Step 3

§ Proof of the Key Proposition

Existence of the maximum

We want to prove: ..

Key Proposition: $\exists\, f \in \mathcal{F}$ s.t. $|f'(z_0)| = s = \sup_{g \in \mathcal{F}} \{|g'(z_0)|\}$

where $\mathcal{F} := \left\{ f : \Omega \to \mathbb{D} \;\middle|\; \begin{array}{l} f \text{ conformal} \\ \text{ie } f \in \mathcal{H}(\Omega), \text{ injec} \\ \text{and } f(z_0) = 0 \end{array} \right\}$

(1) Recall: For a bounded set $U \subset \mathbb{R}$ with $s = \sup U$, there exists a non-decreasing sequence $(a_n) \subset U$ with $\lim_{n\to\infty} a_n = s$.

Hence if $(f_n) \subset \mathcal{F}$ with $|f_n'(z_0)| \to s$ then we want to show that this sequence has a limit $f \in \mathcal{F}$.

(2) Recall we've seen that a sequence of holom functions that converge unif. on compact sets has a limit which is also holomorphic.

But we cannot expect that an arbitrary sequence (f_n) to be unif conv on compacta.
But may be a subsequence has this property.

<u>Recall</u>: In finite dim'l vector space $\mathbb{R}^n$ every bounded sequence has a convergent subsequence

So we're looking for an analog of this for $\mathcal{F}$. This is provided by Montel's thm.

<u>Thm 3.3 (Montel's thm)</u> Let $\Omega \subset \mathbb{C}$ be open; $(f_n)_n$ a sequence in $\mathcal{H}(\Omega)$.

Suppose that for any compact set $K \subset\subset \Omega$ $\exists$ $M_K > 0$ s.t. $|f_n(z)| < M_K$ $\forall n \geq 1$ and $z \in K$.
Then $\exists$ a subsequence (f_{n_k}) which converges uniformly on compact ssets of Ω

We can apply this in the proof of Riemann mapping thm since we have a sequence $(f_n)_n \subset \mathcal{F}$, each $f_n : \Omega \to \mathbb{D}$. So

$|f_n(z)| \leq 1$ $\forall z \in \Omega$ and all n, (not only on compact ssets.

Hence by Montel's thm we can find a subsequence $(f_{n_k}) \subset \mathcal{F}$ which converges unif. on compact ssets of Ω. This will give $f = \lim f_{n_k}$
$f \in \mathcal{H}(\Omega)$

We still need to show that $f \in \mathcal{F}$.
ie f is injective (Clearly $f(z_0) = 0$ since $f_{n_k}(z_0) = 0$)

For this we use the following

Proposition Let (f_n) be a sequence in $\mathcal{F}$ and suppose that $f_n(z) \to f(z)$ for $z \in \Omega$ uniformly on compact ssets $K \subset \Omega$.
Then either f is constant or $f \in \mathcal{F}$ and $\lim f_n'(z_0) = f'(z_0)$

Proof First note since $f_n \to f$ unif on compacta $f \in \mathcal{H}(\Omega)$ and $\lim f_n'(z_0) = f'(z_0)$.

We need to show (a) $f(\Omega) \subset \mathbb{D}$
(b) f is injective or constant

For (a) note that $|f_n(z)| < 1$ since $f_n : \mathcal{H} \to \mathbb{D}$
Hence $|f(z)| = \lim |f_n(z)| \leq 1$

If $|f(z)| = 1$ for some $z \in \Omega$, then

$f(z)$ attains its maximum at a pt inside Ω which contradicts max. modulus principle unless f is constant.

Hence if f is not constant, $f = \lim f_n$ is holomorphic and $f(\Omega) \subset \mathbb{D}$.
ie $f : \Omega \to \mathbb{D}$.

What is left to prove is (b) ie if f is not constant, then it is injective. To finish the proof, we use the following lemma

Lemma 3.5 $\Omega \subset \mathbb{C}$ open connected
$f_n : \Omega \to \mathbb{C}$ conformal.
If $(f_n) \to f : \Omega \to \mathbb{C}$ unif. on compacta then f is injective or constant.

Proof of lemma Suppose f is not injective
we'll show f is constant

Suppose $\exists\ z_1 \neq z_2 \in \Omega$, s.t $f(z_1) = f(z_2)$

If f is not constant, then since the zeroes of holom functions are isolated, we can find a disc $D_\delta(z_2) \subset \Omega$ s.t

$f(z) - f(z_2) \neq 0$ in $z \in \tilde{D}_\delta(z_2)$

In particular, $f(z) - f(z_2) \neq 0 \quad \forall z \in C_{\delta/2}(z_2)$

Note in particular this says that $z_1 \notin D_{\delta/2}(z_2)$ since we assumed $f(z_1) - f(z_2) = 0$.

We now apply the argument principle to the function $f(z) - f(z_1)$ which has a zero, namely z_2, in $D_{\delta/2}(z_2)$ to get

$$\frac{1}{2\pi i} \int_{C_{\delta/2}(z_2)} \frac{f'(z)}{f(z) - f(z_1)} \, dz \geq 1$$

On the other hand $f_n \to f$ unif on compacts. Hence also on the compact set $C_{\delta/2}(z_2)$

Moreover $f_n(z) \neq f_n(z_1) \quad \forall n, \ \forall z \in C_{\delta/2}(z_2)$ since f_n's are injective and $z_1 \notin C_{\delta/2}(z_2)$.

Hence $\dfrac{f_n'(z)}{f_n(z) - f(z_1)} \longrightarrow \dfrac{f'(z)}{f(z) - f(z_1)}$ unif on $C_{\delta/2}(z)$

and we get

$$\frac{1}{2\pi i}\int_{C_{\delta/2}(z_2)} \frac{f'(z)}{f(z)-f(z_2)}\,dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{C_{\delta/2}(z_2)} \frac{f_n'(z)}{f_n(z)-f_n(z_1)}\,dz$$

The integrals on the right count zeroes of holom func $f_n(z) - f_n(z_1)$ inside $D_{\delta/2}(z_2)$, which is none by

injectivity of f_n's and the fact that $z_1 \notin D_{\delta/2}(z_2)$

But the integral on the left ≥ 1 which is a contradiction.
Hence f must be constant

∎

To recap what we've done till now for the proof of the key Proposition

① We used the defn of supremum to find a sequence $(f_n) \in \mathcal{F}$ s.t

$$\lim |f_n'(z_0)| = s = \sup_{g\in\mathcal{F}} |g'(z_0)|$$

② We used Montel's thm, to find a subseq. f_{n_k} s.t $f_{n_k} \to f$ on compacta to obtain $f \in \mathcal{H}(\Omega)$,

③ We used the last prop. to show f ...

$f : \Omega \to \mathbb{D}$ and if f is not constant

then f must be injective and

$\lim f_n'(z_0) = f'(z_0)$. Hence

By renaming the subseq. (f_{n_k}) as (f_n) we

have a sequence $(f_n) \in \mathcal{F}$ with $\lim f_n = f$

$\lim |f_n'(z_0)| = s = |f'(z_0)|$ and

$f \in \mathcal{H}(\Omega)$ and $f(z_0) = 0$, Either f is constant or f injective i.e. $f \in \mathcal{F}$.

Now $f_n'(z_0) \neq 0$ $\forall n$ since

f_n's are injective holom maps, by Prop 1.1 Chap 8, $f_n'(z) \neq 0$ $\forall z \in \Omega$.

We have $|f_n'(z_0)|$ is a non-dec. sequence

$0 < |f_1'(z_0)| < |f_2'(z_0)| \cdots < |f_n'(z_0)|$

Hence $\lim |f_n'(z_0)| = s > 0$

Hence $|f'(z_0)| \neq 0$ and f is not constant.

which finishes the proof of the key Propositio

Finally before giving the proof of Montel's thm note we found $f \in \mathcal{F} := \{ g : \Omega \to \mathbb{D} \mid g(z_0) = 0,\ g \text{ conformal} \}$

s.t $|f'(z_0)| = \sup_{g \in \mathcal{F}} |g'(z_0)| = s > 0$

This f which solves the extremal problem we've seen is also surjective.

Finally note apriori $|f'(z_0)| = s > 0$

$f'(z_0)$ need not be a real number

but if $f'(z_0) = s e^{i\theta}$, and $f : \Omega \to \mathbb{D}$

Let $R_{-\theta}(z) = z e^{-i\theta}$ be a rotation by $-\theta$

then $\tilde{f} := R_{-\theta} \circ f : \Omega \to \mathbb{D}$ is a biholom

and $\tilde{f}'(z_0) = \underbrace{R_{-\theta}'(f(z_0))}_{e^{-i\theta}} \underbrace{f'(z_0)}_{s e^{i\theta}} = s > 0$

is the function we're looking for.