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Applications of Residue thm

The first application is called the Argument principle. It uses the residue theorem applied to f'/f, the logarithmic derivative of f to count the zeroes and poles of f inside a curve.

To this end we note the following simple Lemma:

Lemma: Let $\Omega \subset \mathbb{C}$ open and connected $f \in \mathcal{M}(\Omega)$, $f \neq 0$. Then f'/f, the "logarithmic derivative" of f, is also meromorphic in Ω. And f'/f has poles of order 1 at $z_0 \in \Omega$ for which $\text{ord}_{z_0} f \neq 0$. ie either z_0 is a zero or a pole of f. The residue of f'/f at z_0 is equal to the $\text{ord}_{z_0} f$.

Proof Since $f \neq 0$, Ω open connected zeroes of f do not have a limit point in Ω, and $1/f \in \mathcal{M}(\Omega)$.

Clearly $f' \in \mathcal{M}(\Omega \setminus S_f)$ where S_f is the set of poles of f.

If $z_0 \in S_f$ a pole of order n of f then

$$f(z) = (z - z_0)^{-n} h(z) \quad \forall z \in D_r^*(z_0)$$

where $h \in \mathcal{H}(D_r(z_0))$ and $h(z_0) \neq 0$.
Then for $z \in D_r^*(z_0)$ we have

$$f'(z) = \frac{-n}{(z-z_0)^{n+1}} h(z) + \frac{h'(z)}{(z-z_0)^n}$$

$$= \underbrace{\left[h'(z)(z - z_0) - n\, h(z)\right]}_{=: \tilde{h}(z)} (z - z_0)^{-(n+1)}$$

$\tilde{h}(z) \in \mathcal{H}(D_r(z_0))$ and $\tilde{h}(z_0) = -n\, h(z_0) \neq 0$.

Hence for $\forall z \in D_r^*(z_0)$

$$f'(z) = (z - z_0)^{-(n+1)} \tilde{h}(z)$$

Hence f' has a pole of order $n+1$ at z_0.
(Similarly if f has a zero of order n at z_0
then f' has a zero of order $n-1$ at z_0.)
Hence $f' \in \mathcal{M}(\Omega)$ and so is f'/f

For any $z \in \Omega$, $\operatorname{ord}_{z_0}(f'/f) = \operatorname{ord}_{z_0} f' - \operatorname{ord}_{z_0} f$

$$= \begin{cases} -(n+1) - n = -1 & \text{if } z_0 \text{ is a pole of } f \text{ of order } n \\ (n-1) - n = -1 & \text{if } z_0 \text{ is a zero of } f \text{ order } n \\ \geq 0 & \text{otherwise} \end{cases}$$

Hence f'/f has a pole of order 1 at the points where $\mathrm{ord}_{z_0} f \neq 0$.

We can also calculate the residue using

$$f(z) = (z-z_0)^n g(z) \quad \forall z \in D_r^*(z_0)$$

where $g(z) \in \mathcal{H}(D_r(z_0))$, $g(z) \neq 0$, $\forall z \in D_r(z_0)$

where $n = \mathrm{ord}_{z_0} f$

Hence $n > 0$ if z_0 is a zero

$n < 0$ if z_0 " " pole of f.

$$f'(z) = n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)$$

and $$\frac{f'}{f} = \frac{n}{(z-z_0)} + \underbrace{\frac{g'(z)}{g(z)}}_{G(z) \in \mathcal{H}(D_r(z_0))} \quad \forall z \in D_r(z_0)$$

Hence

$$\boxed{\mathrm{Res}_{z_0}\left(\frac{f'}{f}\right) = n = \mathrm{ord}_{z_0} f}$$

This lemma immediately gives using the residue thm,

Thm 4.1 (Argument principle) Let $\Omega \subset \mathbb{C}$, $\gamma \subset \Omega$, a circle (or any curve s.t the residue formula holds), $f \in \mathcal{M}(\Omega)$.
If f has no zeroes or poles on γ then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{z_0 \in Z_f \cap (\text{int}\,\gamma)} \text{ord}_{z_0} f$$

$$+ \sum_{z_0 \in S_f \cap (\text{int}\,\gamma)} \text{ord}_{z_0} f$$

where Z_f = the set of zeroes of f
S_f = " " " poles of f.

Proof. This follows from prev lemma and $\text{Res}_{z_0}\left(\frac{f'}{f}\right) = \text{ord}_{z_0} f$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{z_i \in \text{int}\,\gamma} \text{Res}_{z_i}\left(\frac{f'}{f}\right)$$

$$= Z - P$$

Z = # of zeroes of f inside γ counted w/ multiplicity

P = # of poles of f inside γ counted w/ multiplicity.

Cor let $f \in \mathbb{C}[z]$ a polynomial. Choose $R > 0$ large enough so that all zeroes of f are inside $D_R(0)$

Then
$$\int_{C_R(0)} \frac{f'}{f}(z) = \deg f$$

We have the following corollary of the Argument principle, which says a holom function when perturbed slightly, it doesn't change its # of zeroes.

Thm 4.3 (Rouché's thm)
Suppose f, g are holomorphic in an open set Ω which contains a circle C and its interior. If
$$|f(z)| > |g(z)| \quad \forall z \in C$$
then f, $f+g$ have the same number of zeroes inside C

Proof: For $t \in [0,1]$ define
$$f_t(z) = f(z) + tg(z)$$
so that $f_0(z) = f(z)$, $f_1(z) = f + g$. Note for $z \in C$

Note
$$|f_t(z)| = |f(z) + tg(z)|$$
$$\geq \big| |f(z)| - t|g(z)| \big|$$

$$|f_t(z)| \geq \big|\, |f(z)| - t|g(z)| \,\big| \qquad (\, |f| > |g| \text{ on } C \text{ and } t \leq 1\,)$$

$$> (1-t)|g(z)| \geq 0$$

Hence $|f_t(z)| > 0$ for $z \in C$, In particular $f_t(z)$ has no zeroes on C and then by argument principle $n_t = \frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)}\, dz$

where n_t is the number of zeroes of f_t in C.

Now note n_t is a continuous function of t since $f_t'(z)/f_t(z)$ is jointly continuous for $t \in [0,1]$ and $z \in C$, since both $f_t'(z)$, $f_t(z)$ are jointly continous and $f_t(z) \neq 0$ on C.

(Recall from real analysis:
$f : [a,b] \times [c,d] \to \mathbb{R}$ continous on $[a,b]\times[c,d]$
then
$$h(t) := \int_c^d f(t, x)\, dx \quad \text{is cont. on } [a,b]$$

But n_t is also integer valued. Hence it must be a constant.
(otherwise the intermediate value thm gives the existence of $t_0 \in [0,1]$ s.t n_{t_0} is not integral.)
In particular $n_0 = n_1$ and

Hence $n_0 = \frac{1}{2\pi i} \int_C \frac{f'}{f} dz = n_1 = \frac{1}{2\pi i} \int_C \frac{(f+g)'}{f+g} dz$

$\boxtimes$

<u>Example</u> Use Rouché's thm to show that the # of zeroes of the polynomial

$p(z) = z^6 + 8z^4 + z^3 + 2z + 3$ inside the unit circle is 4.

We need to express $p(z) =$ Big + small

on $|z| = 1$. In this case Big $= 8z^4 = f$

small $= z^6 + z^3 + 2z + 3 = g$

Note $g(z) = |z^6 + z^3 + 2z + 3| < |8z^4|$ on C

Since for $|z|=1$, $|z^6 + z^3 + 2z + 3| \leq |z|^6 + |z|^3 + 2|z| + 3$
$= 1 + 1 + 2 + 3$
$= 7 < 8$
$= 8|z|^4$

hence by Rouché thm

$8z^4 = f$ and $f+g = p(z)$ have the same number of zeroes inside the unit circle, since $8z^4$ has 4 zeroes so does p. ∎

Example Rouché's thm also gives a simple proof of fund. thm of algebra

Let $p(z) = z^d + a_{d-1} z^{d-1} + \dots + a_0$

For $|z|$ large enough the term z^d dominates

Take $f(z) = z^d$, $g(z) = a_{d-1} z^{d-1} + \dots + a_0$

Then $|f(z)| > |g(z)|$ on $|z| = R$

and hence $p(z) = f+g$ and $f = z^d$ has same number of zeroes inside $|z| = R$ since f has d zeroes so does p. ∎