Last week -

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ extended complex plane

$\infty \pm z = z \pm \infty = \infty$ $\quad \forall z \in \mathbb{C}$

$\infty \cdot z = z \cdot \infty = \infty$ $\quad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$

$z/\infty = 0$ $\quad \forall z \in \mathbb{C}$

$z/0 = \infty$ $\quad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$

$\infty \pm \infty$, ∞/∞, $0/0$, $0 \cdot \infty$ are not assigned a meaning in $\hat{\mathbb{C}}$.

- A sequence $(z_n) \in \mathbb{C}$ converges to ∞ if $\lim |z_n| = \infty$, where $(|z_n|)_n$ is a sequence in $\mathbb{R}$

- $\lim_{z \to z_0} f(z) = \infty$ if $\lim_{z \to z_0} |f(z)| = \infty$.

We now define

Defn A function $f : \Omega \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\Omega \subseteq \mathbb{C}$ open is called meromorphic if the following conditions are satisfied

(1) The set $S_f = \{ z \in \Omega \mid f \text{ is not holom at } z \}$

has no limit point in Ω (ie S_f is discrete in Ω)

(2) The points in S_f are poles of f

(3) The restriction of f to $\Omega \setminus S_f$ is holomorphic

ie $f \in \mathcal{H}(\Omega \setminus S_f)$

Let $\mathcal{M}(\Omega)$ = set of all meromorphic functions in Ω.

Example Let $P(z), Q(z) \in \mathbb{C}[z]$ 2 polynomial with no common zeroes.

Note any rational function $\frac{p(z)}{q(z)}$ can be reduced to $P(z)/Q(z)$ with no common zeroes.

Let $f(z) = \begin{cases} P(z)/Q(z) & \text{if } Q(z) \neq 0 \\ \infty & \text{if } Q(z) = 0 \end{cases}$

Then $f \in \mathcal{M}(\mathbb{C})$
Since f is holomorphic outside the finite zero set of $Q(z)$
If z_0 is a zero of $Q(z)$, then f has a pole at z_0 since

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|P(z_0)|}{|Q(z_0)|} = \infty$$

since we assumed $P(z_0) \neq 0$
Hence z_0 is a pole of f.

(2) The function $f(z) = \cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$

is a meromorphic function in $\mathbb{C}$ with $S_f = \mathbb{Z}$

(3) Let $f = \dfrac{e^{1/z}}{z^2-1}$. Then f is meromorphic in $\mathbb{C} - \{0\}$, with $S_f = \{\pm 1\}$ but not in $\mathbb{C}$.

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If we have 2 functions $f, g \in \mathcal{M}(\Omega)$ with pole sets S_f, S_g

then $(f+g)$ is holomorphic in $\Omega \setminus (S_f \cup S_g)$
to define $f+g$ at pts in $\Omega \setminus (S_f \cup S_g)$
we just use the usual defn
$$(f+g)(z) = f(z) + g(z) \qquad \forall z \in \Omega \setminus (S_f \cup S_g)$$

So we really only have to worry about pts $z \in S_f \cup S_g$

But we can extend $f+g : \Omega \setminus (S_f \cup S_g) \to \mathbb{C}$ to a meromorphic function

$$f+g : \Omega \to \hat{\mathbb{C}} \quad \text{as}$$

follows: if $z_0 \in S_f \cup S_g$, write

$$\left.\begin{aligned} f(z) &= P_f(z) + \tilde{f}(z) \\ g(z) &= P_g(z) + \tilde{g}(z) \end{aligned}\right\} \forall\ z \in D_r^*(z_0)$$

where P_f, P_g are the principal part of f and g at z_0 (one of them can be zero if f, or g does not have a pole at z_0)

$$\tilde{f}, \tilde{g} \in \mathcal{H}(D_r(z_0))$$

then $f+g = \underbrace{P_f(z) + P_g(z)}_{\substack{\text{lin. combination} \\ \text{of terms } \frac{1}{(z-z_0)^l}}} + \underbrace{\tilde{f}+\tilde{g}}_{\in \mathcal{H}(D_r(z_0))}$

So $f+g$ has a pole of order ≥ 1 unless $P_f(z) + P_g(z) = 0$ (which can happen)

Hence: $f+g \in \mathcal{M}(\Omega)$ with

$$S_{f+g} \subseteq S_f \cup S_g$$

We've proved part (2) of the following proposition.

Proposition $\Omega \subset \mathbb{C}$ open

① $\mathcal{M}(\Omega) \supseteq \mathcal{H}(\Omega)$

② If $f, g \in \mathcal{M}(\Omega)$ then $af + bg \in \mathcal{M}(\Omega)$ for any $a, b \in \mathbb{C}$

Hence $\mathcal{M}(\Omega)$ is a $\mathbb{C}$-vector space

③ $f, g \in \mathcal{M}(\Omega)$, then $fg \in \mathcal{M}(\Omega)$

④ If $0 \not\equiv f \in \mathcal{M}(\Omega)$ and zeroes of f do not have a limit pt in Ω then $1/f \in \mathcal{M}(\Omega)$.

Proof ① Obvious but note we identified a holom func $f: \Omega \to \mathbb{C}$ with the corresponding function $\tilde{f}: \Omega \to \hat{\mathbb{C}}$ where $\tilde{f} = i \circ f$, $i: \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$.

② The same argument for $f+g$ works with $af + bg$

③ $f = P_f + \tilde{f}$, $g = P_g + \tilde{g}$. Let $z_0 \in S_f \cup S_g$

then $fg = (P_f + \tilde{f})(P_g + \tilde{g})$

$= P_{fg} + G$ where P_{fg} is a lin comb of $\frac{1}{(z-z_0)^\ell}$

G holom in $D_r(z_0)$.

$$fg = \left(\sum_{k=-n}^{\infty} a_k (z-z_0)^k\right)\left(\sum_{\ell=-m}^{\infty} b_\ell (z-z_0)^\ell\right)$$

$$\sum_{N=-(n+m)}^{\infty}\left(\sum_{\substack{k,\ell \\ k+\ell=N}} a_k b_{N-\ell}\right)(z-z_0)^N$$

e.g. if $f = \dfrac{a_{-1}}{z-z_0} + \sum_{n\geq 0} a_n (z-z_0)^n$

$$g = \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{z-z_0} + \sum_{\ell=0}^{\infty} b_\ell (z-z_0)^\ell$$

with $a_{-1}\neq 0$, $b_{-2}\neq 0$

$$fg = \frac{b_{-2}a_{-1}}{(z-z_0)^3} + \frac{b_{-2}a_0 + a_{-1}b_{-1}}{(z-z_0)^2} + \frac{a_{-1}b_0 + b_{-1}a_0 + b_{-2}a_1}{(z-z_0)} + G$$

where G is holomorphic and $b_{-2}a_{-1}\neq 0$ if $b_{-2}\neq 0$, $a_{-1}\neq 0$.

~~Hence~~ fg has a pole of order 3.

Similar to $f+g$, we can define

$$fg = \begin{cases} f(z)g(z) & \text{if } z\in\Omega\setminus(S_f\cup S_g) \\ \infty & \text{if } z\in(S_f\cup S_g) \end{cases}$$

Then: fg is meromorphic in $\mathcal{M}(\Omega)$

with $S_{fg} \subseteq S_f \cup S_g$.

(2) If $f \in \mathcal{M}(\Omega)$. If $z_0 \in \Omega \setminus S_f$

and $f(z_0) \neq 0$ then

$\frac{1}{f}$ is holom. at z_0. If $z_0 \in \Omega \setminus S_f$

and $f(z_0) = 0$ then $1/f$ has a pole of order k = order of zero of f at $z_0 \geq 1$.

If $z_0 \in S_f$ then $\left|\frac{1}{f(z)}\right| \xrightarrow[z \to z_0]{} 0$

hence $\frac{1}{f}$ has a removable singularity

at z_0. If zeroes of f has

no limit point in Ω then the poles

of $\frac{1}{f}$ have no limit point in Ω

and hence $1/f \in \mathcal{M}(\Omega)$.

Rmk. If we assume that $f \not\equiv 0$ in any connected component of Ω, then $1/f \in \mathcal{M}(\Omega)$.

Recall: If $f : \Omega \to \mathbb{C}$, Ω open connected $f \in \mathcal{H}(\Omega)$. Then the zeroes of f do not have a limit point in Ω. (Thm 4.8).
For open connected set Ω
The same is true for $f \in \mathcal{M}(\Omega)$.

Proposition Let Ω open connected, $f \in \mathcal{M}(\Omega)$
let $Z = \{z \in \Omega \mid f(z) = 0\}$, $f \not\equiv 0$
then Z has no limit point in Ω.

Proof. Assume on the contrary that
$\exists$ (z_n) distinct points in Z
s.t $\lim z_n = b \in \Omega$, $f(z_n) = 0$

Let S_f = poles of f. Then
$f \in \mathcal{H}(\Omega \setminus S_f)$ and $\Omega \setminus S_f$ is open, connected, $f \not\equiv 0$ Hence
by the above result we've recalled $b \notin \Omega \setminus S_f$
(S_f is a countable set. see exercise 9.2 (a)).
B.A now $b \notin S_f$ either since if
b is a pole of f then $\lim_{z \to b} |f(z)| = \infty$
means that $|f(z)| > 0$ for $\forall z \in \Omega$ with
$|z - b| < \varepsilon$
But this is impossible since
if $z_n \to b$ then $|z_n - b| < \varepsilon$ for $n \geq n_0$
and $f(z_n) = 0$.

$\boxtimes$

Remark

Let $f \in \mathcal{M}(\Omega)$, z_0, a pole of f

Since S_f has no limit point in Ω

$\exists$ a punctured nbhd $D_r^*(z_0)$ of z_0

s.t

$$D_r^*(z_0) \cap S_f = \phi .$$

If the order of the pole of f at z_0 is k

then $f(z) = (z - z_0)^{-k} g(z)$ with a

analytic function $g(z) \in \mathcal{H}(D_r(z_0))$

Hence locally every meromorphic function is the quotient of 2 holom. functions

~~Here~~ for $f = \dfrac{g(z)}{(z - z_0)^k}$

It is a non-trivial result that if Ω is open and connected, i.e. a domain then this is globally possible.

ie For any $f \in \mathcal{M}(\Omega)$, for Ω open connected

$\exists$ $g, h \in \mathcal{H}(\Omega)$ s.t $f = g/h$.

Algebraically we can state this as follows:

Recall if Ω open connected then $\mathcal{H}(\Omega)$ has no zero divisors, it is an integral domain. It has a quotient field

$$Q(\mathcal{H}(\Omega)) = \left\{ \frac{g}{h} : g, h \in \mathcal{H}(\Omega), \ h \neq 0 \right\}$$

and this quotient field (or field of fractions)

is $\mathcal{M}(\Omega)$.

(This is similar to the construction of $\mathbb{Q}$ as field of fractions of the integral domain $\mathbb{Z}$).

Defn Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$. $f \in \mathcal{M}(\Omega)$, $f \neq 0$. Define the valuation (or order)

of f at z_0, denoted by $\text{ord}_{z_0} f$, $\nu_{z_0}(f)$

to be the integer $k \in \mathbb{Z}$ s.t

i) If z_0 is not a pole of f, ie $f(z_0) \neq \infty$ then $k \geq 0$ is the order of vanishing of f at z_0

(ii) If $f(z_0) = \infty$ ie z_0 is a pole then $k \leq -1$ is minus the order of the pole at z_0.

(ie if $\text{ord}_{z_0} f > 0$ then z_0 is a zero

$\text{ord}_{z_0} f < 0$ then z_0 is a pole

$\text{ord}_{z_0} f = 0$ then $f(z_0) \neq 0$, $f(z_0) \neq \infty$.

Combining what we know about (Thm 1.1, and 1.2) the behaviour of functions near zeroes and poles we get

Proposition If $f \in \mathcal{M}(\Omega)$, $f \neq 0$, $z_0 \in \Omega$

1) $\mathrm{ord}_{z_0} k \iff \exists\, r > 0$, $h \in \mathcal{H}(D_r(z_0))$ s.t. $h(z_0) \neq 0$ and

$$f(z) = (z - z_0)^k h(z).$$
$$\forall z \in D_r^*(z_0).$$

($k < 0$ if z_0 a pole, $k > 0$, if z_0 a zero)

2) $\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0} f + \mathrm{ord}_{z_0} g$

3) If $f + g \neq 0$ then

$$\mathrm{ord}_{z_0}(f+g) \geq \min\left[\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\right]$$

__Example__ $f(z) = \dfrac{z}{(e^z - 1)^2}$, z has zero of order 1 at 0

$(e^z - 1)^2$ has zeroes of order 2 at $z = 2\pi i n$. $n \in \mathbb{Z}$.

$\mathrm{ord}_0 f = \mathrm{ord}_0 z - \mathrm{ord}_0 (e^z - 1)^2$
$= 1 - 2 = -1$ Hence f has a pole of order 1 at $z = 0$. For $n \neq 0$
$\mathrm{ord}_{2\pi i n} f = \mathrm{ord}_{2\pi i n} z - \mathrm{ord}_{2\pi i n} (e^z - 1)^2 = 0 - 2 = -2$
Hence f has pole of order 2 at $2\pi i n$.

Remark $\hat{\mathbb{C}}$ and the stereographic Projection

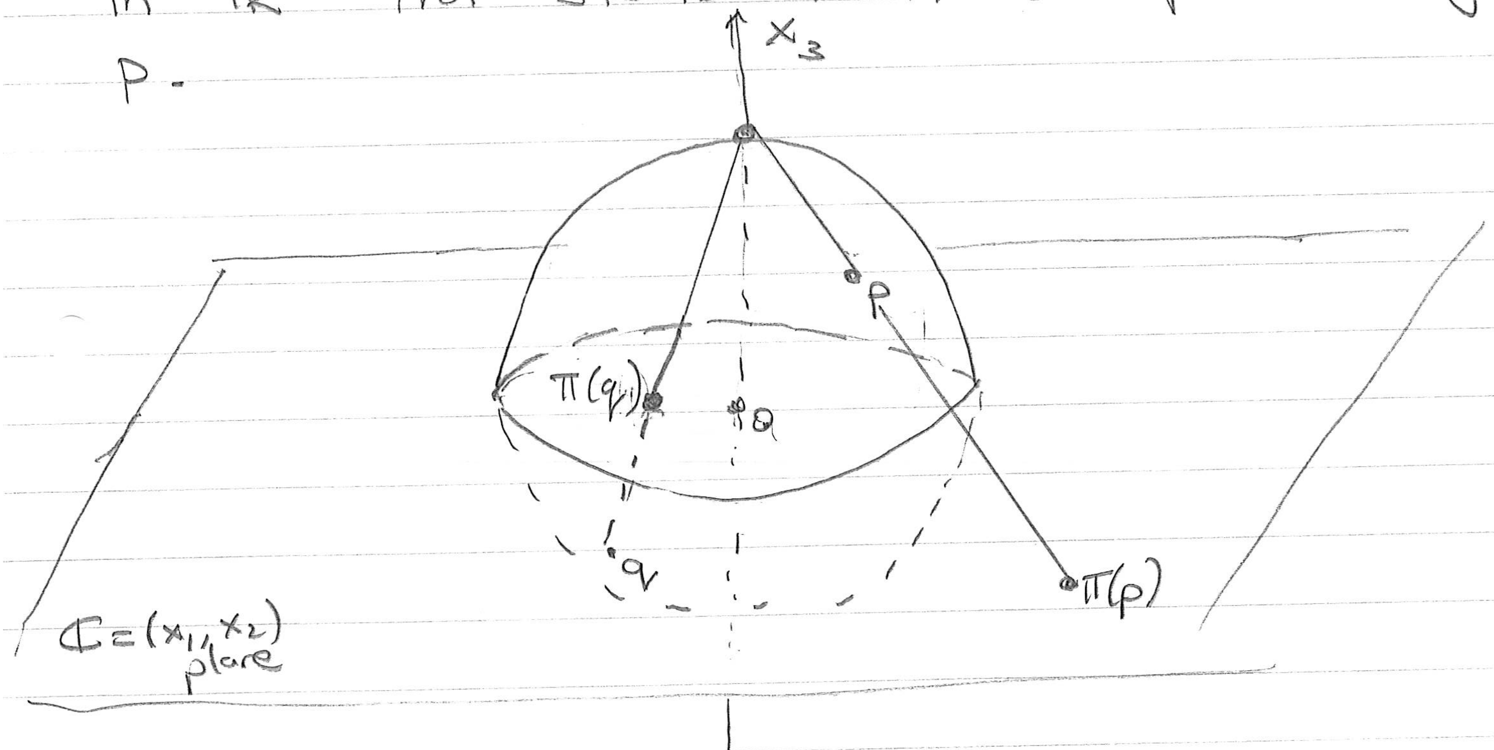
Let $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.

Identifying $(x_1, x_2, 0)$ with $\mathbb{C}$ we can think of $\mathbb{C}$ sitting in $\mathbb{R}^3$, as the (x_1, x_2)-plane.

Set $N = (0,0,1)$, and define the map

$\pi : S^2 \setminus \{N\} \to \mathbb{C}$ as follows

For $p \in S^2$, $p \neq N = (0,0,1)$, let $\pi(p)$ be the intersection of $\mathbb{C}$ with the ray in $\mathbb{R}^3$ that starts at N and passes through p.



π is called the stereographic projection of $S^2 \setminus \{N\}$ onto $\mathbb{C}$

Explicitly π is given by

$$\pi(p) = \pi(x_1, x_2, x_3) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0\right)$$

$$= \frac{x_1}{1-x_3} + \frac{x_2}{1-x_3} i$$

Note the eqn of the ray that starts at N and go through p is : $N + t(p - N)$, $t \geq 0$
and $\pi(p) = N + t_0(p - N)$ where t_0 is unique positive real number so that $(0,0,1) + t_0(x_1, x_2, x_3 - 1)$ has 3rd coordinate 0.
Solving for t_0 gives the formula for $\pi(p)$ above.

Defining $\pi(N) = \infty$ gives a bijection

$$\pi : S^2 \to \hat{\mathbb{C}}$$

Conversely given $z \in \mathbb{C}$ one checks that

$$\pi^{-1}(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \in S^2 \setminus \{N\}$$

$\pi^{-1}(\infty) = N$ gives the inverse map
~~Hence~~ we get
S^2 is homeomorphic to $\hat{\mathbb{C}}$.
Since both maps are continuous.

Before we study the values of holomorphic functions using Residue formula let me mention that we can also talk about meromorphic functions on $\hat{\mathbb{C}}$ (as opposed to $\mathcal{M}(\Omega)$, $\Omega \subset \mathbb{C}$)

We have already allowed ∞ as a value of meromorphic functions. We can also allow ∞ in the definition domain and study functions $f : \tilde{\Omega} \longrightarrow \hat{\mathbb{C}}$

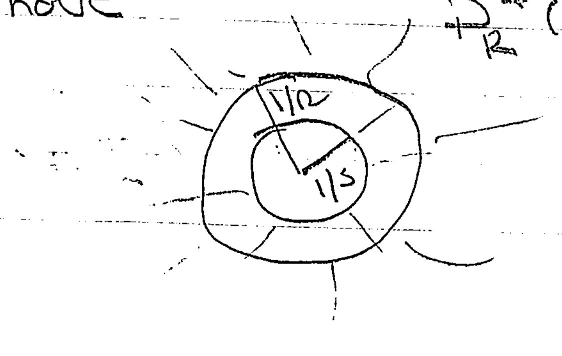
where $\tilde{\Omega} \subset \hat{\mathbb{C}}$.

If a function f is analytic for large values of z, ie $|z| > \frac{1}{R}$ for some $R > 0$, then the function

$g(z) := f(\frac{1}{z})$ is holomorphic in a

deleted nbhd of 0, $D_R^*(0)$.

(We will denote $\{z \in \mathbb{C} \mid |z| > R^{-1}\}$ by $D_R^*(\infty)$. This notation is designed to have $D_R^*(\infty) \subset D_S^*(\infty)$ when $R < S$.)



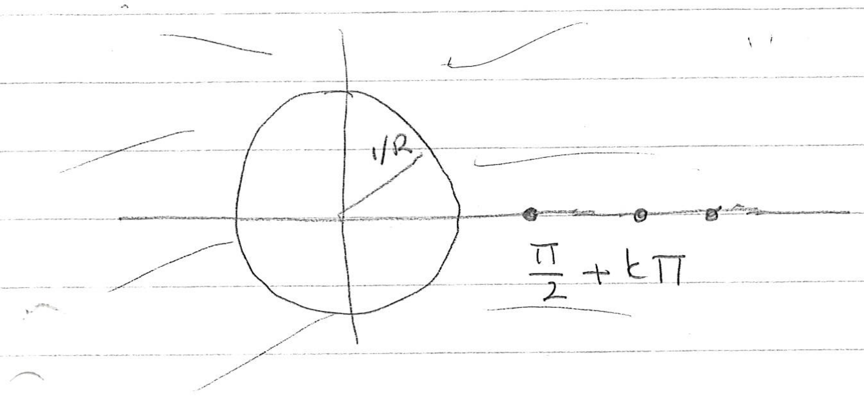
Defn For a function f which is analytic for $|z| > 1/R$ for some $R > 0$ we say f has an isolated singularity at ∞ (which will be called removable, a pole or essential if $g(z) := f(\frac{1}{z})$ has an isolated singularity at 0 (which is removable, a pole or essential resp.)

A meromorphic function in the complex plane that is either holomorphic at ∞ or has a pole at ∞ is called meromorphic in $\hat{\mathbb{C}}$

Example ① An entire function is analytic in $D_R^*(\infty)$ for every $R > 0$.
For example $f(z) = e^z$
e^z has an isolated singularity at ∞ which is essential, because $e^{1/z}$ has an essential singularity at 0.
Hence e^z is not meromorphic in $\hat{\mathbb{C}}$.
(It is meromorphic on $\mathbb{C}$, in fact holom in $\mathbb{C}$.

② $p(z) \in \mathbb{C}[z]$ has a pole at ∞, $p = a_n z^n + \dots + a_0$
since $p(\frac{1}{z}) = \frac{a_n}{z^n} + \dots \frac{a_1}{z} + a_0$ has a pole of order n at 0.

(2) $f(z) = \tan z$ does not have an isolated singularity at ∞: Each $D^*_R(\infty)$ includes poles of f; $z = \frac{\pi}{2} + k\pi$



Also note $g(z) = \tan\left(\frac{1}{z}\right)$ have singularities at $S = \left\{\left(\frac{\pi}{2} + k\pi\right)^{-1} \mid k \in \mathbb{Z}\right\}$ which accumulate at $z = 0$. The singularity of $\tan\frac{1}{z}$ at $z = 0$ is not isolated.

We have the following theorem for meromorphic functions on $\hat{\mathbb{C}}$

Thm 3.4. If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic in the extended complex plane, then it is a rational function.

Clearly each rational function is a meromorphic function on $\hat{\mathbb{C}}$.

Hence $\mathcal{M}(\hat{\mathbb{C}}) = \left\{ \frac{P(z)}{Q(z)} \,\middle|\, P(z), Q(z) \in \mathbb{C}[z] \right\}$ polynomials

Proof. Exercise