22-10-24

Our next application is the Morera's theorem, which is a converse to Goursot's theorem.

Recall Goursot's thm says: let $f : \Omega \to \mathbb{C}$ (Ω open) be a holomorphic function. Let $T \subset \Omega$ be a triangle whose interior is also contained in Ω then

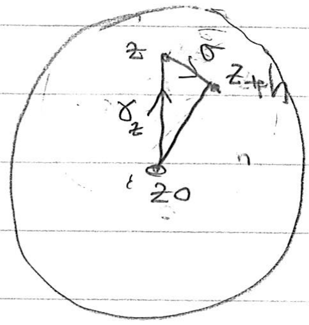
$$\int_T f(z)\,dz = 0.$$

Thm (5.1 #) (Morera's theorem)
Let $\Omega \subset \mathbb{C}$ open and $f : \Omega \to \mathbb{C}$ continuous. Assume that for any open disc $D \subset \Omega$ and any triangle T whose inside contained in D we have that

$$\int_T f(z)\,dz = 0.$$

Then f is holom. on Ω.

Proof let $z_0 \in \Omega$, $D_r(z_0) \subset \Omega$



For $z \in D_r(z_0)$ define

$$F(z) := \int_{\gamma_z = [z_0, z]} f(w)\,dw$$

where $\gamma_z : [0,1] \to \mathbb{C}$

$t \to z_0(1-t) + zt$

the line segment joining z_0 to z

Then for a small h so that $z+h \in D_r(z_0)$,

$$F(z+h) - F(z) = \int_{\sigma = [z, z+h]} f(w)\,dw$$

Since $\int_T f(w)\,dw = 0 \quad \forall\, T \subset D_r(z_0)$ by assumption

In particular for $T = \langle z_0, z, z+h \rangle$

Then using continuity of f at z one can show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

$$F(z+h) - F(z) = \int_{[z, z+h]} \big(f(w) - f(z) + f(z)\big)\,dw$$

$$= f(z) \underbrace{\int_{[z, z+h]} dw} + \int_{[z, z+h]} \big(f(w) - f(z)\big)\,dw$$

$$\left| \int_{[z,z+h]} \big(f(w) - f(z)\big)\, dw \right| \leq \sup_{w \in [z, z+h]} |f(w) - f(z)|\ h$$

so $$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \sup_{w \in [z, z+h]} |f(w) - f(z)|$$

But f is continuous. Hence

$$\sup_{w \in [z, z+h]} |f(w) - f(z)| \to 0 \quad \text{as } h \to 0$$

and $$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

So F is holomorphic on $D_r(z_0)$.
But then F' is also holom on $D_r(z_0)$.
Since $F' = f$, it follows that
f is holom on $D_r(z_0)$.
But then f is holom on all of Ω as $z_0 \in \Omega$ was orbitrary.

(See also Exercise 4.4)

§ 5.2 Sequences of holomorphic functions

It is known from real analysis that pointwise convergence of a sequence of functions lead to pathologies, such as ~~the~~ the pointwise limit of a sequence of continuous functions is not necessarily continuous. eg $f_n : [0,1] \to \mathbb{R}$, $x \mapsto x^n$ conv to $f(x) = \begin{cases} 0 & x \in [0 \\ 1 & x = 1 \end{cases}$

To avoid this we used a stronger form of convergence ; uniform convergence. For example The limit of a uniformly convergent sequence of continuous functions is continuous.

We also have that uniformly conv. seq. of integrable functions converges to an integrable function.
Hence uniform convergence of sequence of functions has better stability properties
But uniformly convergent seq. of differentiable functions does not necessarily have differentiable limits.

eg $f_n(x) = \sqrt{\dfrac{nx^2+1}{n}}$ $x \in [-1,1]$, $f_n(x) \to |x|$
cont but not diff.

We'll see that sequence of complex functions have much better stability properties

As in the real case, uniform limit of a sequence of continuous functions is continuous and similarly line integrals of a uniform. conv. sequence of functions converge to the line integral of the limit function

In contrast to the situation in real analysis we'll see that complex differentiability is also stable with respect to uniform convergence

Recall: A sequence $f_1, f_2, \ldots : \Omega \to \mathbb{C}$ of functions defined on an open set $\Omega \subseteq \mathbb{C}$ is called uniformly convergent (in Ω) to the limit $f : \Omega \to \mathbb{C}$ if

$\forall \varepsilon > 0, \ \exists N > 0$ s.t

$$|f(z) - f_n(z)| < \varepsilon \quad \forall n \geq N, \ \forall z \in \Omega$$

$$\iff \lim_{n \to \infty} \sup\{|f(z) - f_n(z)| : z \in \Omega\} = 0.$$

(N does not depend on z, only on ε)

In fact we only need uniform convergence locally, or equivalently uniform convergence on compact subsets.

Defn Let $\Omega \subset \mathbb{C}$ be open. $f_n : \Omega \to \mathbb{C}$ a sequence of functions. $(f_n)_{n \geq 1}$ is called locally uniformly convergent or compactly convergent or uniformly convergent on compact sets if the following equivalent conditions are satisfied

① $\forall a \in \Omega \; \exists \varepsilon > 0$ s.t $B_\varepsilon(a) \subset \Omega$ s.t $\left(f_n\big|_{B_\varepsilon(a)}\right)$ converges uniformly

② For every compact set $K \subset \Omega$ $(f_n|_K)$ converges uniformly.

Note ① $\Rightarrow$ ② Since K is covered by finitely many discs in ①

② $\Rightarrow$ ① Since Ω is open, $\forall a \in \Omega$ there is a closed disc, (i.e. compact) $a \in D \subset \Omega$.

Remark Note that since continuity is a local property even in the case of real valued functions, local uniform convergence of continous functions will imply continuity of the limit function

Hence similar to the real case one can show

Prop. $(f_n)_{n\geq 1}$, $f_n : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$ open
f_n continuous
If (f_n) converges uniformly on compact sets to f then f is continuous.

The main theorem we have is

Thm 5.2 Let $(f_n)_{n\geq 1}$ be a sequence of holomorphic functions on Ω, $\Omega \subseteq \mathbb{C}$ open. If (f_n) converges uniformly to a function f in every compact set of Ω. Then f is also holomorphic.

Proof. Since f_n are each holom, they're also continuous. Hence by above Prop. their limit f is also continuous

To show f is also holomorphic we'll use Morera's theorem, and the fact that any triangle T is compact.

By Morera's thm, since f is continuous to show f is holom, it is enough to show $\int_T f(w)dw = 0$ for any open disc D, $T \subset D \subset \Omega$ and T triangle contained in D.

Let $D = D_r(z_0) \subset \Omega$ an open disc in Ω

T any triangle with inside contained in D.

By Goursat's thm $\int_T f_n(w)\,dw = 0 \quad \forall n \geq 1$

Since $f_n(z) \to f(z)$ uniformly on compact sets

and T is compact

$f_n(z) \to f(z)$ uniformly $\forall z \in T$

$$\left| \int_T f_n(z)\,dz - \int_T f(z)\,dz \right|$$

$$\leq \int_T |(f_n(z) - f(z))|\,|dz|$$

$$\leq \underbrace{\sup_{z \in T} |f_n(z) - f(z)|}_{\downarrow \, 0} \text{ (length of } T)$$

0 since $f_n(z) \to f(z)$ unif on T

Hence $\underbrace{\lim \int_T f_n(z)\,dz}_{0} = \int_T f(z)\,dz$

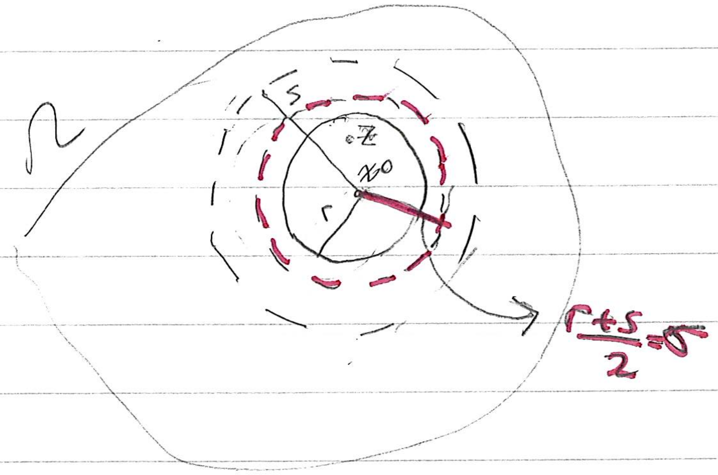
Hence $\int_T f(z)\,dz = 0 \Rightarrow f$ is holom on Ω ☒

We have the following generalization

Thm 5.3 $\{f_n\}_{n=1}^{\infty}$ a seq. of holomorphic functions in $\Omega \subset \mathbb{C}$ (Ω open) such that $f_n \to f$ unif. on every compact sset of Ω, then $\{f_n'\}_{n\geq 1}$ converges uniformly to f' on every compact sset.

Proof. Let $z_0 \in \Omega$, $r>0$ s.t $\overline{D_r(z_0)} \subset \Omega$. f_n converges unif to f on $\overline{D_r(z_0)}$

Let $s>r$ s.t $D_s(z_0) \subset \Omega$



Let $\sigma = \frac{r+s}{2} \in (r, s)$

we then have

By CIF for derivatives II (Cor 4.2)

$$f'(z) = \frac{1}{2\pi i} \int_{C_\sigma(z_0)} \frac{f(w)}{(w-z)^2}\, dw$$

and $$f_n'(z) = \frac{1}{2\pi i} \int_{C_\sigma(z_0)} \frac{f_n(w)}{(w-z)^2}\, dw$$

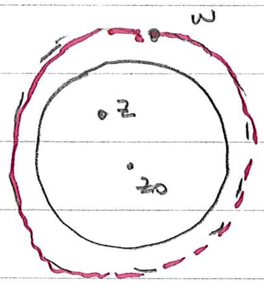
for every $z \in \overline{D_r(z_0)} \subset D_\sigma(z_0)$

Hence for $z \in \overline{D_r(z_0)}$ we have

$$\left|f_n'(z) - f'(z)\right| = \left|\frac{1}{2\pi i}\int_{C_\sigma(z_0)} \frac{f_n(w) - f(w)}{(w-z)^2}\,dw\right|$$

$$\leq \frac{1}{2\pi}(2\pi\sigma) \sup_{w \in C_\sigma(z_0)} \left|\frac{f_n(w) - f(w)}{(w-z)^2}\right|$$

But for $w \in C_\sigma(z_0)$ and $z \in \overline{D_r(z_0)}$, $|w - z_0| = \sigma$, $|z - z_0| \leq r$



$$|w - z| = |w - z_0 - (z - z_0)|$$

$$\geq \big||w - z_0| - |z - z_0|\big|$$

$$\sigma - r$$

which gives $|w - z|^{-2} \leq (\sigma - r)^{-2}$

Hence $$|f_n'(z) - f'(z)| \leq \frac{\sigma}{(\sigma - r)^2} \underbrace{\sup_{C_\sigma(z_0)} |f_n(w) - f(w)|}_{\downarrow\, 0}$$

Since $f_n(w) \to f(w)$ unif. on the compact set $C_\sigma(z_0)$

We have that $f_n' \to f'$ unif on $\overline{D_r(z_0)}$

Since every compact set is contained in a union of finitely many such discs we're done

$\blacksquare$

Remarks

(1) These theorems are often used to prove holomorphicity of functions defined by infinite series. Let f_n be a seq. of holom. functions.

If $F(z) = \sum_{n=1}^{\infty} f_n(z) \qquad z \in \Omega$

let $S_N(z) = \sum_{n=1}^{N} f_n(z)$. Then $S_N(z)$ is holom

if $\{S_N(z)\}_{N=1}^{\infty}$ converges uniformly on compact ssets of Ω

then $\lim S_N(z) = F(z)$ is also holomorphic

(2) For series of functions, we have also the following useful theorem of Weierstrass, called **Weierstrass M-test**

Thm Let $f_n : \Omega \to \mathbb{C}$ a sequence of functions $U \subset \Omega$ a non-empty sset.
Suppose $\exists$ a sequence of real numbers $M_n \geq 0$ s.t

$|f_n(z)| \leq M_n \;\forall n \in \mathbb{N}, \forall z \in U$ and $\sum_{n=0}^{\infty} M_n < \infty$

Then $\sum_{n=1}^{\infty} f_n$ converges abs. and unif on U

Proof Exercise.

Example For $s \in \mathbb{C}$, $s = \sigma + it$
$\sigma, t \in \mathbb{R}$, $n \in \mathbb{N}$

the function $s \longmapsto n^s := \exp(s \log n)$

is an analytic function on $\mathbb{C}$

$$|n^s| = \left| e^{(\sigma+it)\log n} \right| = e^{\sigma \log n} = n^\sigma$$

Then we have

Proposition (2.1. Chapt 6) The series $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely and uniformly on every half plane

$$U_\delta := \{ s \in \mathbb{C} \mid \operatorname{Re} s \geq 1 + \delta \}, \quad \delta > 0$$

and is holomorphic in $\{ s \in \mathbb{C} \mid \operatorname{Re} s > 1 \}$.

Proof. For each $\delta > 0$ we have
If $\operatorname{Re} s = \sigma \geq 1 + \delta > 1$
Then the series $\zeta(s)$ is uniformly bounded

by $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$ since

$$|\zeta(s)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum \frac{1}{|n^s|} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on every half plane $\text{Re}\, s \geq 1+\delta > 1$, $\forall \delta$ and hence defines a holomorphic function in $\text{Re}\, s > 1$

(Every compact sset of $\{s \mid \text{Re}\, s > 1\}$ is contained in such a half plane $\text{Re}\, s \geq 1+\delta$)

Example For $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}\, z > 0\} \subset \mathbb{C}$

(Note $\mathbb{H}$ is open)

we define the theta function

$$\Theta(z) := \sum_{n=0}^{\infty} e^{2\pi i n^2 z}$$

Prop. $\forall z \in \mathbb{H}$, $\Theta(z)$ is well-defined ie the series converges and defines a holomorphic function there

Proof: We'll show that it converges uniformly on any subset of the form $\mathbb{H}_\delta := \{z \in \mathbb{C} \mid \text{Im}\, z \geq \delta\}$ with $\delta > 0$ since any compact sset of $\mathbb{H}$ is contained in such a set, this will imply the result