Then as in the proof of the Proposition since f is analytic in $D_r(a) \subset \Omega$ expanding f in a power series there

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k, \quad |z-a| < r$$

we have that

$$f(z) = (z-a)^n g(z) \quad \text{with } g(a) \neq 0$$

and g is analytic in $D_r(a)$

Since g is continuous, $\exists\ \varepsilon$ $D_\varepsilon(a) \subset D_r(a)$ $(0 < \varepsilon < r)$ s.t $g(z) \neq 0$ on $D_\varepsilon(a)$

Then $f(z) = \underbrace{(z-a)^n}_{\neq 0} \underbrace{g(z)}_{\neq 0 \ D_\varepsilon(a)}$

$\forall z \in D_\varepsilon(a) \setminus \{a\}$ ie $f(z) \neq 0$ in $D_\varepsilon(a) \setminus \{a\}$

Hence $Z \cap (D_\varepsilon(a) \setminus \{a\}) = \emptyset$

But this says a is not a limit point of Z.

Hence $f^{(n)}(a) = 0 \quad \forall n \geq 0$.

II $\boxed{(b) \Rightarrow (a)}$, let $A := \{ z \in \Omega \mid f^{(n)}(z) = 0 \ \forall n \geq 0 \}$

By assumption $a \in A$, hence $A \neq \emptyset$.

We'll show that $A = \Omega$, hence $f \equiv 0$.

Recall!

for an open Ω, connected means that the only both open and closed sets of Ω are ϕ and Ω.

(It is not possible to find 2 disjoint non-empty open sets Ω_1, Ω_2 s.t. $\Omega = \Omega_1 \cup \Omega_2$)

Since $A \neq \phi$, if we can show that A is both open and closed then A will be Ω.

A is open: To see this let $c \in A$ let $r > 0$ s.t $D_r(c) \subset \Omega$. Then

$$f(z) = \sum a_n (z-c)^n, \quad \forall z \in D_r(c)$$

and $a_n = \dfrac{f^{(n)}(c)}{n!} = 0$ since $c \in A$

Hence $f(z) = 0$ in $D_r(c)$

But this means $D_r(c) \subset A$

Hence for an arbitrary $c \in A$, we found a nbhd $D_r(c) \subset A$, which shows A is open.

A is closed: To w.t.s. if $\{z_k\}$ is a sequence of points in A s.t. $\lim_{k \to \infty} z_k = c \in \Omega$ then $c \in A$. i.e. A contains all its limit points.

let $c \in \Omega$ be a limit point of a sequence $\{z_k\} \subset A$.

Then for any n, and any k

$$f^{(n)}(z_k) = 0 \quad \text{by defn of the set } A$$

But $f^{(n)}$ is continuous. Hence

$$0 = \lim f^{(n)}(z_k) = f^{(n)}(\lim z_k) = f^{(n)}(c)$$

Hence $f^{(n)}(c) = 0 \quad \forall n$, and therefore $c \in A$
Hence A is closed in Ω.

$\boxtimes$

Rmk The identity thm makes it clear that the real functions $\sin, \cos, \exp : \mathbb{R} \to \mathbb{R}$ can be uniquely extended to $\mathbb{C}$.

The func'l eqns can also be transfered from reals to complex numbers.

eg (1) We know $\sin z, \cos z$ are entire funcs satisfying $\sin^2 x + \cos^2 x = 1 \quad \forall x \in \mathbb{R}$
Then we have necessarily that $\sin^2 z + \cos^2 z = 1$ $\forall z \in \mathbb{C}$. This follows by taking $f(z) = \sin^2 z + \cos^2 z$, $g(z) \equiv 1$. Since f and g agree on $\mathbb{R} \subset \mathbb{C}$, they agree on all of $\mathbb{C}$.

(2) From $\exp(x+y) = \exp(x)\exp(y)$:

$\forall\, x, y \in \mathbb{R}$, we get using identity thm

for fixed $y \in \mathbb{R}$ (but arbitrary) that

$$\exp(z+y) = \exp(z)(\exp y) \quad \forall z \in \mathbb{C}$$

Then another application of the identity thm gives

$$\exp(z+w) = \exp z + \exp w \quad \forall z, w \in \mathbb{C}$$

Rmk The example above is a special case of

Thm Let Ω be open connected, $U \subset \Omega$ which contains a sequence of distinct pts with limit point also in U. Let $F(z,w)$ be a function defined for $z, w \in \Omega$ such that $F(z,w)$ is analytic in z for each fixed $w \in \Omega$ and analytic in w " " " $z \in \Omega$.
If $F(z,w) = 0$ whenever both $z, w \in U$, then

$$F(z,w) = 0 \quad \forall z, w \in \Omega.$$

Example let $f(z) = \sum_{n=0}^{\infty} z^n \quad \forall\, z \in D_1(0) = \Omega$

f converges for $z \in D_1(0)$ and defines a holom function there.
Note for $z = 1$ $f(z)$ does not converge
Hence for any ε, we cannot
define $f(z)$ with $\sum z^n$ on $D_{1+\varepsilon}(0)$
since any such disc contains
$z = 1$.

Let $\tilde{\Omega} = \mathbb{C} \setminus \{1\}$ then $\Omega \subset \tilde{\Omega}$

and $F(z) = \dfrac{1}{1-z}$ is defined on all of
$\tilde{\Omega}$ and it agrees with $\sum z^n$
whenever $z \in D_1(0)$.

$F(z) = \dfrac{1}{1-z}$ is <u>the</u> analytic continuation of f to $\mathbb{C} \setminus \{1\}$.

Warning! This does <u>not</u> say that

for $z \in \mathbb{C} \setminus \overline{D_1(0)}$ $f = \sum_{n=0}^{\infty} z^n$ represents F.

Note in the identity thm we have
2 holom functions defined on the
<u>same</u> set Ω. Here we have

$$f : D_1(0) \to \mathbb{C} \qquad\qquad F : \mathbb{C} \setminus \{1\} \to \mathbb{C}$$
$$z \mapsto \sum z^n \qquad\qquad z \mapsto \frac{1}{1-z}$$

We now come back to our remark from earlier

Remark Not every holom function $f : \Omega \to \mathbb{C}$ can be extended to $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$!

$\Omega \subset \tilde{\Omega}$.

Example $f(z) = \sum_{n=0}^{\infty} z^{n!}$

If $|z| < 1$ then $|z|^{n!} < |z|^n$

and by comparison w/ geom. series

$\sum z^{n!}$ converges abs. and unifom. on compact ssets of $D_{1-\varepsilon}(0)$

Hence defines a holom func on $D_1(0)$

Claim: f cannot be extended anywhere beyond $D_{1-\varepsilon}(0)$.

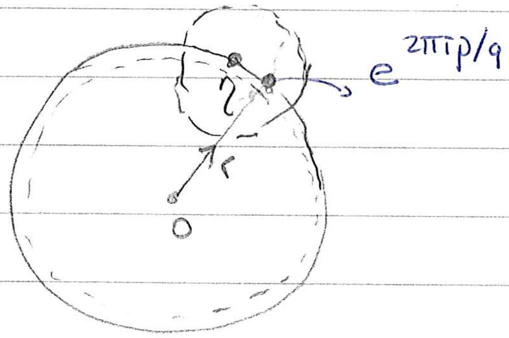
Proof. let $p/q \in \mathbb{Q}$, $p, q \in \mathbb{Z}$, $q > 0$
$z = re^{2\pi i p/q}$

$$f(z) = \sum_{n=0}^{q-1} z^{n!} + \underbrace{\sum_{n=q}^{\infty} r^{n!} \underbrace{e^{2\pi i \frac{p}{q} \cdot n!}}_{\substack{1 \\ \forall n \geq q}}}_{\substack{\downarrow \text{ as } r \nearrow 1 \\ \infty}}$$

Hence $f(z) \to \infty$ if $r \nearrow 1$.

Hence $f(re^{2\pi i p/q}) \to \infty$ as $r \to 1$

Let $\eta \in S^1$. If there were a $\tilde{f}$ holom extention of f to $D_\delta(\eta)$ for some $\delta > 0$. Then we can choose p/q s.t $e^{2\pi i p/q} \in D_\delta(\eta)$



But then for $z = e^{2\pi i p/q} \in D_\delta(\eta)$ we have seen $f(rz) \to \infty$ as $r \to 1$ hence no such $\tilde{f}$ exists.

Here is another corollary of the Identity Thm

Thm Let $f, g \in \mathcal{H}(\Omega)$, Ω open connected

If $fg \equiv 0$ then $f \equiv 0$ or $g \equiv 0$.

Proof Suppose $f \not\equiv 0$. W.t.s $g \equiv 0$.

$f \not\equiv 0$ so $\exists\, a \in \Omega$ s.t $f(a) \neq 0$

By continuity of f

$\exists$ a nbhd of a, $D_\varepsilon(a) \subset \Omega$ s.t

$f(z) \neq 0 \quad \forall z \in D_\varepsilon(a)$.

The assumption $f(z)g(z) = 0 \quad \forall z \in \Omega$

This implies that $g(z) = 0 \quad \forall z \in D_\varepsilon(a)$

But then $g\big|_{D_\varepsilon(a)} = \vartheta\big|_{D_\varepsilon(a)} = 0$

Using Identity theorem applied to $g : \Omega \to \mathbb{C}$ and the zero function $\vartheta : \Omega \to \mathbb{C}$

gives $g(z) = \vartheta(z) = 0 \quad \forall z \in \Omega$.

Rmk The analytic functions on an non-empty open subset $\Omega \subset \mathbb{C}$ form a commutative ring with 1 since the sum and product of holomorphic functions are holomorph

Recall: $(R, +, \cdot)$ is a ring with 1

① $(R, +)$ is an abelian gp.

② R is a monoid under $\cdot$
ie (i) $(ab)c = a(bc)$ $\forall a, b, c \in R$
(ii) $\exists\, 1 \in R$ such that $a.1 = 1.a$ $\forall a \in R$

③ $\cdot$ distributes over $+$
$a.(b+c) = ab + ac$
$(b+c)a = ba + ca$.

The last thm says that if Ω is open connected then this ring has no zero divisors!