Last week.

We've seen applications of Cauchy's thm and CIF: $f: \Omega \to \mathbb{C}$, $\overline{D} \subset \Omega$.

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)} dw \quad \text{for any } z \in D.$$

- **Thm** $f: \Omega \to \mathbb{C}$ (Ω open) holom. $z_0 \in \Omega$
  $r > 0$ s.t $\overline{D_r(z_0)} \subset \Omega$. Then
  $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in D_r(z_0)$$
  where $\boxed{n! a_n = f^{n}(z_0) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw}$

Cauchy inequalities: $|f^{(n)}(z_0)| \leq \frac{n! \, \|f\|_{C_r(z_0)}}{r^n}$

$$\|f\|_{C_r(z_0)} := \sup_{w \in C_r(z_0)} |f(w)|$$

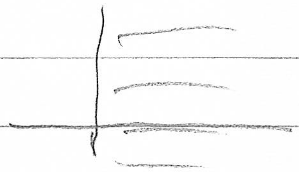
- **Thm** (Liouville's thm): If $f \in \mathcal{H}(\mathbb{C})$, and bounded then f is a constant

Thm (Fundamental thm of algebra) Every poly $P(z) = a_n z^n + \dots + a_0$ of degree $n \geq 1$ ($a_n \neq 0$) has precisely n roots $w_1, \dots w_n \in \mathbb{C}$ and $P(z) = a_n (z - w_1) \dots (z - w_n)$

We've already seen that $P(z)$ has a root, say $w_1 \in \mathbb{C}$

Rmk Liouville's thm is special to functions holom on <u>all</u> of $\mathbb{C}$.

let $\Omega = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$



$f(z) = \dfrac{1}{z+1}$ then $f \in \mathcal{H}(\Omega)$

and bdd; $|f(z)| = \dfrac{1}{|z+1|} \leq 1$ $(|z+1| \geq 1$

if $\operatorname{Re} z > 0$.

But f is not constant

Next we discuss the principle of analytic continuation (of identities) which says that: if Ω is open and connected, $f \in \mathcal{H}(\Omega)$ and f vanishes on an infinite set Z of distinct points with a limit point $z_0 \in \Omega \setminus Z$ then $f \equiv 0$.

Remark ① Holomorphic functions can have ∞ly many zeroes.
e.g $f(z) = \cos z$ (or $\sin z$) has zeroes for $z = (2k+1)\frac{\pi}{2}$ (or $z = \pi k$)

But we'll see that the zeroes are isolated, ie for each zero z_0 of f $\exists$ a neighbourhood of z_0 with no other zeroes.

② There are holomorphic functions with no zoroes, eg: constant function, e^z.

We start by the defn of a limit point

Defn $z_0 \in \mathbb{C}$ is a limit point of a set Ω if $\exists$ a sequence $(z_n)_{n\geq 1}$ in $\Omega \setminus \{z_0\}$ (ie $z_n \neq z_0$) s.t $\lim z_n = z_0$.
Hence $\forall \varepsilon > 0$, $\Omega \cap (D_r(z_0) \setminus \{z_0\}) \neq \phi$.

Rmk If $\Omega = [-1, 1] \cup \{2i\}$ then the $z_n \neq z_0$ condition avoids the case $2i$ is a limit point of Ω. Since otherwise we could take $z_n = 2i$ $\forall n$.

We next define order of zero of f at z_0.

Defn Ω open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$.
The order of zero of f at z_0, (or order of vanishing at z_0) denoted by $\operatorname{ord}_{z_0}(f)$ or $n_{z_0}(f)$ or $\nu_{z_0}(f)$

is either ∞ if $f^{(k)}(z_0) = 0$ $\forall k \geq 0$
or it is the smallest integer k
s.t $f(z_0) = f'(z_0) = \dots = f^{(k-1)}(z_0) = 0$
$f^{(k)}(z_0) \neq 0$.
If $f(z_0) \neq 0$ then $k = 0$.

$$\operatorname{ord}_{z_0}(f) = \min\{k \geq 0 \mid f^{(k)}(z_0) \neq 0\}$$

We have the following

Proposition Let Ω be open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$
Then (1) if $\operatorname{ord}_{z_0} f = \infty$ then $f(z) = 0$

for any $z \in D_r(z_0)$ s.t $D_r(z_0) \subset \Omega$.
i.e f is locally zero

(2) If $ord_{z_0}(f) \neq \infty$ then $\exists!\, h \in \mathcal{H}(D_r(z_0))$

and $n \in \mathbb{Z}, n \geq 0$ s.t

$$f(z) = (z - z_0)^n h(z) \quad \forall z \in D_r(z_0)$$

where $h(z_0) \neq 0$, $n = ord_{z_0}(f)$

(3) For any $f, g \in \mathcal{H}(\Omega)$ we have

$$ord_{z_0}(f+g) \geq \min\left(ord_{z_0} f, ord_{z_0} g\right)$$

$$ord_{z_0}(fg) = ord_{z_0}(f) + ord_{z_0}(g)$$

Proof (1) f is holomorphic on Ω, Ω open

Hence by Thm 4.4, $\exists\, r > 0$ s.t

$\forall z \in D_r(z_0) \subset \Omega$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Since $ord_{z_0}(f) = \infty$, $f^{(n)}(z_0) = 0 \quad \forall n$

Hence $f(z) = 0 \quad \forall z \in D_r(z_0) \subset \Omega$.

(2) If $ord_{z_0}(f) \neq \infty$, then by defn

$\exists\ k \geq 0$ s.t $f(z_0) = \dots = f^{(k-1)}(z_0) = 0$

and $f^{(k)}(z_0) \neq 0$

Again using Thm 4.4, $\exists\ r > 0$ s.t $D_r(z_0) \subset U$,

and $\forall z \in D_r(z_0)$ we have the power series repn

$$f(z) = \frac{f^{k}(z_0)}{k!}(z - z_0)^k + \sum_{n=k+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$= (z - z_0)^k \left[\frac{f^{k}(z_0)}{k!} + \sum_{m=1}^{\infty} \frac{f^{m+k}(z_0)}{(m+k)!}(z - z_0)^m \right]$$

$$= (z - z_0)^k \left[\sum_{m=0}^{\infty} \frac{f^{(m+k)}(z_0)}{(m+k)!}(z - z_0)^m \right]$$

Hence if we define

$$h(z) := \sum_{m=0}^{\infty} \frac{f^{(m+k)}(z_0)}{(m+k)!}(z - z_0)^m \quad \forall z \in D_r(z_0)$$

Then $h(z) \in \mathcal{H}(D_r(z_0))$ since it is given by a convergent power series and $h(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$.

Note Since $h \in \mathcal{H}(D_r(z_0))$ it is also continuous there and since $h(z_0) \neq 0$

$\exists\ 0 < \varepsilon < r$ s.t $h(z) \neq 0\ \forall z \in D_\varepsilon(z_0)$.

Moreover h, n are unique since if

$$f(z) = (z - z_0)^n h(z) = (z - z_0)^m g(z)$$

with h, g holom. and $h(z_0) \neq 0$, $g(z_0) \neq 0$

Then if $m > n$ we get

$$f(z) = (z - z_0)^n (z - z_0)^{m-n} g(z)$$
$$= (z - z_0)^n h(z)$$

For $z \neq z_0$

$$h(z) = (z - z_0)^{m-n} g(z)$$

but now taking lim on both sides as $z \to z_0$ gives $h(z_0) = 0$ which is a contradiction unless $m = n$, and then $h(z) = g(z)$

(3). Note for any k

$$f^{(k)}(z_0) + g^{(k)}(z_0) = (f+g)^{(k)}(z_0)$$

Hence if $f^{(k)}(z_0) = 0 = g^{(k)}(z_0)$ then $(f+g)^{(k)}(z_0) = 0$

This imply that $\mathrm{ord}_{z_0}(f+g) \geq \min(\mathrm{ord}_{z_0} f, \mathrm{ord}_{z_0} g)$

By part (2) we write $f(z) = (z-z_0)^{\mathrm{ord}_{z_0} f}\, h_1(z)$

$$g(z) = (z-z_0)^{\mathrm{ord}_{z_0} g}\, h_2(z)$$

with $\forall z \in D_r(z_0)$ $h_1(z_0) \neq 0$, $h_2(z_0) \neq 0$

then $fg = (z-z_0)^{\mathrm{ord}_{z_0} f + \mathrm{ord}_{z_0} g}\, h_1(z)\, h_2(z)$

with $(h_1 h_2)(z_0) \neq 0$

From this, using the power series expansion of fg or the uniqueness of n, h in part (2)

we get $\mathrm{ord}_{z_0} f + \mathrm{ord}_{z_0} g = \mathrm{ord}_{z_0}(fg)$

$\blacksquare$

As a corollary we get that the zeroes of a holom function are isolated. More precisely we have

Thm Let $\Omega \subset \mathbb{C}$ open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$. Assume $f(z_0) = 0$ ie $\mathrm{ord}_{z_0} f \geq 1$. If $\mathrm{ord}_{z_0} f \neq \infty$ then

$\exists\ \delta > 0$ s.t $f(z) \neq 0$ if $z \in D_\delta(z_0)$ and $z \neq z_0$

Pf. We write
$$f(z) = (z - z_0)^n h(z)$$
with $n = \mathrm{ord}_{z_0} f$, $h(z_0) \neq 0$ $\forall z \in D_r(z_0)$.

 $\rightarrow$ f is not zero in $D_\delta(z_0)$ except at z_0.

Let $z \neq z_0$, $z \in D_r(z_0)$. Then

$$f(z) = 0 \Leftrightarrow h(z) = 0.$$

But $h(z_0) \neq 0$ and $h(z)$ is continuous on $D_r(z_0)$ so $\exists\ 0 < \delta \leq r$ s.t $h(z) \neq 0$ for $|z - z_0| < \delta$.

Hence $f(z) \neq 0$ $\forall z \in D_\delta(z_0)$

Now we can state the principle of analytic continuation.

Thm (II. 4-8) Let $\Omega \subset \mathbb{C}$ open and connected.

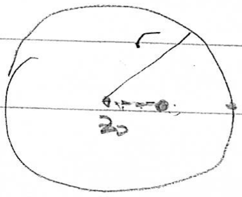
Let $f \in \mathcal{H}(\Omega)$. Let Z be an infinite set with a limit point $z_0 \in \Omega$, $z_0 \notin Z$. If $f(z) = 0$ $\forall z \in Z$, then $f \equiv 0$.

Before we give the proof, we record the following immediate corollary.

Corollary (4-9) Suppose f, g holom in Ω (open, connected) and $f(z) = g(z)$ for all z in some non-emty open sset $U \subset \Omega$ (or more generally for $z \in Z$, a sequence of distinct points with limit point in Ω) then $f(z) = g(z)$ $\forall z \in \Omega$.

Proof of cor: Apply Thm 4-8 to $f - g$ $\blacksquare$

Note If $U \subset \Omega$ open, $U \neq \emptyset$, then $\exists$ $D_r(z_0) \subset U$ for some $z_0 \in U$, $r > 0$ and the sequence $\left\{ z_0 + \frac{r}{n+1} \right\}_{n=1}^{\infty} \subset D_r(z_0) \subset U$ has limit point $z_0 \in \Omega \setminus \left\{ z_0 + \frac{r}{n+1} \right\}_{n=1}^{\infty}$



(Hence every open set Ω contain a sequence z_n of distinct pts w/ limit pt in $\Omega \setminus Z$)

Remark

1. The reason this result is called principle of analytic continuation is the following: If $f \in \mathcal{H}(\Omega)$, Ω open connected and $\Omega \subset \tilde{\Omega}$ open connected then there is <u>at most one</u> $\tilde{f} \in \mathcal{H}(\tilde{\Omega})$ s.t $f(z) = \tilde{f}(z)$ $\forall z \in \Omega$. When such $\tilde{f}$ exists we say f has analytic continuation to $\tilde{\Omega}$.

(Note if $g(z) \in \mathcal{H}(\tilde{\Omega})$ is s.t $g(z) = f(z)$ $\forall z \in \Omega$ then $\tilde{f}(z) - g(z) = 0$ $\forall z \in \Omega$. Hence $\tilde{f}(z) = g(z)$ $\forall z \in \tilde{\Omega}$ by above theorem. Hence $\tilde{f}$ is unique).

2. The assumption that Ω is connected is essential. Since if $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_i \neq \phi$, $\Omega_1 \cap \Omega_2 = \phi$ then one can define $f, g : \Omega \to \mathbb{C}$ by $f|_{\Omega_1} = 1$ and $f|_{\Omega_2} = 0$ and $g = 0$.

Then even though $f|_{\Omega_2} = g|_{\Omega_2}$ coincide f and g do not coincide in Ω.

3) The condition that the limit point of zeroes is in Ω is also crucial.

For example take $f : \mathbb{C}\setminus\{0\} =: \Omega \to \mathbb{C}$

$$z \mapsto \sin(\pi/z)$$

$$\sin\left(\frac{\pi}{z}\right) = \frac{e^{i\pi/z} - e^{-\pi i/z}}{2i}$$

$$f \in \mathcal{H}(\mathbb{C}\setminus\{0\}), \quad f \not\equiv 0$$

$$\because \; f(i) = \frac{e^{\pi} - e^{-\pi}}{2i} \neq 0.$$

$$f\left(\frac{1}{n}\right) = \sin(\pi n) = 0 \quad \forall n \geq 1$$

with $\lim \frac{1}{n} = 0$

But the limit point of zeroes is not in Ω.

This example shows that the zeroes can converge to a boundary point.

Note we do have that the zeroes $\{\frac{1}{n}\}$ are isolated

We'll prove the following thm which proves Thm 4.8.

Thm (4.8)' Let Ω be open connected $f \in \mathcal{H}(\Omega)$. Then the following are equivalent

(a) $f \equiv 0$

(b) $\exists$ a point $a \in \Omega$ s.t $f^{(n)}(a) = 0$ $\forall n \geq 0$

(c) $\{z \in \Omega \mid f(z) = 0\}$ has a limit point in Ω.

An immediate corollary of thm 4.8' is

Cor 4.9' (Identity thm) Let $f, g \in \mathcal{H}(\Omega)$ Ω open connected, $\Omega \neq \emptyset$

Then TFAE

(a) $f = g$

(b) $\exists$ a point $a \in \Omega$ s.t $f^{(n)}(a) = g^{(n)}(a)$ $\forall n \geq 0$

(c) $\{z \in \Omega \mid f(z) = g(z)\}$ has a limit point in Ω

Prof of thm (4.8)'

Clearly (a) $\Rightarrow$ (c). Since then

$\{z \in \Omega \mid f(z) = 0\} = \Omega$.

We'll prove (c) $\Rightarrow$ (b) $\Rightarrow$ (a)

I (c) $\Rightarrow$ (b)

let $Z := \{z \in \Omega \mid f(z) = 0\}$

By assumption Z has a limit point $a \in \Omega$. let $r > 0$ s.t $D_r(a) \subset \Omega$ (Ω open). f is continuous, and a is a limit point in Z. ie $\exists\, z_n \in Z \setminus \{a\}$ s.t $\lim z_n = a$. But then

$$0 = \lim f(z_n) = f(\lim z_n) = f(a)$$

Claim $f^{(n)}(a) = 0 \quad \forall n \geq 0$ Hence (b).

Pf of claim: Suppose on the contrary $\exists\, n > 0$ s.t $f(a) = 0 = \ldots f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$.

Then as in the proof of the Proposition since f is analytic in $D_r(a) \subset \Omega$ expanding f in a power ~~series~~ tree

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k, \quad |z-a| < r$$

we have that

$$f(z) = (z-a)^n g(z) \quad \text{with } g(a) \neq 0$$

and g is analytic in $D_r(a)$

Since g is continuous, $\exists\ \varepsilon$ $D_\varepsilon(a) \subset D_r(a)$ $(0 < \varepsilon < r)$ s.t $g(z) \neq 0$ on $D_\varepsilon(a)$

Then $f(z) = \underbrace{(z-a)^n}_{\neq 0} \underbrace{g(z)}_{\neq 0\ D_\varepsilon(a)}$

$\forall z \in D_\varepsilon(a) \setminus \{a\}$ ie $f(z) \neq 0$ in $D_\varepsilon(a) \setminus \{a\}$

Hence $Z \cap (D_\varepsilon(a) \setminus \{a\}) = \phi$

But this says a is not a limit point of Z.

Hence $f^{(n)}(a) = 0 \quad \forall n \geq 0$.

II $\boxed{(b) \Rightarrow (a)}$ Let $A := \{ z \in \Omega \mid f^{(n)}(z) = 0\ \forall n \geq 0 \}$

By assumption $a \in A$, hence $A \neq \phi$.