23.10.24.

For $z \in \mathbb{H}_\delta$, $z = x+iy$, $y \geq \delta > 0$.

$$\left|e^{2\pi i n^2 z}\right| = \left|e^{2\pi i n^2 x}\right| \left|e^{-2\pi n^2 y}\right|$$

$$= e^{-2\pi n^2 y} \leq e^{-2\pi n y} \quad \forall n.$$

Since $y \geq \delta$, $e^{-2\pi y} \leq e^{-2\pi\delta} < 1$

Hence $\left|\sum_{n=0}^{\infty} e^{2\pi i n^2 z}\right| \leq \underbrace{\sum_n e^{-2\pi n\delta}}_{\text{geometric series}} < \infty$

Hence $\sum_{n=0}^{\infty} e^{2\pi i n^2 z}$ converges uniformly on $\mathbb{H}_\delta$ for any $\delta > 0$

Hence it defines a holomorphic function on $\mathbb{H}$

$\boxtimes$

Remark We hope to come back to $\zeta(s)$ and $\Theta(z)$, and use $\Theta(z)$ to show that $\zeta(s)$ (which is defined by the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}\, s > 1$) has an analytic continuation to $\mathbb{C} \setminus \{1\}$. $\pi^{-s/2} \zeta(s) \Gamma(s/2) = \frac{1}{2}\int_0^\infty (\Theta(it) - 1) t^{s/}$

Finally we also a similar theorem for functions defined in terms of integrals ie similar to the thms for functions defined in terms of infinite series.

Many special functions in mathematics are defined in terms of integrals of the type

$$f(z) := \int_a^b F(z,t)\,dt$$

or as limits of such integrals

For example: $\Gamma(z) := \lim_{\varepsilon \to \infty} \int_{1/\varepsilon}^{\varepsilon} e^{-t}\, t^{z} \frac{dt}{t}$

We have the following Thm.

Thm 5.4 let $\Omega \subset \mathbb{C}$ open, $I = [a,b] \in \mathbb{R}$ a closed bounded interval

let $F : \Omega \times I \to \mathbb{C}$ be a function with the following properties

(a) $F : \Omega \times I \to \mathbb{C}$ is continuous on $\Omega \times I$

(b) For each $t_0 \in I$, the function $f_{t_0}(z) := F(z, t_0) : \Omega \to \mathbb{C}$ is holomorphic.

Then the function $f(z)$ defined by the integral

$$f(z) := \int_a^b F(z,t)\,dt$$ is

holomorphic on Ω.

<u>Proof</u> The idea is to use the Riemann sums to approximate the integral: let $f_n(z) := \frac{(b-a)}{n} \sum_{j=0}^{n-1} F(z, a + \frac{b-a}{n} j)$

Then $f_n(z)$ is a finite sum of holom. functions hence holomorphic.

We want to show that $f_n(z)$ converges to f uniformly on compact ssets. Then using Thm 5.3 we can conclude that f is holom.

let $K \subset \Omega$ be compact.

We use that a continuous function $F: \Omega \times I \to \mathbb{C}$ when restricted to the compact set $K \times I$ is unif continuous

Hence $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall (z_i, t_i) \in K \times I$, $i = 1, 2$

If $|z_1 - z_2| < \delta$ and $|t_1 - t_2| < \delta$ then

$$\left| F(z_1, t_1) - F(z_2, t_2) \right| < \frac{\varepsilon}{b-a}.$$

Let n be large enough so that $\frac{b-a}{n} < \delta$

Then $\forall z \in K$

$$f_n(z) - f(z) = \sum_{j=0}^{n-1} \int_{a + j\left(\frac{b-a}{n}\right)}^{a+(j+1)\frac{b-a}{n}} \left[F\left(z, a + j\left(\frac{b-a}{n}\right)\right) - F(z, t) \right] dt$$

Using $f(z) = \int_a^b F(z,t)\,dt = \int_a^{a+\left(\frac{b-a}{n}\right)} F + \int_{a+\frac{b-a}{n}}^{a+2\left(\frac{b-a}{n}\right)} F + \cdots \int_{a+(n-1)\frac{b-a}{n}}^{b} F$

and $f_n(z) = \left(\frac{b-a}{n}\right) \sum_{j=0}^{n-1} F\left(z, a + j\left(\frac{b-a}{n}\right)\right) =$

$$= \sum_{j=0}^{n-1} \underbrace{\int_{a+j\left(\frac{b-a}{n}\right)}^{a+(j+1)\left(\frac{b-a}{n}\right)} F\left(z, a + j\left(\frac{b-a}{n}\right)\right) dt}_{= F\left(z, a+j\left(\frac{b-a}{n}\right)\right) \cdot \left(\frac{b-a}{n}\right)} =$$

since the integrand is indep of t

Now since $t \in \left[a + j\left(\frac{b-a}{n}\right), a + (j+1)\frac{b-a}{n}\right]$

$$\left|t - \left(a + j\left(\frac{b-a}{n}\right)\right)\right| < \frac{b-a}{n} < \delta$$

Since z-arguments are equal we also have
$0 = |z - z| < \delta$

Hence $\left|F\left(z, a + j\left(\frac{b-a}{n}\right)\right) - F(z,t)\right| < \frac{\varepsilon}{b-a}$

and $|f_n(z) - f(z)| \leq \frac{\varepsilon}{b-a} \sum_{j=0}^{n-1} \left(\frac{b-a}{n}\right) = \varepsilon$

$\forall z \in K$ which gives the uniform convergence of f_n to f on K.

Hence f is holomorphic

$\blacksquare$

Remark One can with some more work can also show that $f'(z)$ is given by

$$f'(z) = \int_a^b \underbrace{F'(z,t)}_{f_t'(z)}\,dt \quad \forall z \in \Omega.$$

Ie. we can interchange $\int$ and $\frac{\partial}{\partial z}$

Remark Many special functions that appear as solns of differential equations, for ex. Bessel functions have integral representations

eg: $J_n(z)$ is defined as solution of Bessel's diff equations

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - n^2) f = 0$$

For $n \in \mathbb{Z}$,

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin t} \cdot e^{-int} \, dt$$

$$J_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin t} \, dt$$

$F(z, t) = e^{iz\sin t}$ is continuous on $\mathbb{C} \times \mathbb{R}$

For each $t \in [-\pi, \pi]$
$f_t(z) = e^{iz\sin t} : \mathbb{C} \longrightarrow \mathbb{C}$
is holomorphic

Hence the function $\int_{-\pi}^{\pi} e^{iz\sin t} \, dt$ is holom. on $\mathbb{C}$.

$$J_0'(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin t} (i\sin t) \, dt$$