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Step 1 Automorphisms of $\mathbb{D}$ and uniqueness

For the automorphisms of $\mathbb{D}$ we have

Thm 2.2 If $f: \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$, then $\exists$ $\theta \in \mathbb{R}$, and $\alpha \in \mathbb{D}$ s.t.

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

Then $f(0) = e^{i\theta}\alpha$, $f'(0) = e^{i\theta}(|\alpha|^2 - 1)$

Conversly all maps of this form are autom of $\mathbb{D}$.

Remark (1) Note an immediate corollary of Thm 2.2 is that only autom of $\mathbb{D}$ that fixes 0 are rotations

Since

$$f(0) = e^{i\theta}\alpha = 0 \Rightarrow \alpha = 0$$

$$\Rightarrow f(z) = -e^{i\theta} z = e^{i\tilde{\theta}} z \quad \text{for some } \tilde{\theta}$$

(2) This thm is enough to prove the uniqueness of conformal equiv.

$$f: \Omega \to \mathbb{D}.$$

If f_1, f_2 are 2 such maps, then with $f_1(z_0) = f_2(z_0) = 0$, $f_1'(z_0), f_2'(z_0) > 0$

then $g = f_2 \circ f_1^{-1}: \mathbb{D} \to \mathbb{D}$ autom of $\mathbb{D}$

Step 1 Automorphisms of $\mathbb{D}$ and uniqueness

For the autom. of $\mathbb{D}$ we have the following theorem

Thm 2.2 If $f: \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$, then $\exists\, \theta \in \mathbb{R}$, and $\alpha \in \mathbb{D}$ such that

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z} \qquad \forall z \in \mathbb{D} = D_1(0).$$

Moreover $f(0) = e^{i\theta}\alpha$, $f'(0) = e^{i\theta}(|\alpha|^2 - 1)$

Conversely all maps of this form are autom. of $\mathbb{D}$.

Remark An immediate corollary of Thm 2.2 is that the only autom.s of $\mathbb{D}$ which fixes 0 are rotations. Indeed if

$f(0) = e^{i\theta}\alpha = 0$, then $\alpha = 0$ and

$f(z) = -e^{i\theta} z = e^{i\tilde{\theta}} z$ for some $\tilde{\theta}$

Another immediate corollary is the uniqueness statement in Riemann's thm.

Cor Let Ω be a proper simply connected domain in $\mathbb{C}$. Let $f_i: \Omega \to \mathbb{D}$ 2 conformal equivalences, $z_0 \in \Omega$ s.t

$f_i(z_0) = 0$ and $f_i'(z_0) > 0$, $i = 1, 2$

Then $f_1 = f_2$

Proof Let $g = f_2 \circ f_1^{-1}$. Then

$g : \mathbb{D} \to \mathbb{D}$ is an autom of $\mathbb{D}$. (25.5)

Hence $g = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ for some $\theta \in \mathbb{R}$

and $\alpha \in \mathbb{D}$.

Since $f_1(z_0) = 0$, $f_2(z_0) = 0$

$g(0) = 0$, so $\alpha = 0$ and

$g(z) = -e^{i\theta} z$ for $z \in \mathbb{D}$ and $g'(z) = -e^{i\theta}$

Then

$-e^{i\theta} = g'(0) = f_2'(f_1^{-1}(0)) \cdot (f_1^{-1})'(0)$

$= f_2'(z_0) \cdot \dfrac{1}{f_1'(z_0)}$

Hence $\dfrac{f_2'(z_0)}{f_1'(z_0)} = -e^{i\theta} > 0$ (since $f_1'(z_0) > 0$)

$\Rightarrow -e^{i\theta}$ is a positive real number

$\Rightarrow \theta = \pi + 2\pi k$, $e^{i\theta} = -1$

Since we also have that $\alpha = 0$

$g(z) = z \Rightarrow f_1 = f_2$ ∎.

The proof of Thm 2.2 uses a simple but important lemma.

Lemma 2.1 (Schwarz's Lemma).

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) =$
Then

(a) $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$

(b) If for some $z_0 \neq 0$, we have $|f(z_0)| = |z_0|$ then f is a rotation

(c) $|f'(0)| \leq 1$ and equality holds if and only if f is a rotation Ie $\exists \theta \in \mathbb{R}$ s.t $f(z) = e^{i\theta} z$.

Rmk (1) Parts (b) and (c) gives conditions on f so that up to a rotation f is the identity map.
Since we assume $f(0) = 0$, the condition in (b) about $|f(z_0)|$ is true for $z_0 = 0$ and we cannot conclude from it that f is a rotation
(c) is the condition at 0 that is necessary to conclude f is a rotation (Ie $|f'(0)| = 1$)

(2) This lemma is once again a statement for holom. functions. One cannot conclude for a differentiable function any of (a), (b), (c)
eg $f : (-1, 1) \to (-1, 1) \quad f(x) = \sin(\frac{\pi x}{2}), \; f(0) = 0$
but $f'(0) = \pi/2 \in (1, 2)$

Proof (Schwarz's lemma) It is a consequence of maximum modulus principle

(a) $f(0)=0 \Rightarrow \text{ord}_0 f \geq 1$; we can define $g(z)=\frac{f(z)}{z}$ for $z \in \mathbb{D}_1(0)$

Since $\text{ord}_0 f \geq 1$ and $\text{ord}_0 z = 1$, in fact g has a removable singularity at $z=0$. So

$g \in \mathcal{H}(\mathbb{D})$. Fix $z \in \mathbb{D}$

let $0 \leq |z| < r < 1$. For $|w|=r$ we have

$$|g(z)| \leq \max_{|w|=r} |g(w)| = \frac{1}{r} \max_{|w|=r} |f(w)| \leq \frac{1}{r}$$ since $|f(w)|<1$

By maximum modulus principle
$|g(z)| \leq \frac{1}{r} \quad \forall z \in \overline{D_r(0)}$

(holomorphic function g cannot attain a maximum inside $D_1(0)$).

Since this is true $\forall z$, $|z|<r<1$, letting $r \to 1$, it follows that

$|g(z)| \leq 1$ and hence

$|f(z)| \leq |z| \quad \forall z \in D_1(0)$

(b) Part (a) gives $\sup_{z \in D_1(0)} |g(z)| \leq 1$

But the assumption $|f(z_0)| = |z_0|$ for some $0 \neq z_0 \in D$ then says g has a local maximum at $z_0 \in D$. By maximum modulus principle

this can only happen if g is constant

Hence $\exists$ $c \in \mathbb{C}$ s.t $f(z) = z g(z) = c z$

$\forall$ $z \in D_1(0)$. Since $|f(z_0)| = |z_0|$, $|c| = 1$

Hence $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

(c) $$g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$$

Hence $|f'(0)| = |g(0)| \leq 1$

If $|f'(0)| = 1$, then again 0 is a local maximum of g and we conclude as in (b) to get $f(z) = e^{i\theta} z$ for some θ

We can now give the proof of classification of autom. of $\mathbb{D}$

Proof of Thm 2.2.

First note that any function φ_α of the form $\varphi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$ for $\alpha \in \mathbb{C}$ with $|\alpha|<1$

is an autom of $\mathbb{D}$.

~~Since~~ ① Since $|\alpha|<1$, $1-\bar{\alpha}z \neq 0$ for $|z|<1$

so $\varphi_\alpha \in \mathcal{H}(\mathbb{D})$.

② φ_α is injective: $\varphi_\alpha(z) = \varphi_\alpha(w) \iff$

$$\frac{\alpha - z}{1-\bar{\alpha}z} = \frac{\alpha - w}{1-\bar{\alpha}w} \iff \cancel{\alpha} - |\alpha|^2 w - z + \cancel{\bar{\alpha}zw} = \cancel{\alpha} - |\alpha|^2 z - w + \cancel{\bar{\alpha}zw}$$

$$\iff (1-|\alpha|^2)z = (1-|\alpha|^2)w \iff z = w.$$

Hence φ_α is a conformal map $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$

③ $\varphi_\alpha(\mathbb{D}) \subset \mathbb{D}$: If $|z|=1$ then $z = e^{i\theta}$

$$\varphi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \bar{\alpha})} = e^{-i\theta}\left(\frac{\alpha - e^{i\theta}}{e^{-i\theta} - \bar{\alpha}}\right)$$

$$= e^{-i\theta}\frac{w}{-\bar{w}}$$

with $w = \alpha - e^{i\theta}$

Hence $\left|\varphi_\alpha(e^{i\theta})\right| = \left|e^{-i\theta} \cdot \frac{w}{-\bar{w}}\right| = 1$

By max. modulus principle $|\varphi_\alpha(z)| < 1 \quad \forall z \in \mathbb{D}$

(Since $\varphi_\alpha(z)$ is not the constant map, it cannot have a local max. inside $\mathbb{D}$.

(4) $(\varphi_\alpha \circ \varphi_\alpha)(z) = \dfrac{\alpha - \dfrac{\alpha - z}{1-\bar{\alpha}z}}{1 - \bar{\alpha}\left(\dfrac{\alpha - z}{1-\bar{\alpha}z}\right)} = \dfrac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \bar{\alpha}z - |\alpha|^2 + \bar{\alpha}z}$

$$= \frac{(1-|\alpha|^2)\, z}{1-|\alpha|^2} = z .$$

Hence φ_α is its own inverse.

Clearly any rotation $R(z) = e^{i\theta} z$ is also an autom of $\mathbb{D}$.

Hence $(R \circ \varphi_\alpha)(z) = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha}z}$ is an autom of $\mathbb{D}$.

Now let f be any autom of $\mathbb{D}$.
Then $\exists$ a unique $\alpha \in \mathbb{D}$ s.t $f(\alpha) = 0$