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**Remark** Note $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

and $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

and $\cos z + i\sin z = e^{iz}$

But this is <u>not</u> a decomposition of $e^{iz}$ into real and imaginary parts
Both $\cos z$, $i\sin z$ are complex valued

When $z = x \in \mathbb{R}$ then $e^{ix} = \cos x + i\sin x$ is a decomposition into real and imaginary parts

When $z = iy \in i\mathbb{R}$ then $e^{iz} = e^{-y}$

and $e^{-y} = \cos iy + i\sin iy$ is a sum of 2 real valued functions

$\cos iy = \frac{e^{-y} + e^{y}}{2}$ , $i\sin iy = \frac{e^{-y} - e^{y}}{2}$

$= \cosh y$ $\qquad = -\sinh y$

Both $\cosh y$ and $\sinh y$ grow as $y \to \infty$
but their difference is

$e^{-y} = \cosh y - \sinh y$ and decays as $y \to \infty$

We now turn to more theoretical applications of residue thm.

We start by giving one more description of an isolated singularity which is a pole.

Namely we have the following

**Proposition** (Cor 3.2) Suppose $f$ has an isolated singularity at the point $z_0$. Then $z_0$ is a pole of $f$ if and only if $\lim_{\substack{z \to z_0 \\ z \neq z_0}} |f(z)| = \infty$

Proof: If $f(z)$ has a pole of order $k \geq 1$ at $z_0$, then
$$f(z) = g(z) \cdot (z - z_0)^{-k} \quad \text{on } D_r^*(z_0)$$

for some $r > 0$ and $g \in \mathcal{H}(D_r(z_0))$ and $g(z_0) \neq 0$

Then $|f(z)| = |g(z)| \, |z - z_0|^{-k} \to \infty$ as $z \to z_0$

Since $|g(z)| \to |g(z_0)| \neq 0$ and $k \geq 1$.

Conversely if $|f(z)| \to \infty$ then we can

find $r > 0$ s.t. $|f(z)| \geq 1$ for $z \in D_r^*(z_0)$

(in particular $f(z) \neq 0$ for $z \in D_r^*(z_0)$)

so that $h(z) := \frac{1}{f(z)}$ is holomorphic in $D_r^*(z_0)$ and $|h(z)| \leq 1$ there.

Furthermore $\lim_{z \to z_0} h(z) = \lim_{z \to z_0} \frac{1}{f(z)} = 0$

By Riemann's thm $h(z)$ extends to a holom function in $D_r(z_0)$ by defining $h(z_0) := \lim_{z \to z_0} \frac{1}{f(z)} = 0$.

If $N$ is the order of zero of $h$ at $z_0$, then $f(z) = 1/h(z)$ has a pole of order $N$ at $z_0$ ☒

We've seen an isolated singularity $z_0$ of $f$ is removable if $f(z)$ is bdd near $z_0$ and $z_0$ is a pole if $|f(z)| \to \infty$ as $z \to z_0$

**Defn** An isolated singularity $z_0$ of $f$ is called an essential singularity if it is neither removable nor a pole.

As we saw in the very beginning the function $e^{1/z}$ near $z=0$ has a more erratic behaviour.

e.g $e^{1/x} \to 0$ as $x \to 0$ from negative reals.

whereas $e^{1/x} \to \infty$ as $x \to 0$ from the +ve reals

In fact any function $f: \Omega \to \mathbb{C}$ behaves erratically near an essential singularity. More precisely we have

**Thm (Casorati – Weierstrass)** Suppose

$f$ is holomorphic in $D_r^*(z_0)$ and has an essential singularity at $z_0$.
Then the image of $D_r^+(z_0)$ under $f$ is dense in $\mathbb{C}$.

**Remark.** The Casorati – Weierstrass thm states that the image of a punctured disc $D_r^+(z_0)$, no matter how small, effectively fills up the whole complex plane. (where $z_0$ is an essential singularity).
In fact a remarkable thm of Picard says

**Thm (Picard) (1879)**
If $f \in \mathcal{H}(D_r^+(z_0))$ and has an essential singularity at $z_0$, then $\mathbb{C} \setminus f(D_r^+(z_0))$ contains at most one point.

The function $f(z) = e^{1/z}$ maps each punctured disc centered at $z=0$ to $\mathbb{C} - \{0\}$

ie it does not take the value $0$

so the "exceptional value" permitted by Picard's thm may in fact exist.

Proof of Casorati – Weierstrass :

w.t.s. $\forall w \in \mathbb{C}$, $\forall \varepsilon > 0$, $\exists z \in D_r^*(z_0)$

s.t $|f(z) - w| < \varepsilon$.

We argue by contradiction and will show that this will force the singularity $z_0$ to be either removable or a pole and hence contradicting the assumption that $z_0$ is essential

Assume on the contrary that $\exists w_0 \in \mathbb{C}$ and $\delta > 0$ s.t $\forall z \in D_r^*(z_0)$

$$|f(z) - w_0| \geq \delta$$

let $g(z) := \dfrac{1}{f(z) - w_0}$ $\forall z \in D_r^*(z_0)$

then on $D_r^*(z_0)$ $g(z)$ is bounded by $1/\delta$

hence has a removable singularity at $z_0$.

by Riemann's thm on remov. singularities (Thm 3.1 III)

Hence there is an extension of $g$, ie

we can define $g$ at $z_0$ so that $g$ becomes holom in $D_r(z_0)$. In particular $\lim_{z \to z_0} g(z)$ exists

Since $|f(z) - w_0| \geq \delta$ and $g(z) = \dfrac{1}{f(z) - w_0}$

clearly $g$ is zero free in $D_r^*(z_0)$. Hence

Its reciprocal $1/g$ has an isolated singularity at $z_0$. This singularity of $1/g$ is

either a pole or removable depending on whether $\lim_{z \to z_0} |g(z)| = 0$ or not (resp.)

This in turn gives that the singularity of $f = w_0 + \dfrac{1}{g}$ at $z_0$ can be

no worse than a pole, giving the anticipated contradiction $\blacksquare$ (Note $\lim_{z \to z_0} g(z)$ exists since $g$ has a removable singularity at $z_0$) $\blacksquare$

## Meromorphic Functions

We now look at functions whose singularities are poles. Since at a pole $\lim_{z \to z_0} |f(z)| = \infty$ this suggests

adding $\infty$ to the values of functions and and hence include the poles in their domain of definition

eg. $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $z \to \dfrac{1}{z}$ can be extended to $\hat{f} : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, $z \to \dfrac{1}{z}$.

**Defn** ① $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$

the "ideal" point at $\infty$
unsigned.

$\hat{\mathbb{C}}$ is called the extended complex plane

We can supplement the rules in $\mathbb{C}$ by

$\infty \pm z = z \pm \infty = \infty$ for $z \in \mathbb{C}$

$\infty . z = z . \infty = \infty$ for $z \in \hat{\mathbb{C}} \setminus \{0\}$

$z/\infty = 0$ for $z \in \mathbb{C}$

$z/0 = \infty$ for $z \in \hat{\mathbb{C}} \setminus \{0\}$.

The expressions $\infty \pm \infty$, $\infty/\infty$, $0/0$, $0 . \infty$ are not assigned a meaning in $\hat{\mathbb{C}}$.

② A sequence $(z_n) \subset \mathbb{C}$ converges to $\infty$ if

$\lim |z_n| = \infty$ where $(|z_n|)$ is a sequence in $\mathbb{R}$.

Similarly we say $\lim_{z \to z_0} f(z) = \infty$ if

$\lim_{z \to z_0} |f(z)| = \infty$.

Remark $\hat{\mathbb{C}}$ is not a field!