17.9.24

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§ 0. Introduction

Our goal this semester is to study functions $f: \mathbb{C} \to \mathbb{C}$ defined on the complex plane $\mathbb{C}$, or on an open set of $\mathbb{C}$.

We will see that the study of complex function theory is not simply the study of functions on $\mathbb{R}^2$.

We will see that many ways the theory of functions of one real variable is more complicated than the theory of of functions of a complex variable.

To give an idea of what I mean let's try to compare and contrast:

1. It is not too difficult to write down a function of a real variable that is n times differentiable but not infinitely differentiable

$$f(x) := \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- The derivative of f exists for every x including $x=0$. $f'(0) = 0$. Hence f is differentiable but its derivative is not continuous. Hence not differentiable twice. By integrating f as many times as you like you can get a function h which is differentiable that many times but not infinitely differentiable.

- In contrast: We'll see that if $f: \mathbb{C} \to \mathbb{C}$ is differentiable once then it is differentiable ∞-ly many times.

2. There are functions $f: \mathbb{R} \to \mathbb{R}$ that are ∞-ly many times differentiable whose Taylor series does not represent f ie f is not analytic.

e.g $$f(x) := \begin{cases} \exp\left(\frac{-1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f is ∞'ly differentiable. Unfortunately at $x=0$ all derivaties are zero. Hence its Taylor series is identically zero and cannot represent f.

In contrast: If $f : \mathbb{C} \to \mathbb{C}$ is a function of a complex variable which is differentiable then f is analytic, i.e. it can be represented by a power series.
(differentiable = analytic)

3. There are plenty of C^∞ functions of a real variable that are bounded e.g $\sin x$, $\cos x$

In contrast: We'll see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable and bounded then it is constant (Liouville's Thm)

4. For two functions of a real variable f, g f and g can agree on an open set without being equal

In contrast If $f, g : \mathbb{C} \to \mathbb{C}$ are 2 differentiable functions which coincides on an arbitrarily small disc (or even on a convergent sequence (z_n)) then $f = g$

(Analytic continuation principle)

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Remark
The power of complex function theory comes from this "robustness" or rigidity. It is a subject in some sense analysis, geometry, algebra come together.

This, we will see, allows one to prove theorems that a priori has nothing to do with complex numbers.

Ex ① The Integral.

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \sqrt{\frac{2\pi}{4}}$$

② Let $\pi(x) := \#\{ p \text{ prime} \mid p \leq x \}$

then $\pi(x) \underset{x \to \infty}{\sim} \dfrac{x}{\log x}$ Prime Number Theorem

$\left(\lim\limits_{x \to \infty} \pi(x) / (x/\log x) = 1 \right).$

(3) If $f \in \mathbb{C}[x]$ a non-zero poly. Then f has a zero in $\mathbb{C}$.

(Fund. thm of Algebra)

(4) Let $r_4(n) := \#\{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 = n\}$

Then
$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$$

In particular $r_4(n) \geq 1 \quad \forall n$.

Before we start with the defn of differentiability of a complex variable function, we quickly recall the defns and basic properties of complex numbers as well as some basic notions of topology and analysis

Chapter I

§ 1 Complex numbers and complex plane (Review)

$$\boxed{\mathbb{C} := \{ x + iy \mid x, y \in \mathbb{R},\ i^2 = -1 \}}$$

$\mathbb{R} \hookrightarrow \mathbb{C}$, $\mathbb{R} \subset \mathbb{C}$

$r \longmapsto r + i.0$

For $z = x + iy \in \mathbb{C}$ we define

Real part of $z := x = Re(z)$

Imaginary part of $z := y = Im(z)$

Complex conjugate of $z := \boxed{x - iy = \bar{z}}$

- $$\boxed{Re(z) = \frac{z + \bar{z}}{2}}$$
- $$\boxed{Im(z) = \frac{z - \bar{z}}{2i}}$$
- $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$

  $z \in i\mathbb{R} \Leftrightarrow z = -\bar{z}$ (purely imaginary)

- Complex numbers can also be represented as ordered pairs of real numbers in $\mathbb{R}^2$ $z = (x, y)$ where we have $z = w$ with $w = (u, v)$ if and only if $x = u$ and $y = v$

Addition in $\mathbb{C}$: if $z = x + iy$, $w = u + iv$

$$z + w := (x + u) + i(v + y)$$

as pairs $z + w = (x + u, y + v)$

Multiplication in $\mathbb{C}$

$$z \cdot w = (x+iy)(u+iv)$$
$$= xu + i(xv+yu) + i^2 vy$$
(using $i^2 = -1$)
$$= (xu - vy) + i(xv+yu)$$

as pairs in $\mathbb{R}^2$: $(x,y)\cdot(u,v) := (xu-vy, xv+yu)$

Note $i = (0,1)$, and $(0,1)\cdot(0,1) = (-1,0) = -1$

$\mathbb{R}^2$, together with these 2 operations $+, \cdot$ becomes a field ie

$(\mathbb{R}^2, +, \cdot)$ satisfies ① $(\mathbb{R}^2, +)$ is a comm. group with additive identity $0 := (0,0)$

② $(\mathbb{R}^2 \setminus (0,0), \cdot)$ is a comm group with mult. identity $1 := (1,0)$

③ $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3 = (z_2+z_3)z_1$

Additive inverse of $z = (x,y)$: $-z = (-x,-y)$

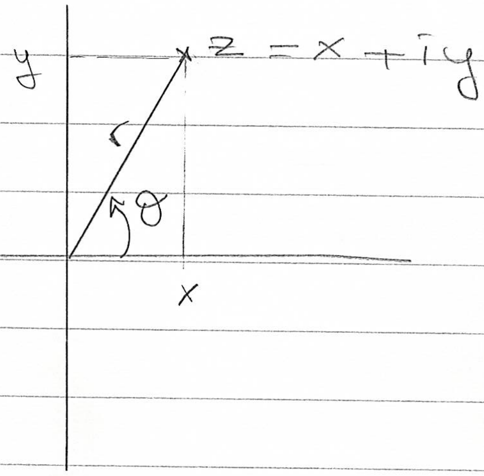
Multip. inverse of z; $0 \neq z = (x,y)$ is $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

$$= \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

Here $\boxed{|z| := \sqrt{z\bar{z}} = \sqrt{x^2+y^2}}$ is called

the norm, or modulus or abs. value of z

We also have the polar coordinate representation of complex numbers



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$x = r\cos\theta$

$y = r\sin\theta$

where $r \geq 0$, $\theta \in \mathbb{R}$

with $r = |z|$

The polar representation is not unique unless $z \neq 0$ and we restrict $\theta \in (-\pi, \pi]$ (or any other interval of length 2π.

θ is called the argument of z which is defined uniquely up to a multiple of 2π and is denoted by $\arg z$

$$\boxed{\arg z = \{\theta \in \mathbb{R} \mid z = |z|e^{i\theta}\}}$$

The argument of z chosen in the interval $(-\pi, \pi]$ is called the principal argument and denoted by $\operatorname{Arg} z$

$\operatorname{Arg}(i) = \frac{\pi}{2}$ $\qquad$ $\operatorname{Arg}(-c) = \pi$ $\quad 0 < c \in \mathbb{R}$

Remark. No assignment of argument is made to $0 \in \mathbb{C}$.

For $z = x + iy \neq 0$ we have

$$\operatorname{Arg} z = \begin{cases} \operatorname{Arcsin}(y/|z|) & \text{if } x \geq 0 \\ \pi - \operatorname{Arcsin}(y/|z|) & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi - \operatorname{Arcsin}(y/|z|) & \text{if } x < 0 \text{ and } y < 0 \end{cases}$$

Here for $t \in [-1, 1]$, $\operatorname{Arcsin} t$ is the unique number $u \in [-\pi/2, \pi/2]$ s.t. $\sin u = t$

Observe that $\arg(z^{-1}) = -\arg z$

$$\arg(zw) = \arg z + \arg w$$

But it is <u>not</u> always the case that

$$\operatorname{Arg}(z^{-1}) = -\operatorname{Arg} z \quad \text{or}$$

$$\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w.$$

for example $\operatorname{Arg}(-\frac{1}{2}) = \pi \neq -\operatorname{Arg}(-2) = -\pi$

and $\pi = \operatorname{Arg}(-1) = \operatorname{Arg}((-i)(-i))$

$$\neq \operatorname{Arg}(-i) + \operatorname{Arg}(-i) = -\frac{\pi}{2} + -\frac{\pi}{2} = -\pi$$

<u>Recall</u>:
- $|z| = 0 \iff z = 0 \quad \forall z \in \mathbb{C}$
- $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2| \quad \forall z_1, z_2 \in \mathbb{C}$
- $|z_1 z_2| = |z_1||z_2| \quad \forall z_1, z_2 \in \mathbb{C}$
- $|\bar{z}| = |z|$
- $|\mathrm{Re}(z)| \leq |z|$, $|\mathrm{Im}(z)| \leq z \quad \forall z \in \mathbb{C}$
- If $z = re^{i\theta}$, $w = se^{i\beta}$ then $zw = rse^{i(\theta+\beta)}$

Representation of $\mathbb{C}$ as 2×2 matrices.

For $z = a + ib$, let $Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$w = c + di$, let $W = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$

then $ZW = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{pmatrix}$

On the other hand

$zw = ac - bd + i(bc + ad)$

Hence the multiplication in $\mathbb{C}$ corresponds to multiplication of corresponding matrices in $M_2(\mathbb{R})$

$\therefore$ We can represent any $z \in \mathbb{C}$ with the matrix

$$Z = \begin{pmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{pmatrix}$$

$$= (\operatorname{Re} z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\operatorname{Im} z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$i \in \mathbb{C}$ corresponds to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In polar form $z = re^{i\theta}$, the corresponding matrix is $Z = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Next we recall some definitions that we need from topology and Analysis

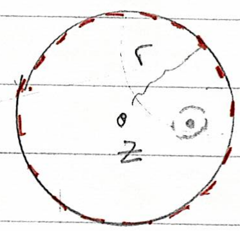
We will denote the open disc of radius r centered at z by $D_r(z)$ or $D(z; r)$

$$D_r(z) := \{w \in \mathbb{C} \mid |w - z| < r\}$$

$\overline{D_r}(z) := \{w \in \mathbb{C} \mid |w - z| \leq r\}$ is the closed disc.

The boundary of $D_r(z)$ is the circle
$C_r(z) := \overline{D_r}(z) \setminus D_r(z)$

$= \{w \in \mathbb{C} \mid |z - w| = r\}$



- A subset $U \subset \mathbb{C}$ is open if $\forall z \in U, \exists r > 0$ such that $D_r(z) \subset U$

  eg: $\emptyset$, $\mathbb{C}$, $D_r(z)$, $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

- A subset $U \subset \mathbb{C}$ is called closed if $\mathbb{C} \setminus U$ is open.

  U is closed $\iff$ $\forall$ sequence $(z_n) \in U$, if $z_n \to z$ then $z \in U$

  e.g $\emptyset$, $\mathbb{C}$, $\overline{D_r(z)}$, $C_r(z)$, $\mathbb{R}$

- A subset $K \subset \mathbb{C}$ is called compact if it is closed and bounded (i.e $\exists M$ s.t $|z| < M$ $\forall z \in K$).

(11)

- $K \subset \mathbb{C}$ is compact $\Leftrightarrow$ every sequence $\{(z_n)\} \subset K$ has a subsequence that converges to a point in K

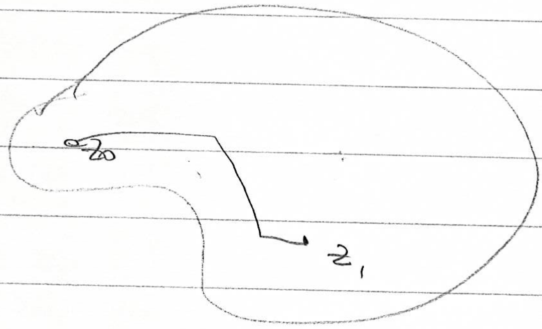
e.g, ϕ, $\overline{D_r(z)}$, $C_r(z)$, $[a,b] \times [c,d]$

A subset A of $\mathbb{C}$ is called disconnected if $\exists$ open sets U and V such that (i) $U \cap V = \phi$ (ii) $A \cap U \neq \phi$, $A \cap V \neq \phi$ and (iii) $A \subset U \cup V$.

A sset A is called connected if it is not disconnected.

A connected open set $\phi \neq U \subset \mathbb{C}$ is called a region or a domain

- Any pair of distinct points z_0, z_1 in an open connected sset $A \subset \mathbb{C}$ can be connected by a polygonal path lying in A



e.g ϕ, $\mathbb{C}$, $D_r(z)$, $\overline{D_r(z)}$, $C_r(z)$, $\mathbb{R}$ are connected

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \cup D_1(2i)$ are disconnected.

Rmk In general metric spaces path connected $\Rightarrow$ connected but connected $\not\Rightarrow$ path connected.

We recall

1. A sequence (z_n) with $z_n = x_n + i y_n$ converges to $z = x + iy$ if one of the following equivalent conditions hold
   (i) $x_n \longrightarrow x$ and $y_n \to y$ in $\mathbb{R}$

   (ii) $|z_n - z| \to 0$ in $\mathbb{R}$

   (iii) $\forall \varepsilon > 0, \exists N ; \forall m, n \geq N, |z_m - z_n| < \varepsilon$

2. Let $U \subset \mathbb{C}$ be an open subset $f : U \longrightarrow \mathbb{C}$ any function.
   For $z_0 \in U$ and $w_0 \in \mathbb{C}$ we have

$$\lim_{\substack{z \to z_0 \\ z \in U}} f(z) = w_0$$

   if one of the following equivalent conditions hold
   (i) $\forall \varepsilon > 0, \exists \delta > 0, \forall z \in U$ with $|z - z_0| < \delta$, we have $|f(z) - w_0| < \varepsilon$

   (ii) If $(z_n) \subset U$ is a sequence with $\lim z_n = z_0$, then $f(z_n)$ converges to w_0.

3. $f : U \to \mathbb{C}$ is continous on U
   iff $\forall z_0 \in U \quad \lim_{z \to z_0} f(z) = f(z_0)$
   iff $\forall (z_n) \subset U$ with $\lim z_n = z_0$, $\lim f(z_n) = f(\lim z_n) = f(z_0)$

§2 Holomorphic Functions

§2.1 Definitions, basic properties

This is a central notion for the rest of the class.

Let $\Omega \subset \mathbb{C}$ an open set
$f : \Omega \to \mathbb{C}$ a complex valued function on Ω.

Defn: f is called holomorphic at $z_0 \in \Omega$ if

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{h} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists}$$

Here $h \in \mathbb{C}$, $z_0 + h \in \Omega$, (so that the quotient is well defined)

If the limit exists, we denote it with $f'(z_0)$.

It is called the derivative of f at z_0.

f is called holomorphic on Ω if it is holomorphic $\forall z_0 \in \Omega$.

If f is holomorphic on all of $\mathbb{C}$, then it is called entire.

Remark Regular or complex differentiable are other words used for holomorphic.

Example 2.1 Let $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z$

Then f is entire. Since

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim \frac{(z_0+h) - z_0}{h} = \lim \frac{h}{h} =$$

Hence $f'(z) = 1 \quad \forall z \in \mathbb{C}$.

As in real variables, we have

Prop 2.2. ① Let $\mathcal{H}(\Omega) := \{ f: \Omega \to \mathbb{C} \mid f \text{ is hol. on } \Omega \}$

then $\mathcal{H}(\Omega)$ is a $\mathbb{C}$-vector space

More precisely

i) If $f, g \in \mathcal{H}(\Omega)$ then $\alpha f + \beta g \in \mathcal{H}(\Omega)$ for every $\alpha, \beta \in \mathbb{C}$.

And

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$

($0: \Omega \to \mathbb{C}$, $0(z) = 0$ is the 0-element)

② $f, g \in \mathcal{H}(\Omega) \Rightarrow fg \in \mathcal{H}(\Omega)$

and $(fg)' = f'g + fg'$

③ If $g(z_0) \neq 0$ then f/g is hol. at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}$$

If $g(z) \in \mathcal{H}(\Omega)$ and $g(z) \neq 0 \quad \forall z \in \Omega$

then $f/g \in \mathcal{H}(\Omega)$.