Review

A point $z_0 \in \mathbb{C}$ is an isolated singularity of $f : \Omega \to \mathbb{C}$ if f is holomorphic in some punctured disc $D_r^*(z_0) \subset \Omega$

z_0 is a removable singularity of $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ if one of the following equivalent conditions hold

1. f is holom extendable to Ω
2. f " cont. " " "
3. f is bdd in $D_r^*(z_0) \subset \Omega$ for some $r > 0$
4. $\lim_{z \to z_0} (z - z_0) f(z) = 0$

z_0 is a pole of order k of $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ if one of the equivalent conditions hold

1. $\exists\, n > 0$ s.t $(z - z_0)^n f(z)$ is bdd near z_0 and $k = \min\{n \in \mathbb{N} \mid (z - z_0)^n f(z)$ is bdd near $z_0\}$

2. $\exists\, r > 0$, $\exists\, g \in \mathcal{H}(D_r(z_0))$ s.t $g(z_0) \neq 0$ and $f(z) = (z - z_0)^{-k} g(z)$ $\forall z \in D_r^*(z_0)$

3. $\exists\, r > 0$ and $\exists\, h \in \mathcal{H}(D_r(z_0))$ s.t $D_r(z_0) \subset$ [illegible], $h(z) \neq 0$ $\forall z \in D_r^*(z_0)$, h has a zero of order k at z_0 and $f(z) = \dfrac{1}{h(z)}$ $\forall z \in D_r^*$

z_0 is an essential singularity if it is neither removable nor a pole!

Thm If f has a pole of order k at z_0
then $\exists r > 0$, $D_r(z_0) \subset \Omega$ s.t.
$\forall z \in D_r^{+}(z_0)$

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)$$

where $G \in \mathcal{H}(D_r(z_0))$
a_{-1} = residue of f at z_0.

$$P_{z_0}(f,z) = P_{z_0}(z) = \frac{a_{-k}}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

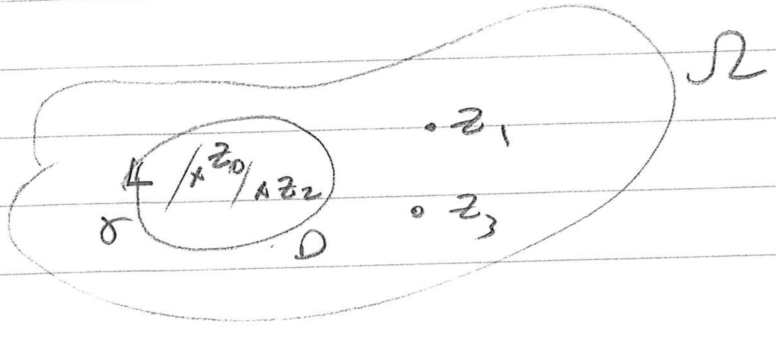
Thm (Residue Formula) let $\Omega \subset \mathbb{C}$ open, $F = \{z_0, \dots z_n\}$ a finite set in Ω

Suppose $f \in \mathcal{H}(\Omega \setminus F)$ holom except for poles at $z_0, z_1, \dots, z_n \in F$
let γ be any circle contained in Ω with ccw orientation and s.t. $\gamma \cap F = \emptyset$
let D be the open disc bdd by γ

Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in F \cap D} \operatorname{res}_{z_i} f$$

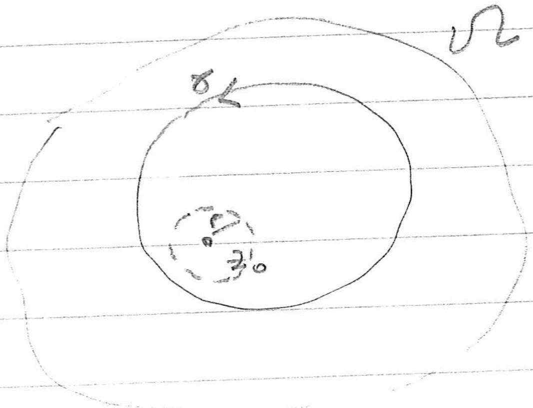
eg



$$\frac{1}{2\pi i}\int_\gamma f\,dz = \operatorname{Res}_{z_0} f + \operatorname{Res}_{z_2} f$$

Proof of the Residue Formula.

Let's first assume f is holomorphic in an open set Ω containing a circle and its interior, except for a single pole at z_0 inside γ. Let D the disc bdd by γ.



By thm 1.3

$$f(z) = P_{z_0}(z) + G(z)$$

where $G(z)$ is holom in a nbhd $D_r(z_0)$ of z_0

and $P_{z_0}(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0}$

is the principal part of f at z_0.

Another way to say this is that the function $f(z) - P_{z_0}(z)$ extends holomorphically to Ω. (Note $P_{z_0}(z)$ is holom on $\mathbb{C}$ except at z_0)

$$g(z) = \begin{cases} f(z) - P_{z_0}(z) & z \in \Omega \setminus \{z_0\} \\ G(z) & z \in D_r(z_0) \end{cases}$$

is the holom extension of $f(z) - P_{z_0}(z)$ to Ω.

Then $$\int_\gamma \left(f(z) - P_{z_0}(z)\right) dz = 0$$

$$\int_\gamma f(z)\,dz = \int_\gamma P_{z_0}(z)\,dz$$

and we're left to prove $\int_\gamma P_{z_0}(z)\,dz = 2\pi i\, a_{-1}$

But this follows Cauchy integral formula applied the constant function, $F(z) = 1$

<u>Recall</u>: CIF for derivatives. Let $C = \partial D$ be any circle whose interior D is contained in Ω. Then for $F \in \mathcal{H}(\Omega)$, any $z \in D$

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{F(w)}{(w-z)^{n+1}}\,dw$$

Here

$$\int_\gamma \frac{dz}{(z-z_0)^n} = \frac{2\pi i}{n!}\frac{d}{dz}^{(n-1)}(1) = \begin{cases} 0 & \text{if } n-1 \geq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

Hence we get $\int_\gamma f(z)\,dz = 2\pi i\, a_{-1}$.

For the general case that f is holom

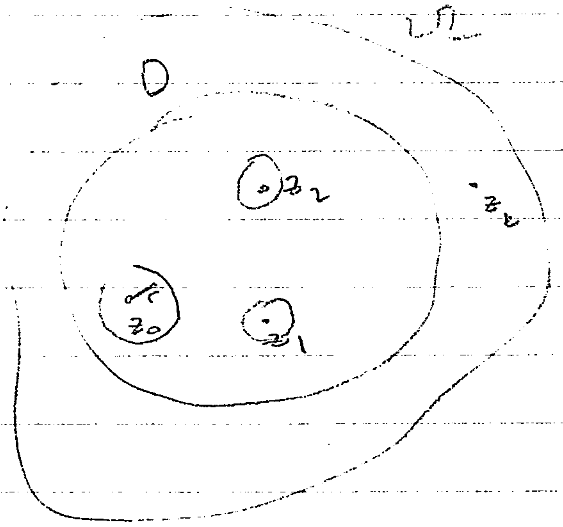
in Ω except for finitely many points $z_0 \ldots z_n$.

For each z_i let P_{z_i} be the principal part at z_i, which is holomorphic in $\mathbb{C} \setminus \{z_i\}$.

Define $g(z) := f(z) - \sum_{z_i \in F} P_{z_i}$ if $z \notin F$.

Then $g \in \mathcal{H}(\Omega \setminus F)$ and in fact g can be extended holomorphically to all Ω.

To see this let $z_0 \in F$, $r > 0$ s.t. $D_r(z_0) \subset D$, $D_r^*(z_0) \cap F = \emptyset$ and $f(z) - P_{z_0}(z)$ is hol. in $D_r(z_0)$



Then for $z \in D_r^*(z_0)$

$$g(z) = \overbrace{\sum_{\substack{z_i \in F \\ z_i \neq z_0}} P_{z_i}(z)}^{\text{hol. in } D_r(z_0)} + \underbrace{f(z) - P_{z_0}(z)}_{\text{extends hol. to } D_r(z_0)}$$

This gives an extension of g to $(\Omega \setminus F) \cup \{z_0\} = \Omega \setminus \{z_1 \ldots z_n\}$

We can do this for each $z_i \in F$ to get a holom. ext. of g to all Ω

and by Cauchy's Thm $\int_\gamma g \, dz = 0$

Which in return gives

$$\int_\gamma f(z)\,dz = \sum_{z_i \in F} \int_\gamma P_{z_i}(z)\,dz$$

If $z_i \in F \cap D$ then as before

$$\int_\gamma P_{z_i}(z)\,dz = \int_\gamma \sum_{j=1}^{k} \frac{a_{-j}}{(z-z_i)^k}\,dz$$

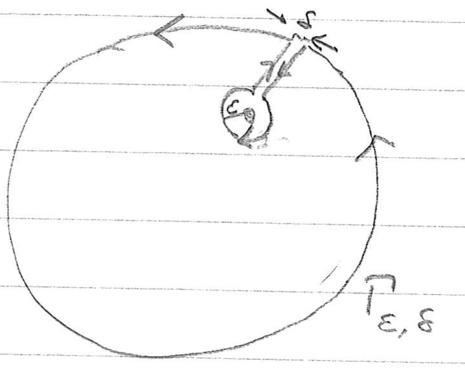
$$= 2\pi i\, a_{-1} = 2\pi i \,\mathrm{Res}_{z_i} f$$

If $z_i \in F$ but not insite D then

$$\int_\gamma P_{z_i}(z)\,dz = 0$$ since then $P_{z_i}(z)$ is holomorphic inside the disc.

Hence we get $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in F \cap D} \mathrm{res}_{z_i} f$

Remark. ① Another way to prove this is the following: First assume there is just one pole inside γ. Consider the following contour $\Gamma_{\varepsilon,\delta}$.



Inside $\Gamma_{\varepsilon,\delta}$ f is holom and can show

$$\int_{\Gamma_{\varepsilon,\delta}} f(z)\,dz = 0$$

Here we went around the pole z_0 with a small circle of radius ε. The width of the corridor is δ. We can then make the width of the corridor narrower by letting $\delta \to 0$ and use continuity of f to show that the 2 sides of the corridor cancel each other. The remaining part consists of 2 curves, the large circle γ and the small circle C_ε with cw orientation, and get

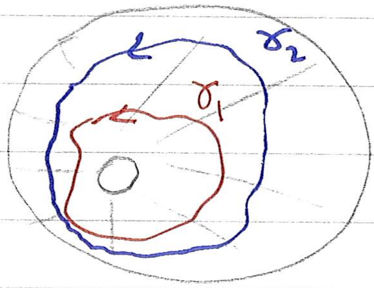
$$\int_\gamma f(z) + \int_{C_\varepsilon} f(z)\,dz = 0.$$

But it takes some effort to make this argument rigorous

Remark The best way to understand and generalize the residue formula (and CIF) is via homotopy.

It is based on the following principle:

Let f be holomorphic in an open set Ω

For example  between 2 circles

Then the principle is that if 2 closed curves can be deformed to each other while remaining in Ω, then $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$

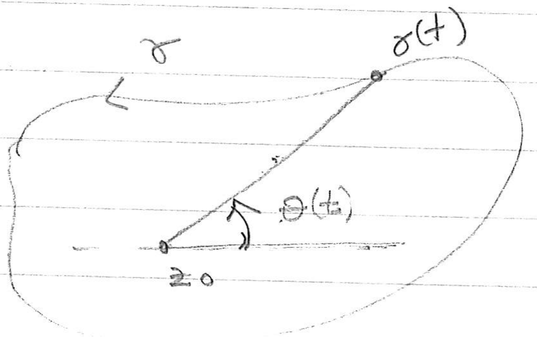
We'll get back to this soon.

Remark 3. If γ is not a circle, but a triangle, or a polygon or any curve γ which has a parametrization of the form

$$\gamma : [a,b] \longrightarrow \mathbb{C} \setminus \{z_0\}$$

$$t \longmapsto z_0 + r(t)\, e^{i\theta(t)}$$

for some C^1 functions r, and $\theta : [a,b] \to \mathbb{R}$
$r(t) > 0$, $r(a) = r(b)$, $\theta(a) = 0$, $\theta(b) = 2\pi$



$r(t) := |\gamma(t) - z_0|$

$\theta(t)$ is a continuous choice of argument along $\tilde{\gamma}(t) = \gamma(t) - z_0$.

$$e^{i\theta(t)} = \frac{\gamma(t) - z_0}{|\gamma(t) - z_0|}.$$

Then $\gamma'(t) = r'(t)\, e^{i\theta(t)} + r(t)\, i\, \theta'(t)\, e^{i\theta(t)}$

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \int_a^b \frac{\gamma'(t)\, dt}{r(t)\, e^{i\theta(t)}}$$

$$= \int_a^b \frac{r'(t)}{r(t)}\, dt + i \int_a^b \theta'(t)\, dt = \log r(t) \Big|_a^b + i\, \theta(t) \Big|_a^b$$

$$= 0 + 2\pi i$$

(This is similar to the parametrization of a circle using a point inside other than the center.)

Note for $\int_\gamma \frac{dz}{(z-z_0)^n} = 0$ for $n > 1$

since $\frac{1}{(z-z_0)^{n-1}} \cdot \frac{1}{1-n}$ is a primitive

of $\frac{1}{(z-z_0)^n}$ in $\mathbb{C} \setminus \{z_0\}$, and γ is closed.

Hence for any such contour γ we have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in \text{inside } \gamma} \operatorname{res}_{z_i} f$$

— o —

Before we give more theoretical applications of residue theorem, let's give some applications to the evaluation of real integrals.

Example Integrals of rational functions

e.g. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$

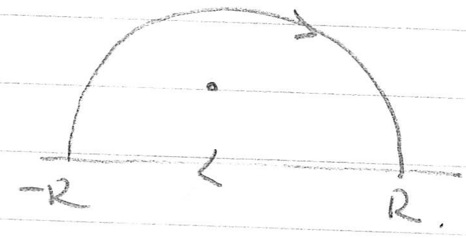
This is of course using arctangent can be evaluated easily. We give another proof using residue theorem.

Idea: To choose a function f and a closed contour so that part of the contour leads to the real integral after taking limits

In this particular case we take

$f(z) = \dfrac{1}{1+z^2}$ and γ_R as the

contour



f has only one pole, at $z=i$ inside γ_R

$$\int_{\gamma_R} f\,dz = 2\pi i \operatorname{Res}_i f = 2\pi i \lim_{z\to i} (z-i)\frac{1}{z^2+1}$$

$$= 2\pi i \lim_{z\to i} \frac{1}{z+i} = \pi.$$

$$\pi = \int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{T_R} \frac{dz}{1+z^2}$$

as $R \to \infty$ the first integral gives $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

and we'll see that over the semicircle T_R

the integral goes to zero as $R \to \infty$

This is ~~because~~ on T_R

$$|z^2+1| > R^2-1$$

Hence $$\frac{1}{z^2+1} < \frac{1}{R^2-1} \sim \frac{1}{R^2}$$

$$\left| \int_{T_R} \frac{dz}{1+z^2} \right| < \frac{1}{R^2-1} \cdot \pi R \approx \frac{1}{R}$$ and this goes

to zero as $R \to \infty$.

Hence $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

The same technique works to evaluate the integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$ where P, Q polynomials where Q has no zeroes on the real axis.

and $\deg Q(x) \geq \deg P(x) + 2$

Note we need this bound for degrees of P, and Q so that we can get

$$\int_{\Gamma_R} \frac{P(x)}{Q(x)}\,dx \to 0 \quad \text{as} \quad R \to \infty$$

If $\deg Q = n$, $\deg P = m$, on the semicircle for R large

$Q(z)$ satisfy : $|Q(z)| > B|z|^n$ for some B

and we can bound $\left|\frac{P(z)}{Q(z)}\right| < C\,\frac{R^m}{R^n}$

$$< C\,\frac{1}{R^{n-m}}$$

Hence $\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz \right| \leq C \frac{1}{R^{n-m}} \cdot R$

$$= C \frac{1}{R^{n-m-1}}$$

For this to go to zero as $R \to \infty$
we need $n-m-1 > 0$

Hence $n > m+1$, ie $n \geq m+2$

Hence $\deg Q \geq \deg P + 2$.

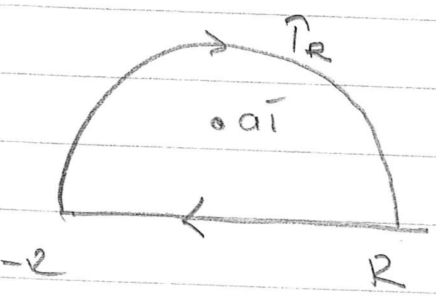
Then $\int_{\gamma_R} \frac{P(z)}{Q(z)} dz = \int_{-R}^{R} \frac{P(x)}{Q(x)} dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz$

gives as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{z_i} \operatorname{Res}_{z_i} \frac{P(z)}{Q(z)}$$

where z_i's are the poles of $\frac{P}{Q}$ inside γ_R

Example $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$



let $f(z) = \dfrac{1}{(z^2+a^2)^2}$

then f has poles at $\pm ai$ of order 2. wlog assume $a>0$

$$\operatorname*{Res}_{z=ai} \frac{1}{(z^2+a^2)^2} = \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 f(z)\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai} \frac{-2}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{-2}{-8a^3 i}$$

$$= \frac{-i}{4a^3}.$$

Hence $\displaystyle 2\pi i \operatorname*{Res}_{z=ai} f = \frac{2\pi i(-i)}{4a^3} = \frac{\pi}{2a^3}.$

As above $\displaystyle\lim_{R\to\infty}\int_{\Gamma_R} \frac{dz}{(z^2+a^2)^2} \to 0$ and we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

Example The same contour can be used to evaluate integrals of rational functions times sin(ax), cos(ax).

Ie integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\, dx .$$

where P, Q are poly in $\mathbb{R}[x]$ with $\deg Q \geq \deg P + 2$.

As $f(z)$ we take $\dfrac{P(z)}{Q(z)} e^{iaz}$

and not $(P(z)/Q(z))\cos(az)$ since $\cos az$ behaves badly on the UHP. On the imaginary axis for example

$$\cos it = \frac{e^{t} + e^{-t}}{2} = \frac{e^{2t} + 1}{2e^{t}}$$

is hyperbolic cosine which grows exponentially.

Where as $|e^{iz}| = |e^{i(x+iy)}| = e^{-y}$ which is bdd by 1 in the UHP

ie. $|e^{iz}| \leq 1$ for $\operatorname{Im} z \geq 0$.

e.g. $$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \qquad a>0.$$

$f(z) = \dfrac{e^{iaz}}{z^2+1}$ which has only one pole on the UHP, at $z=i$

$$\operatorname*{Res}_{z=i} \frac{e^{iaz}}{z^2+1} = \lim_{z\to i} \frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

Hence $$\int_{\gamma_R} f(z)\,dz = 2\pi i \, \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Since $|e^{iaz}| < 1$ on UHP

$$\left| \frac{e^{iaz}}{z^2+1} \right| \le \frac{1}{R^2-1}$$ Hence

$$\left| \int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\,dz \right| \le \frac{\pi R}{R^2-1} \to 0 \quad \text{as } R\to\infty$$

and $$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a} \qquad a>0$$

Now we note $\dfrac{\cos ax}{1+x^2} = \mathrm{Re}\left(\dfrac{e^{iax}}{x^2+1}\right)$

Hence taking real parts we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\, dx = \pi e^{-a}$$

This also shows $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\, dx = 0$

which can also be seen directly as $\dfrac{\sin ax}{x^2+1}$ is an odd function.

Example Integrals of Trigonometric functions.

Residue thm can be used to evaluate real integrals of the form

$$\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt$$

where P, Q are polys and $Q(x,y) \neq 0$ for $x^2+y^2 = 1$ $\forall x, y \in \mathbb{R}$.

Ex. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ $\quad a>1$

The idea is to convert it to a contour integral around the unit circle γ

The usual parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$
$\theta \longmapsto e^{i\theta}$
gives that $d\theta = \frac{dz}{iz}$ for z on the unit cir

The trig. functions $\cos\theta$, $\sin\theta$ can also be written in terms of z on the unit circle as follows

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-1/z}{2}$$

Hence we can write $a+\cos\theta = a + \frac{1}{2}(z+z^{-1})$

and $$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \int_{|z|=1} \frac{1}{a+\frac{1}{2}(z+z^{-1})}\frac{dz}{iz}$$

$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+2az+1}$$

The poles of the integrand are at $-a \pm \sqrt{a^2-1}$, Only one of these roots is inside the unit circle $z_0 = -a + \sqrt{a^2-1}$

$$\operatorname{Res}\left(\frac{1}{z^2+2az+1}, \underbrace{-a+\sqrt{a^2-1}}_{=z_0}\right) = \lim_{z \to z_0} (z-z_0)\frac{1}{(z-z_0)(z-z_1)}$$

$$= \lim_{z \to z_0} \frac{1}{2z+2a} = \frac{1}{2(-a+\sqrt{a^2-1})+2a}$$

L'hospital

$$= \frac{1}{2\sqrt{a^2-1}}$$

Hence $$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2}{i}\left(2\pi i \frac{1}{2\sqrt{a^2-1}}\right) = \frac{2\pi}{\sqrt{a^2-1}}$$

∎