Consider $g = f \circ \varphi_\alpha$, $g : \mathbb{D} \to \mathbb{D}$.

then $g(0) = f(\alpha) = 0$.

Schwarz lemma (a) applied to g then gives

$|g(z)| \leq |z| \quad \forall z \in \mathbb{D}$

Since $g^{-1}(0) = 0$ we can apply Schwarz's lemma to g^{-1} to get

$|g^{-1}(w)| \leq |w| \quad \forall w \in \mathbb{D}$.

Using this for $w = g(z)$ gives

$|z| = |g^{-1}(g(z))| \leq |g(z)| \quad \forall z \in \mathbb{D}$.

Combined with $|g(z)| \leq |z|$ we get

that $|g(z)| = |z|$. Once again by

Schwarz's lemma (b) $g(z) = e^{i\theta} z$ for some

$\theta \in \mathbb{R}$. Hence $e^{i\theta} z = (f \circ \varphi_\alpha)(z) = g(z)$

Replacing z with $\varphi_\alpha(z)$ now gives

$e^{i\theta} \varphi_\alpha(z) = g(\varphi_\alpha(z)) = (f \circ \varphi_\alpha)(\varphi_\alpha(z)) = (f \circ \varphi_\alpha \circ \varphi_\alpha)(z)$

$= f((\varphi_\alpha \circ \varphi_\alpha)(z)) = f(z)$ using $\varphi_\alpha \circ \varphi_\alpha = \mathrm{id}$.

Rmk. Combining autom. of $\mathbb{D}$ with the Cayley map

$$F : \mathbb{H} \longrightarrow \mathbb{D}$$
$$z \longrightarrow \frac{z-i}{z+i}$$

allows one to find all autom of $\mathbb{H}$

Thm 2.4 Every autom $g : \mathbb{H} \to \mathbb{H}$ of $\mathbb{H}$ is the form $g(z) = \dfrac{az+b}{cz+d}$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ st $ad-bc>0$

Proof (Exercise) Read in the book.
Note using the map

$$\gamma : \mathrm{Aut}(\mathbb{D}) \longrightarrow \mathrm{Aut}(\mathbb{H})$$
$$\varphi \longmapsto F^{-1} \circ \varphi \circ F$$

any autom of $\mathbb{D}$ leads to an autom of $\mathbb{H}$.

Moreover γ is an isom with inverse $\gamma^{-1} : \mathrm{Aut}(\mathbb{H}) \longrightarrow \mathrm{Aut}(\mathbb{D})$

$$\beta \longrightarrow F \circ \beta \circ F^{-1}$$

Using γ we can pull the autom of $\mathbb{D}$ to autom of $\mathbb{H}$ and show that they are of the above form.

Now we move to Step 2 in the proof of Riemann mapping thm.

Step 2 There is a conformal map

$f: \Omega \to \mathbb{D}$.

(ie Ω is conformally equivalent to a subset of $\mathbb{D}$).

We have the following

Proposition (This is given as step 1 in sec. 3.3 on page 228 in the book)

Let $\Omega \subset \mathbb{C}$, $\emptyset \neq \Omega \neq \mathbb{C}$, simply connected and open. Then there exists a conformal map $f: \Omega \to \mathbb{D}$ s.t

$0 \in f(\Omega)$. ie Ω is conformally equivalent to a subset of $\mathbb{D}$ which contains the origin

Proof. By assumption Ω is proper, hence $\exists$ $\alpha \in \mathbb{C}$ s.t $\alpha \notin \Omega$

By replacing Ω with $\Omega - \alpha := \{z - \alpha \mid z \in \Omega\}$

we can assume $\alpha = 0$. Hence $\Omega \subset \mathbb{C} - \{0\}$

Since Ω is simply connected, there is

$\log_\Omega : \Omega \longrightarrow \mathbb{C}$, $\log_\Omega \in \mathcal{H}(\Omega)$

Note $\log_\Omega$ is also injective, since if

$\log_\Omega z = \log_\Omega w$ then exponentiating

both sides we get that $z = \exp(\log_\Omega z)$

$= \exp(\log_\Omega w) = w$.

Hence $\log_\Omega$ is a conformal map

Now let $w \in \Omega$. Then note that for any $z \in \Omega$

$$\log_\Omega(z) \neq \log_\Omega(w) + 2\pi i$$

Since otherwise, exponentiating we get
$z = \exp(\log_\Omega z) = \exp(\log_\Omega w) \cdot \exp(2\pi i) = w$
Hence $z = w$ but then $\log_\Omega(z) = \log_\Omega(w)$
a contradiction

In fact, $\log_\Omega(z)$ stays away from $\log_\Omega(w) + 2\pi i$

in the sense that $\exists\, \delta > 0$ s.t

(*) $D_{2\delta}(\log(w) + 2\pi i) \cap \log_\Omega(\Omega) = \emptyset$

Indeed otherwise $\forall\ n > 0$ ($\delta = 1/2n$) we get a sequence $z_n \in \Omega$ s.t

$$\left| \log_\Omega z_n - (\log_\Omega w + 2\pi i) \right| < 1/n$$

Hence $\log_\Omega z_n \longrightarrow \log_\Omega w + 2\pi i$

and exponentiating and using the fact that exp is continuous we get

$z_n \longrightarrow w$ and hence

$\log_\Omega z_n \to \log_\Omega w$ using cont. of $\log_\Omega$

which is a contradiction to $\log_\Omega z_n \to \log_\Omega w + 2\pi i$

Now we can consider the map:

$$F : \Omega \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{\log_\Omega z - (\log_\Omega w + 2\pi i)}$$

Since $\log_\Omega$ is injective so is F

Note $F \in \mathcal{H}(\Omega)$ since

$\log z \neq \log w + 2\pi i$ for any $z \in \Omega$.

and hence F is a conformal map

$\circledast \Rightarrow \left| \log_{\Omega} z - (\log(w) + 2\pi i) \right| \geq 2\delta \quad \forall z \in \Omega$

Hence $\left| F(z) - 0 \right| = \left| \dfrac{1}{\log_{\Omega}(z) - (\log w + 2\pi i)} \right| \leq \dfrac{1}{2\delta} < \dfrac{1}{\delta}$
$\forall z \in \Omega$

Hence $F(\Omega) \subset D_{1/\delta}(0)$. We can now translate and rescale F to obtain a function $f : \Omega \to \mathbb{D}$ which contains origin in its image

Let $f(z) := \dfrac{\delta}{4}\left(F(z) - F(w)\right)$

then $f : \Omega \to \mathbb{C}$ is conformal

we have $f(w) = 0$ and

$$|f(z)| \leq \frac{\delta}{4}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) \leq \frac{1}{2} \quad \forall z \in \Omega$$

Hence $f(\Omega) \subset D_1(0)$ $\forall z \in \Omega$ and since $f(w) = 0$, $0 \in f(\Omega)$.

$\boxtimes$

Step 3. An extremal Problem.

Let Ω proper, simply connected sset of $\mathbb{C}$, $z_0 \in \Omega$
By step 2

$$\exists \quad f : \Omega \longrightarrow D_1(0) \quad \text{s.t}$$

$$f(z_0) = 0$$

Let $\mathcal{F} := \left\{ f : \Omega \to D_1(0) \;\middle|\; \begin{array}{l} f \text{ conformal} \\ f(z_0) = 0 \end{array} \right\}$.

Then $\mathcal{F} \neq \emptyset$

We start by noting the following
(See step 2 in section 3.3 in the book)

Lemma The set of values $|f'(z_0)|$, $f \in \mathcal{F}$ is bounded in $[0, \infty)$. Hence

$$\sup_{f \in \mathcal{F}} |f'(z_0)| =: s < \infty.$$

Proof Let $\delta > 0$ s.t $D_{2\delta}(z_0) \subset \Omega$
Let $f \in \mathcal{F}$. Cauchy integral formula gives

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_\delta(z_0)} \frac{f(z)}{(z - z_0)^2} \, dz$$

Hence using the standard estimates

$$|f'(z_0)| \leq \frac{1}{2\pi} \frac{2\pi\delta \cdot \max_{z\in C_\delta}|f(z)|}{\delta^2}$$

$$\leq \frac{1}{\delta} \quad \text{since} \quad |f(z)| \leq 1 \text{ for all } z \in \Omega.$$

Hence $|f'(z_0)|$ is bounded.

$\boxtimes$

The next proposition is the key and says that the supremum

$$s = \sup_{f\in\mathcal{F}} |f'(z_0)| \text{ is taken}$$

Key Proposition: $\exists\ f \in \mathcal{F}$ s.t. $|f'(z_0)| = s$

The proof of this Prop uses a compactness argument which we will come to

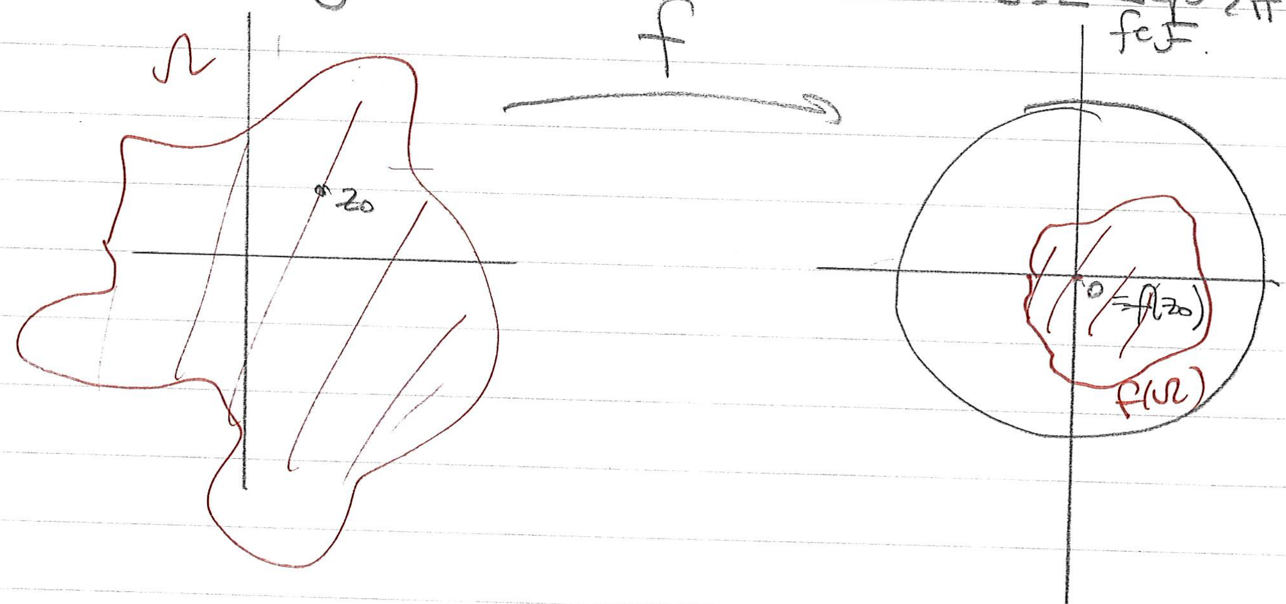
But we first see why this is key in the sense that it gives the conformal equivalence we're looking for between Ω and $\mathbb{D}$.

(See §3.3, step 3 in the book)
p. 231

Remark. The step 2 shows that

Ω is conformally equivalent to an open sset of $\mathbb{D}$, which contains 0

why are we looking for an extremal function realizing the extremal value $S := \sup_{f\in\mathcal{F}} \{|f'(z_0)|\}$



We can assume wlog that Ω is an open sset of $\mathbb{D}$ that contains 0. So can assume $z_0 = 0$

We want to conformally strech Ω to fill $\mathbb{D}$.

$$\mathcal{F} = \left\{ f : \Omega \to \mathbb{D} \;\middle|\; \begin{array}{l} f \text{ holom. injective} \\ f(0) = 0 \end{array} \right\}$$

We want to choose a function in $\mathcal{F}$ with "maximal expansion". What does expanding mean.

$f(0) = 0 \Rightarrow f(z) \sim f'(0)z$ for z near 0.

So if $|f'(0)| > 1$ we say f is expanding since the distances between nearby points are expanding

$$|f(z_1) - f(z_2)| \approx |f'(0)||z_1 - z_2| > |z_1 - z_2|.$$

Step 4 f from Key Prop in Step 3 is surjective

Prop. Let $f \in \mathcal{F}$ be such that $|f'(z_0)| = s$. Then f is a conformal equivalence $f : \Omega \to \mathbb{D}$.
(ie f is also onto $\mathbb{D}$)

Proof. We want to show that f is surject.
Assume not. Then $\exists\ \alpha \in \mathbb{D}$ which is not in $f(\Omega)$, ie $f(z) \neq \alpha$, $\forall z \in \Omega$. We'll construct a $g \in \mathcal{F}$ with $|g'(z_0)| > |f'(z_0)|$ which will be a contradiction to $|f'(z_0)| = s = \sup_{g \in \mathcal{F}} |g'(z_0)|$

To do this we will use φ_α and the squareroot map.

Let $\varphi = \varphi_\alpha : \mathbb{D} \to \mathbb{D}$ be the autom of $\mathbb{D}$

$$z \longmapsto \frac{\alpha - z}{1 - \bar{\alpha} z}$$

$\varphi_\alpha(0) = \alpha$, $\varphi_\alpha(\alpha) = 0$.

Then $\varphi_\alpha \circ f : \Omega \to D_1(0)$ is conformal

$0 \notin (\varphi_\alpha \circ f)(\Omega)$ since if for some $z \in \Omega$

$(\varphi_\alpha \circ (f(z))) = 0$, then $f(z) = \alpha$ which we assumed is not the case.

Since $0 \notin (\varphi \circ f)(\Omega)$, and Ω is simply connected, a logarithm, and square root of ~~of~~ $\varphi \circ f$ exists. ie $\exists$ a holom. map $\tilde{f} : \Omega \to \mathbb{C}$ s.t. $\tilde{f}^2(z) = (\varphi \circ f)(z)$ $\forall z \in \Omega$. [Take $\tilde{g}$ a primitive of $\dfrac{(\varphi_\alpha \circ f)'}{(\varphi_\alpha \circ f)}$

so that $\exp(\tilde{g}(z)) = \varphi_\alpha \circ f$

Then take $\tilde{f} := \exp\left(\frac{1}{2}\tilde{g}(z)\right)$.]

Note $\tilde{f}$ is also injective: If $\tilde{f}(z) = \tilde{f}(w)$ then $(\varphi \circ f)(z) = (\varphi \circ f)(w)$. Since $\varphi \circ f$ is conformal we have $z = w$.

Now $\tilde{f}$ is not yet the function we want

Since $\tilde{f}(z_0) \neq 0$ as $\varphi(f(z_0)) \neq 0$

(Since $0 \notin (\varphi \circ f)(\Omega)$)

let $\tilde{f}(z_0) = \beta$ and consider the

autom of $\mathbb{D}$, $\varphi_\beta : \mathbb{D} \longrightarrow \mathbb{D}$

$$z \longrightarrow \frac{\beta - z}{1 - \bar{\beta} z}$$

$$\varphi_\beta(\beta) = 0$$

Finally let $g(z) := \varphi_\beta \circ \tilde{f} : \Omega \longrightarrow \mathbb{D}$

Then $g(z_0) = 0$.

g is holom, since $\Rightarrow \varphi_\beta, \tilde{f}$ are
g is infective since $\varphi_\beta, \tilde{f}$ are.

Claim: $|g'(z_0)| > |f'(z_0)|$. This will give the contradiction we are looking for

Recall: we first looked at $\varphi_\alpha \circ f$

$$\varphi_\alpha \circ f : \Omega \xrightarrow{f} \mathbb{D} \xrightarrow{\varphi_\alpha} \mathbb{D}$$

Then we took the $\sqrt{\ }$ function, call it h

$$h : (\varphi_\alpha \circ f)(\Omega) \longrightarrow \mathbb{D}$$
$$w \longmapsto \exp\left(\tfrac{1}{2} w\right)$$ and composed w/ $\varphi_\alpha \circ f$

$$\tilde{f} = h \circ \varphi_\alpha \circ f : \Omega \longrightarrow \mathbb{D}$$

so that $\tilde{f}^2 = \varphi_\alpha \circ f$

Then we composed with φ_β, to get

$$g : \Omega \longrightarrow \mathbb{D}$$

$$g = \varphi_\beta \circ \tilde{f} = \varphi_\beta \circ \underbrace{h \circ \varphi_\alpha \circ f}_{\tilde{f}}.$$

Hence $\varphi_\beta^{-1} \circ g = \tilde{f}$

$$\Rightarrow (\varphi_\beta^{-1} \circ g)^2 = \varphi_\alpha \circ f$$

$$\Rightarrow \varphi_\alpha^{-1} \circ (\varphi_\beta^{-1} \circ g)^2 = f.$$

Let $S(z) = z^2$ be the squaring map.
$S : \mathbb{D} \to \mathbb{D}$

Then $f = \underbrace{\varphi_\alpha^{-1} \circ S \circ \varphi_\beta^{-1}}_{:= \Phi} \circ g = \Phi \circ g.$

Note Φ is not injective.

Now $\Phi : \mathbb{D} \to \mathbb{D}$ is holomorphic

$$\Phi(0) = (\varphi_\alpha^{-1} \circ S \circ \varphi_\beta^{-1})(0)$$
$$= \varphi_\alpha^{-1}(\beta^2)$$

But $\beta^2 = (\tilde{f}(z_0))^2 = (\varphi_\alpha \circ f)(z_0)$ (Since $\beta = \tilde{f}(z_0)$)

Hence $\Phi(0) = (\varphi_\alpha^{-1} \circ \varphi_\alpha \circ f)(z_0)$

$$= f(z_0) = 0.$$

Hence we can apply Schwarz's lemma.

Part ⓒ to get $|\Phi'(0)| < 1$.

(Note $|\Phi'(0)| \neq 1$ since if it were then

$\Phi(z) = e^{i\theta} z$ for some θ would mean

Φ is injective but Φ is not injective

since squaring function is not injective)

Hence using the chain rule applied to
$f = \Phi \circ g$ we have

$$f'(z_0) = \Phi'(g(z_0)) \cdot g'(z_0) = \Phi'(0) \cdot g'(z_0)$$

Hence $|f'(z_0)| < |g'(z_0)|$ which is a contradiction ☒