^k$

We also have that

$$\left| G(z)(z - z_0)^{k+1} \right| \leq C s^{k+1}$$

So if $|a| \left(\frac{s}{2}\right)^k > C s^{k+1}$ ie $s < \frac{|a|}{2^k} \frac{1}{C}$

then we can apply Rouché and get that

$g(z) = f(z) - f(z_0) - w$ has the same number of zeroes in $|z - z_0| < s$

as the equation
$$a(z-z_0)^k - w \quad \text{for} \quad |w| < |a| \left(\frac{s}{2}\right)^k$$

(and $s < \min\left(\frac{|a|}{C 2^k}, \frac{r}{2}\right)$) as wanted.

Note if $w = |w| e^{i\theta}$ then the zeroes of
$a(z-z_0)^k - w$ are at z_n, $n = 0, \dots k-1$
where
$$z_n - z_0 = \left|\frac{w}{a}\right|^{1/k} e^{i\left(\frac{\theta + 2\pi n}{k}\right)}, \quad n = 0, \dots k-1$$

But then $|z_n - z_0| = \left|\frac{w}{a}\right|^{1/k} < \frac{s}{2} < s$. Hence all k roots of $a(z-z_0)^k - w$ are indeed inside $D_s(z_0)$!

The rest is straightforward: $f: U \to f(U)$ is clearly bijective. wlog: assume $f(U) = V$
$f^{-1}: V \to U$ is continuous since
$f: U \to V$ is an open map.

Let $w_0 \in V$, $w \in V$ close to w_0. Write $w = f(z)$
$w_0 = f(z_0)$. If $w \neq w_0$ then

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}$$

Since $f'(z_0) \neq 0$, and f' continuous, we have
$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} 1 \Big/ \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{f'(z_0)}$$

Hence $f^{-1} \in \mathcal{H}(V)$ ∎

Remark ① Prop 1.1 says that if $f: U \to V$ is a conformal equivalence then $f^{-1}: V \to U$ is automatically holomorphic

② The conformal equivalence is an equivalence relation

$U \sim_c U$ since $f: U \to U$ identity
$u \mapsto u$

map is bijective and holomorphic

If $U \sim_c V$ with $f: U \to V$ then

$V \sim_c U$ with $f^{-1}: V \to U$

and if $U \sim_c V$, $V \sim_c W$ w/ $f: U \to V$
$g: V \to W$

then $g \circ f: U \to W$ gives a conformal equivalance between U and W.

③ Conformal equivalence allows the transfer the holom. functions on one set to the holomorphic functions on the other set.

Corollary If $f : U \to V$ is a conformal equivalence then the map

$$T : \mathcal{H}(V) \to \mathcal{H}(U)$$
$$\phi \longrightarrow \phi \circ f$$

(where $\phi : V \to \mathbb{C}$ is a holom. func. on V)

is a linear isomorphism with inverse

$$T^{-1} : \mathcal{H}(U) \to \mathcal{H}(V)$$
$$\varphi \longmapsto \varphi \circ f^{-1}$$

($\varphi : U \to \mathbb{C}$ holom func on U).

Ie T is an isom. of vector spaces ie

$$T(a\phi_1 + b\phi_2) = aT(\phi_1) + bT(\phi_2)$$

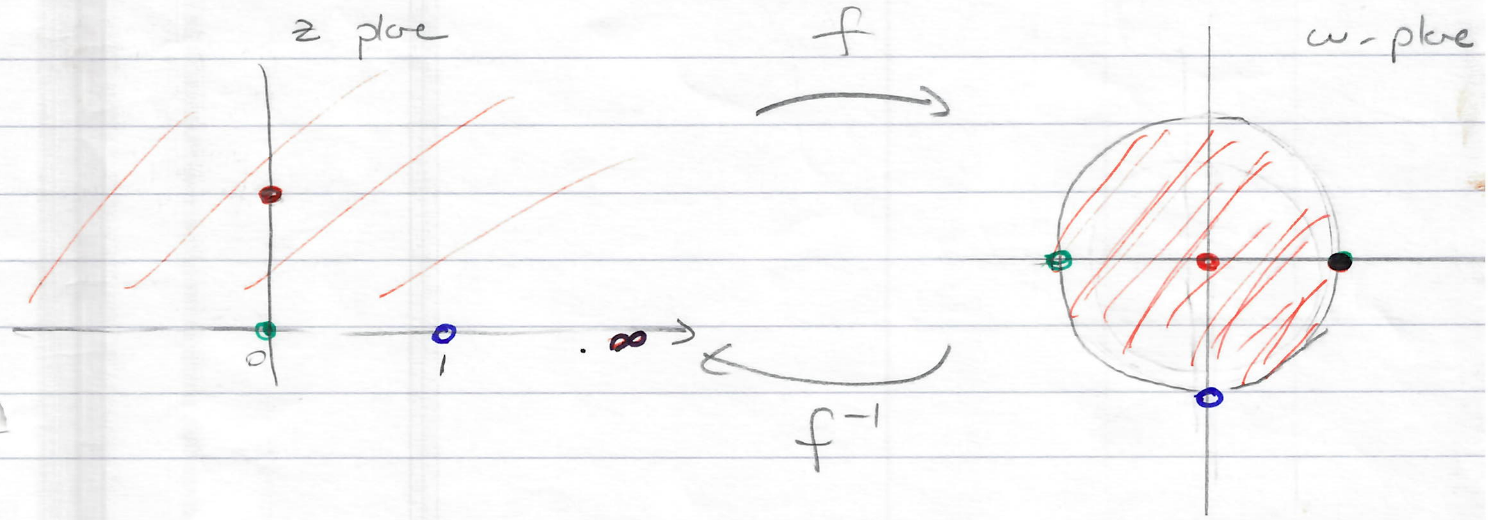
Important

Example (§ 8.1.1) The disc and the UHP.

Let $\mathbb{H} := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ be the upper half plane.

$\mathbb{D} = D_1(0) := \{ z \in \mathbb{C} \mid |z| < 1 \}$ the unit disc

Then the map $f : \mathbb{H} \to \mathbb{D}$, $z \mapsto \dfrac{z - i}{z + i}$ is a conformal equivalence, $f^{-1}(w) = i \, \dfrac{1 + w}{1 - w}$,



Note this example shows that the property that a set is bounded is not preserved under conformal equivalence

Proof. First note that for any $z \in \mathbb{H}$, $|f(z)| = \left|\dfrac{z-i}{z+i}\right| < 1$

Since the distance from z to i is always larger than the distance from z to $-i$, which is in the lower half-plane.

f is clearly holomorphic since $z + i \neq 0$ $\forall z \in \mathbb{H}$

Similarly the map $g(w) = i\,\dfrac{1+w}{1-w}$ is holomorphic for $w \in D_1(0)$

to see that $g(w) \in \mathbb{H}$, we look at the

$$\operatorname{Im} g(w) = \frac{i\left(\frac{1+w}{1+w}\right) - \overline{\left(i\left(\frac{1+w}{1-w}\right)\right)}}{2i}$$

$$= \frac{1}{2}\left(\frac{1+w}{1-w} + \frac{1+\overline{w}}{1-\overline{w}}\right) = \frac{1}{2}\left(\frac{(1-\overline{w})(1+w) + (1-w)(1+\overline{w})}{|1-w|^2}\right)$$

$$= \frac{1-|w|^2}{|1+w|^2} > 0 \quad \text{since } |w| < 1$$

Hence g indeed goes from $D_1(0)$ to $\mathbb{H}$.

Finally direct calculation verifies that $f(g(w)) = w$, $(g \circ f)(z) = z$.

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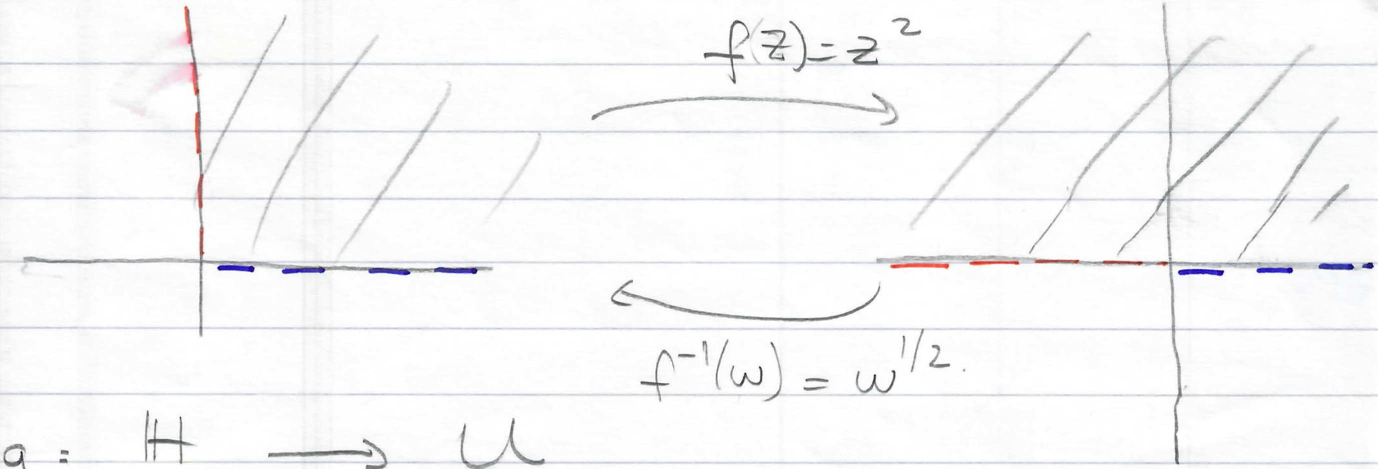
(Note the map takes the real line to the boundary of the disc.

with $f(0) = -1$, $f(1) = -i$, $f(\text{"}\infty\text{"}) = 1$

Example The map $z \to z^2$

$$f : U := \{ z \in \mathbb{C} \mid 0 < \arg z < \pi/2 \} \longrightarrow \mathbb{H}$$
$$z \longrightarrow z^2$$

maps the first quadrant to $\mathbb{H}$



$$g : \mathbb{H} \longrightarrow U$$
$$z \longrightarrow z^{1/2} = \exp\left(\tfrac{1}{2} \operatorname{Log} z\right)$$

Note f is injective since if $z_1^2 = z_2^2$ then $z_1 = \pm z_2$ and only one of z_2, $-z_2$ can be in U. Since $z_1, z_2 \in U$ we have that $z_1 = z_2$.

To show surjectivity let $w = re^{i\theta}$ $0 < \theta < \pi$ (2.49)

$w \in \mathbb{H}$ then $z^2 = w$ has 2 solutions

$$z_{1/2} = \pm w^{1/2} = \pm r^{1/2} e^{i\theta/2}$$

and $z = r^{1/2} e^{i\theta/2}$ is in $\mathcal{U}$.

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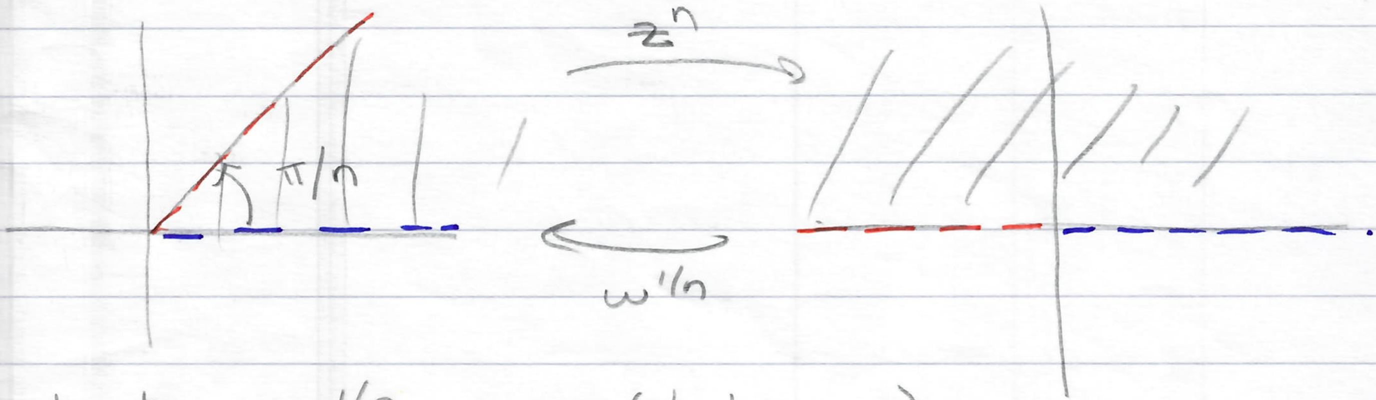
In general the map $z \longrightarrow z^n$ maps a sector $S = \{ z \in \mathbb{C} \mid 0 < \arg(z) < \pi/n \}$



And the $w^{1/n} = \exp(\frac{1}{n} \operatorname{Log} w)$

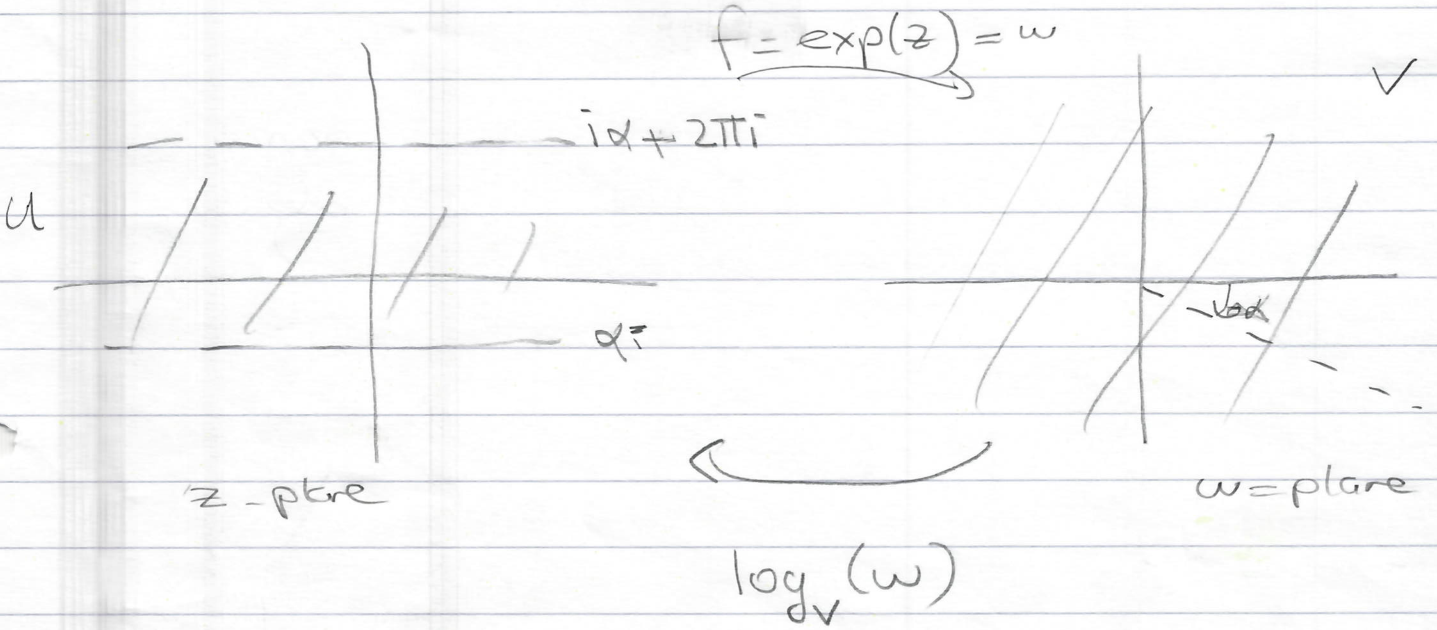
Example The map $f : \mathbb{H} \longrightarrow \mathbb{C}^-$
$z \longrightarrow -z^2$

maps $\mathbb{H}$ to $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$

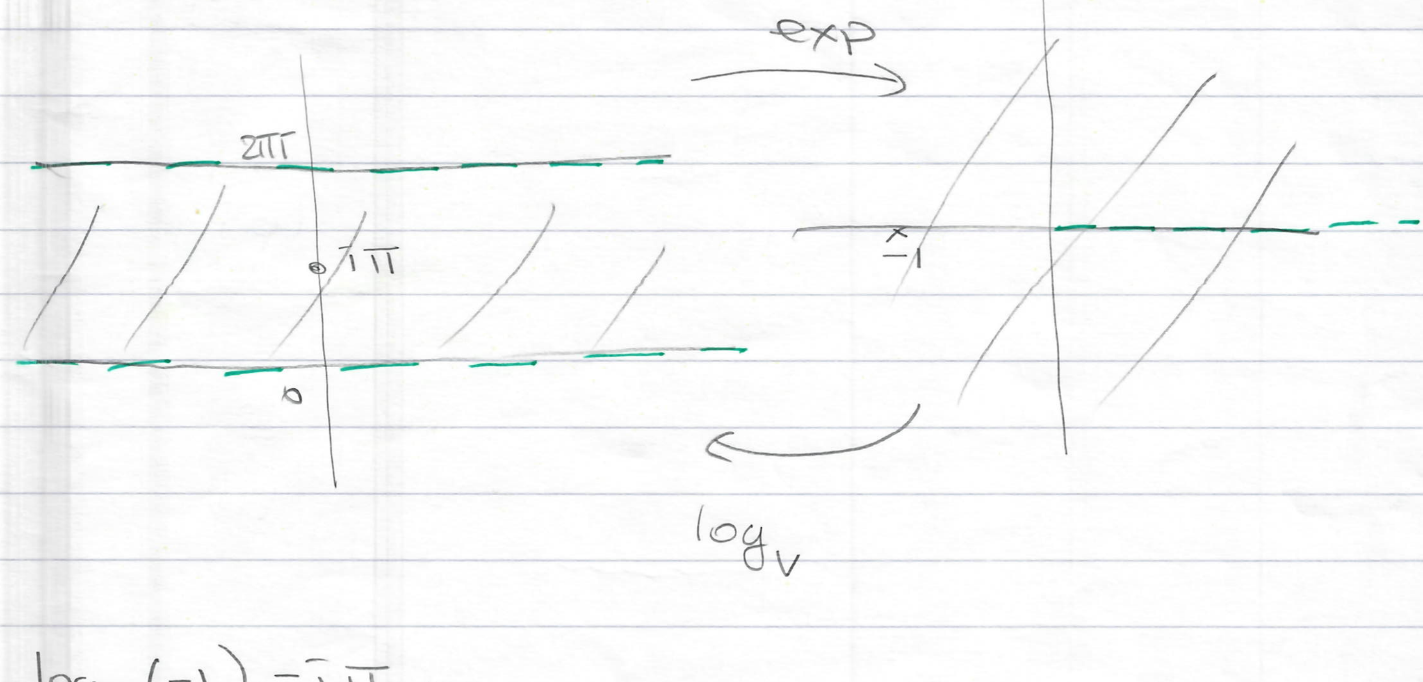
and the map $\tilde{f} : \mathbb{H} \longrightarrow \mathbb{C} \setminus [0, \infty)$
$z \longrightarrow z^2$

maps $\mathbb{H}$ to slit plane cut at the positive reals

Any horizontal strip of length 2π is conformally equivalent to a slit plane



In particular



$\log_v(-1) = i\pi$.

Important non-example $U = \mathbb{C}$, $V = D_1(0)$

Then there is NO biholomorphic map between U and V, since if there were a map
$$f : \mathbb{C} \longrightarrow D_1(0)$$
which is holomorphic. Then f is bounded since $|f(z)| < 1$. Hence by Liouville's thm it is constant, hence is not injective.

Hence $\mathbb{C} \not\simeq_c \mathbb{D}$.

Riemann's thm says that any simply connected domain U which is a proper sset of $\mathbb{C}$, ie $U \neq \emptyset$, and $U \neq \mathbb{C}$ is conformally equivalent to $\mathbb{D}$.

In fact we will prove

Thm (Riemann)(8.3.1) Suppose Ω is proper and simply connected. If $z_0 \in \Omega$ then there is a unique conformal map
$$F : \Omega \longrightarrow \mathbb{D} \quad \text{st} \quad F(z_0) = 0$$
$$F'(z_0) > 0.$$

Cor - Any 2 proper simply connected open ssets of $\mathbb{C}$ are conformally equivalent

Remark Riemann's mapping thm is remarkable
It classifies all simply connected open ssets $\Omega \subseteq \mathbb{C}$, up to conformal equivalence, there are 3 of them $\emptyset$, $\mathbb{C}$, $\mathbb{D}$.
But the proof is not constructive as we'll see. In general it is not easy to find an explicit map.
The rest of the course we'll prove this thm.
The strategy of the proof is as follows.

① **Step 1** - **Uniqueness** : This is going to be easy. It boils down to finding all automorphisms of the unit disc.
Since if we have 2 conformal equivalences

$$f_1 : \Omega \longrightarrow \mathbb{D} \quad , \quad f_2 : \Omega \longrightarrow \mathbb{D}$$

then $f_2 \circ f_1^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ is an autom of $\mathbb{D}$

Step 2 If $\Omega \neq \mathbb{C}$, we'll show there is a conformal map $f : \Omega \to \mathbb{D}$. with $f(z_0) = 0$

Hence Ω is conformally equivalent to an open sset of $D_1(0)$. Hence $\Omega \sim_c f(\Omega) \subset \mathbb{D}_1(0)$

Step 3. Step 2 shows that

The set $\mathcal{F} = \{ f : \Omega \to \mathbb{D} \mid f \text{ conformal}, f(z_0) = 0 \}$

is not empty. We'll see $S := \sup_{f \in \mathcal{F}} |f'(z_0)|$ exists

We'll show that $\exists\, f \in \mathcal{F}$ s.t

$|f'(z_0)|$ is maximal, ie the supremum

S is taken. This f has "maximal expansion speed"

Step 4. The f we found in step 3 is surjective.

If this is the case writing $f'(z_0) = s e^{i\theta}$ and $g(z) = e^{-i\theta} f$ gives the map we are looking for:

$g : \Omega \to \mathbb{D}$ with $g(z_0) = e^{-i\theta} f(z_0) = 0$

$g'(z_0) = s > 0$.