24-9-23

Last time:

let $f : \Omega \to \mathbb{C}$ is diff at $z_0 = (x_0, y_0)$

if we write

$$f(z) = f(x,y) = u(x,y) + i v(x,y)$$

then (a) looking at the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ along $z = x + iy_0$ with $x \to x_0$

we get $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$

Hence (A)

$$\boxed{f'(z_0) = u_x(z_0) + i v_x(z_0) = f_x(z_0)}$$

(b) looking at the limit along $z = x_0 + iy$ with $y \to y_0$

we get

(B)

$$\boxed{f'(z_0) = v_y(z_0) - i u_y(z_0) = -i f_y(z_0)}$$

let

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

(A) = (B)

$\Rightarrow$

$$\boxed{\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}}$$

Cauchy Riemann eqns

What we've shown can be summarized in

**Prop** (Prop 2.3 in the book) If $f$ is holom. at $z_0$, $f(z) = u + iv$. Then

$$\frac{\partial f}{\partial \bar z}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0)$$

$$= 2\frac{\partial u}{\partial z}(z_0)$$

If we write $f(z) = \tilde f(x, y)$ with $\tilde f : \mathbb{R}^2 \to \mathbb{R}^2$ then $\tilde f$ is differentiable with Jacobian $J_{\tilde F}(x_0, y_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}$

$$= \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

and $\det J_{\tilde F} = |f'(z_0)|^2$

$$= u_x^2 + u_y^2$$

$$= u_x^2 + v_x^2$$

**Proof** $\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)$

$$= \frac{1}{2}\left[u_x + iv_x + iu_y - v_y\right]$$

$$= \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right] = 0 \quad \text{using CR}$$

$$\frac{(A)+(B)}{2} \Rightarrow f'(z_0) = \frac{1}{2}\left[f_x(z_0) - i f_y(z_0)\right]$$

$$= \frac{\partial f}{\partial z}(z_0)$$

CR also gives $\frac{\partial f}{\partial z}(z_0) = u_x + iv_x = u_x - iu_y$.

since $= 2\frac{\partial u}{\partial z}$

If $z_0 = x_0 + iy \in U$ $h = h_1 + h_2 i \in \mathbb{C}$

then $f$ hol. at $z_0$ means

$$f(z_0 + h) = f(z_0) + f'(z_0)h + h\,\mathcal{E}(h)$$

with $\lim_{h \to 0} \mathcal{E}(h) = 0$

If $f'(z_0) = a + ib$ then $\left[ f'(z_0) = u_x + iv_x = v_y - iu_y \right]$

$$f'(z_0)h = (a+ib)(h_1 + h_2 i)$$
$$= ah_1 - bh_2 + i(bh_1 + ah_2)$$

Hence if we write $\tilde{f}(x,y) = f(z)$, $H = (h_1, h_2)$ we have that

$$\frac{\left(\tilde{f}\left((x_0, y_0) + (h_1, h_2)\right) - \tilde{f}(x_0, y_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right)}{|H|} \to 0$$

as $|H| \to 0$

This means $\tilde{f} = \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with $J_{\tilde{f}}(x_0, y_0)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

Using $a = u_x = v_y$, $b = v_x = -u_y$
we get

$$J_{\tilde{f}} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}, \quad \det J_{\tilde{f}} = u_x^2 + v_x^2 = |f'(z_0)|^2$$

$\blacksquare$

<u>Remark</u> Recall we can represent any complex number $z = a + ib$ with a $2 \times 2$ matrix:

$$\boxed{a+bi \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}}$$

If $f = u + iv$ is diff at $z_0$, then

$f'(z_0) = u_x + iv_x$ has matrix representation

$$\boxed{f'(z_0) \longleftrightarrow \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}}$$

On the other hand the corresponding function

$$\tilde{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longrightarrow (u(x,y), v(x,y))$$

has a Jacobian matrix $\boxed{J_{\tilde{f}} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}$

comparing these 2 matrices we get exactly the CR eqns

$$\boxed{\begin{aligned} u_y &= -v_x \\ v_y &= u_x \end{aligned}}$$

ie if you remember the general form of Jacobian of a function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and matrix repn of a complex #, you can remember the CR eqns!

The previous prop. shows that

f holom $\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$ (CR. equations)

we also have the following converse

**Thm 2.4** Suppose $f = u + iv$, $f : \Omega \to \mathbb{C}$ $\Omega$ open set. If $u, v$ are continuously differentiable and satisfy CR equations on $\Omega$, then $f$ is holomorphic on $\Omega$ and $f'(z) = \partial f / \partial z$.

**Proof** let $z_0 = (x_0, y_0) \in \Omega$, $h = (h_1, h_2) \in \mathbb{C}$

$u, v$ continuously differentiable imply

$$u(z_0 + h) - u(z_0) = \partial_x u(z_0)\, h_1 + \partial_y u(z_0) h_2 + |h|\, \varepsilon_1(h)$$

with $\varepsilon_1(h) \to 0$ as $h \to 0$

Similarly

$$v(z_0 + h) - v(z_0) = (\partial_x v)\, h_1 + (\partial_y v) h_2 + |h|\, \varepsilon_2(h)$$

with $\lim \varepsilon_2(h) \to 0$.

Then $f = u + iv$ satisfy

$$f(z_0 + h) - f(z_0) = (u + iv)(z_0 + h) - (u + iv)(z_0)$$

$$= (\partial_x u + i \partial_x v)\, h_1 + (\partial_y u + i \partial_y v) h_2$$

$$+ |h|\, \varepsilon(h)$$

where $\varepsilon(h) = (\varepsilon_1 + i\varepsilon_2)(h) \to 0$ as $|h| \to 0$

$$f(z_0+h)-f(z_0) = (\partial_x u - i\partial_y u)\, h_1 +$$

using CR

$$(\partial_y u + i\partial_x u)\, h_2 + \varepsilon(h)|h|$$

$$= (\partial_x u - i\partial_y u)(h_1 + h_2 i) + \varepsilon(h)|h|$$

Hence $f(z_0+h) - f(z_0) = (\partial_x u - i\partial_y u)h + \varepsilon(h)|h|$

with $\varepsilon(h) \to 0$

which says $\left| \frac{(f(z_0+h)-f(z_0)) - (\partial_x u - i\partial_y u)h}{h} \right|$

goes to zero as $h \to 0$.

Hence $f'(z_0)$ exists and equal to

$$\partial_x u - i\partial_y u = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}$$

Example let $f(z) = \overbrace{x^2+y^2}^{u} + \overbrace{2ixy}^{iv}$

Then $\partial_x u(x,y) = 2x \qquad \partial_x v = 2y$

$\partial_y u(x,y) = 2y \qquad \partial_y v = 2x$

$\partial_x u = \partial_y v \quad \forall z \in \mathbb{C}$ $\qquad 2y = \partial_y u = -\partial_x v = -2y$

$2x = 2x$ $\qquad$ only if $y = 0$

Hence $f(z)$ is holomorphic for points only on the real axis. And for these points

$$f'(x_0) = \partial_x u(x_0) + i\,\partial_x v(x_0) = 2x_0$$

A quick summary: $\Omega$ open sset of $\mathbb{C}$

① $f:\Omega \to \mathbb{C}$, $f(z) = u+iv$

$f$ is holom on $\Omega \Rightarrow u, v$ satisfy CR eqns

$$u_x = v_y$$
$$u_y = -v_x$$

$$f'(z) = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y - iv_x$$

② If $u, v$ are real diff. and satisfy CR-eqns then $f = u+iv$ is holomorphic.

③ If we write $\tilde{f}(x,y) = f(z)$, for $f:\Omega \to \mathbb{C}$ holomorphic identifying $\mathbb{C}$ with $\mathbb{R}^2$ then $\tilde{f}:\mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with

$$J_{\tilde{f}}(z_0) = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}, \quad \det J_{\tilde{f}} = |f'(z_0)|^2$$

Remark A matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ defines a linear map $\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

which preserves angles and orientation (ie it is a rotation and a dilation)
If $a+bi \neq 0$, $a+ib = |a+ib|e^{i\theta}$, then it is a rotation by the angle $\theta$, and dilation by $|a+ib|$

Our next result gives important examples of holomorphic functions

## §2.3 Power series

Recall a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}$$

Thm Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series
(Thm 2.5)
Then $\exists R \in \mathbb{R}$, $0 \leq R \leq \infty$ such that

(i) if $|z| < R$ the series converges absolutely

(ii) if $|z| > R$ " " diverges.

Moreover with the convention that $1/0 = \infty$ and $1/\infty = 0$, $R$ is given by

$$1/R = \limsup |a_n|^{1/n}$$

$R$ is called the radius of convergence

$D_R(0) := \{ z \in \mathbb{C} \mid |z| < R \}$ disc of convergence

Proof of thm 2.5 Exercise
(same as in real analysis)

Important example of a power series is the complex exponential function

$$\boxed{e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}}$$

It converges abs $\forall z \in \mathbb{C}$ since $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$

and $|e^z| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$

In fact $e^z$ conv. uniformly on compact ssets of $\mathbb{C}$.

The following thm shows that $e^z$ in particular and power series in general give examples of holomorphic functions in their disc of convergence.

**Thm 2.6** The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

$f'(z)$ has the same radius of convergence as $f$.

**Proof** is similar to the one in real variables but we'll repeat it here since it is an important Thm.

Let $R$ be the radius of convergence of $f(z)$

Since

$$\lim_{n\to\infty} n^{1/n} = 1$$

$$\limsup |a_n n|^{1/n} = \limsup |a_n|^{1/n} = R$$

Hence $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence. Repeated application shows that the sum $\sum_{n=0}^{\infty} n(n-1)\cdot(n-k) a_n z^{n-k}$ has radius of convergence $R$ for any $k$

Let $z \in \mathbb{C}$, $|z| < R$ choose $\delta$ s.t

$|z| + \delta < R$ eg can take $\delta \leq \frac{R - |z|}{2}$

Let $h \in \mathbb{C}$, $|h| < \delta$, w.t.s.

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| \to 0 \text{ as } h \to 0.$$

But

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right|$$

$$= \left| \sum_{n=0}^{\infty} \left( \frac{a_n (z+h)^n - a_n z^n}{h} - n a_n z^{n-1} \right) \right|$$

$$\leq \sum_{n=0}^{\infty} |a_n| \left| \frac{1}{h}\left( \sum_{k=0}^{n} \binom{n}{k} h^k z^{n-k} - z^n \right) - n z^{n-1} \right|$$

$$= \sum_{n=2}^{\infty} |a_n| \left| \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k} \right|$$

$$\leq \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |h|^{k-1} |z|^{n-k}$$

$$\leq \sum_{n=2}^{\infty} |a_n| \; n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} |h|^{k-2} |z|^{n-k} |h|$$

Using, for $k=0$

$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

$= \frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}$

$\leq n(n-1)\binom{n-2}{k-2}$

$(k \geq 2)$

$$\leq |h| \sum_{n=2}^{\infty} |a_n| \, n(n-1) \left( |z| + |h| \right)^{n-2}$$

$$\leq |h| \underbrace{\left( \sum_{n=2}^{\infty} |a_n| \, n(n-1) \left( \frac{R+|z|}{2} \right)^{n-2} \right)}_{\text{indep of } h}$$

since $|h| < \frac{R - |z|}{2} = \delta$. Since $|z| < R$,

Note $\frac{R+|z|}{2} < R$. Hence the infinite sum converges and LHS $\to 0$ as $h \to 0$

Hence $\frac{f(z+h) - f(z)}{h} \xrightarrow[h \to 0]{} \sum_{n=0}^{\infty} n a_n z^{n-1}$