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Examples

① $\exp : \mathbb{C} \to \mathbb{C}$

$$z \to \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{conv. } \forall z$$

As such is holom. on all $\mathbb{C}$.

$$\exp(z)' = \sum_{n=0}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

We will write e^z instead of $\exp(z)$ mostly.

② Trigonometric functions

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n!} \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos i = \frac{e^{i^2} + e^{-i^2}}{2} = \frac{e^{-1} + e}{2} = \frac{e^2+1}{2e}$$

$$\sin i = \frac{e^{i^2} - e^{-i^2}}{2i} = i\left(\frac{e^2-1}{2e}\right)$$

③ $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has conv. radius 1.

converges $\forall z, |z| \leq 1$ since $\sum \frac{1}{n^2} < \infty$

④ Geometric series: $\sum_{n=0}^{\infty} z^n$ converges for $|z| < 1$

⑤ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$ conv for $|z| < 1$

for $z = 1$ conv (Leibniz' criteria).

$z = -1$ div Harmonic series

§3 Complex line integrals (Integrals along curves).

We start by recalling the main definitions and properties of curves.

Defn ① A parametrized curve in $\mathbb{C}$ is a continuous function $\gamma : [a,b] \longrightarrow \mathbb{C}$, where $[a,b]$ is a closed interval of real numbers.

② A smooth curve is a curve $\gamma : [a,b] \to \mathbb{C}$
$\gamma(t) = x(t) + i y(t)$ if its derivative
$\gamma'(t) = x'(t) + i y'(t)$ exists $\forall t \in [a,b]$
and if $\gamma'(t)$ is continuous on $[a,b]$.
and $\gamma'(t) \neq 0$ for $t \in [a,b]$

Here
$$\gamma'(a) := \lim_{\substack{h \to 0 \\ h > 0}} \frac{\gamma(a+h) - \gamma(a)}{h},$$

$$\gamma'(b) := \lim_{\substack{h \to 0 \\ h < 0}} \frac{\gamma(b+h) - \gamma(b)}{h}$$

are the right and left hand derivatives resp.

③ A piecewise smooth curve is a curve $\gamma : [a,b] \to \mathbb{C}$, γ is cont. on $[a,b]$
and $\exists$ points $a = a_0 < a_1 < \cdots < a_n = b$
s.t. $\gamma(t)$ is smooth on each interval $[a_k, a_{k+1}]$.

(4) A closed curve is a curve $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$

(5) A curve is simple if it is not self intersecting ie $\gamma(t) \neq \gamma(s)$ unless $s=t$ or $s=a$ and $t=b$.

Rmk For us in this course the curves will always be piecewise smooth. From now on when we say a curve we mean a piecewise smooth one even if I forget to write.

(6) $\tilde{\gamma} : [c,d] \to \mathbb{C}$ is called reparametrization of $\gamma : [a,b] \to \mathbb{C}$ if there exists a continuously differentiable function

$\sigma : [c,d] \to [a,b]$ which is bijective with $\sigma'(t) > 0$ $\forall t$, and

$\tilde{\gamma} = \gamma \circ \sigma$. $\sigma'(t) > 0$ means the orientation is preserved.

(γ, $\tilde{\gamma}$ represents the same geometric object with different parametrizations.

Remark

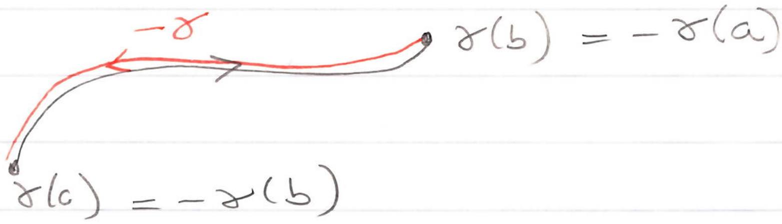
We will often work with a particular parametrization since most important notions will be independent of parametrization (for example path integrals) Because of this independence, we often describe curves by drawing them as geometric objects in the plane

There are 2 elementary methods to modify or combine paths to obtain new paths.

① If $\gamma : [a, b] \to \mathbb{C}$ is a path
$t \to \gamma(t)$

the reverse path $-\gamma$ (or γ^-)
is the path $-\gamma : [a, b] \to \mathbb{C}$
$t \to \gamma(b + a - t)$

i.e. $(-\gamma)(t) = \gamma(a + b - t)$



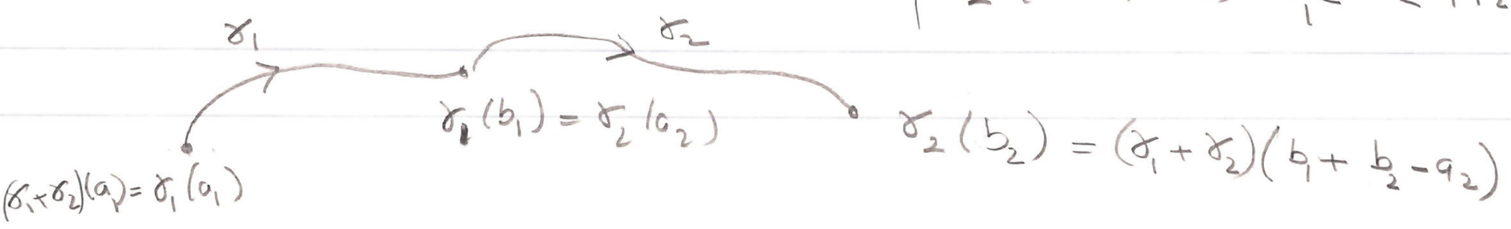
② If $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, $\gamma_2 : [a_2, b_2] \to \mathbb{C}$

are 2 path s.t $\gamma_1(b_1) = \gamma_2(a_2)$ then

the concatenation or sum of the paths γ_1, γ_2

is a path $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$

defined as $$[\gamma_1 + \gamma_2](t) := \begin{cases} \gamma_1(t) & \text{if } a_1 \le t \le b_1 \\ \gamma_2(t - b_1 + a_2) & b_1 \le t \le b_1 + b_2 - a_2 \end{cases}$$



Examples ① Given 2 points $z_1, z_2 \in \mathbb{C}$ the path

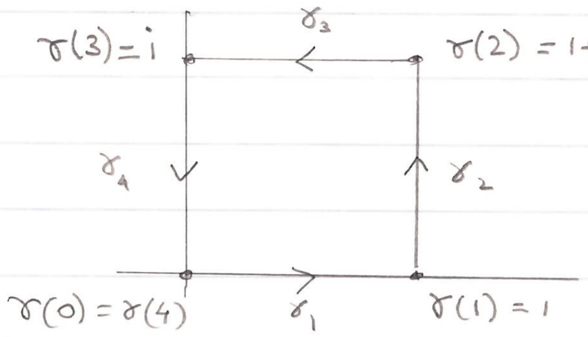
$$\gamma : [0,1] \longrightarrow \mathbb{C}$$
$$t \longmapsto (1-t)z_1 + z_2 t$$

is the parametrization of the line segment between z_1 and z_2

smooth
simple
not closed.

② $\gamma : [0,4] \longrightarrow \mathbb{C}$

$$\gamma(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 + i(t-1) & \text{if } 1 \le t \le 2 \\ (3-t) + i & \text{if } 2 \le t \le 3 \\ i(4-t) & \text{if } 3 \le t \le 4 \end{cases}$$



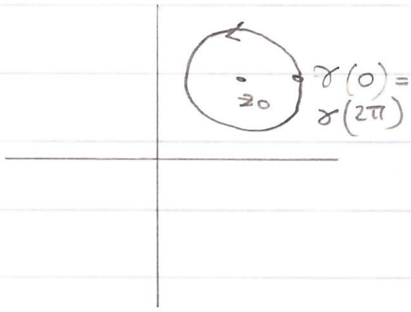
piecewise smooth
simple closed

γ is sum of 4 paths: $\gamma_1 : [0,1] \longrightarrow \mathbb{C}$, $t \longrightarrow t$ $\qquad \gamma_2 : [0,1] \longrightarrow \mathbb{C}$, $t \longrightarrow 1+t$

$\gamma_3 : [0,1] \longrightarrow \mathbb{C}$, $t \longrightarrow i + (1-t)$ $\qquad \gamma_4 : [0,1] \longrightarrow \mathbb{C}$, $t \longrightarrow (1-t)\, i$

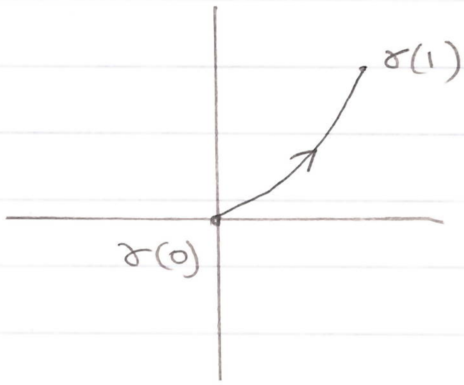
(3) A circle with center at z_0, and radius r has a parametrization

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto z_0 + re^{it}$$



smooth, closed, simple

(4) $$\gamma : [0,1] \longrightarrow \mathbb{C}$$
$$t \longrightarrow t + it^2$$



smooth, non closed, simple

To define the complex line integrals recall that continuous function of a real valued function g on an interval $[a, b]$ is Riemann integrable, ie $\int_a^b g(t)\,dt$ exists

For a complex valued function $g : [a, b] \to \mathbb{C}$ we can define the integral

$$\int_a^b g(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

where $g(t) = u(t) + iv(t)$

Defn Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path, and $f : \Omega \to \mathbb{C}$ is a complex valued function which is defined and continuous on γ. We define **the integral of f along γ** by

$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$

Since $g(t) = [f(\gamma(t))]\gamma'(t) : [a,b] \to \mathbb{C}$ is continuous on $[a,b]$, the integral on the right is meaningful, as long as we show that it is independent of the parametrization of γ.

Let $\tilde{\gamma} : [c,d] \to \mathbb{C}$ be another parametrization s.t

$\tilde{\gamma}(s) = (\gamma \circ \sigma)(s)$ for some $\sigma : [c,d] \to [a,b]$ with $\sigma \in C^1$, $\sigma'(s) > 0$

Then $$\int_{\tilde{\gamma}} f(z)\,dz = \int_c^d f(\tilde{\gamma}(s))\,\tilde{\gamma}'(s)\,ds$$

$$= \int_c^d f(\gamma(\sigma(s)))\,\gamma'(\sigma(s)).\,\sigma'(s)\,ds$$

letting $t = \sigma(s)$, $dt = \sigma'(s)\,ds$ gives $$= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_\gamma f(z)\,dz$$

The following properties of path integrals follow easily from the properties of the Riemann integral.

Prop 3.1 Let $f, g : \Omega \to \mathbb{C}$ continuous, $\gamma, \gamma_1, \gamma_2$ are piecewise smooth curves in Ω, $a, b \in \mathbb{C}$. Then

① $$\int_\gamma (af + bg)(z)\,dz = a\int_\gamma f(z)\,dz + b\int_\gamma g(z)\,dz$$

② If γ^- is the curve γ with reverse orientation then

$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz$$

③ $$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

④ $$\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z\in\gamma}|f(z)|\ \text{length}(\gamma)$$

where $$\sup_{z\in\gamma}|f(z)| = \sup_{t\in[0,b]}|f(\gamma(t))|$$

and $$\text{length}(\gamma) = \sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}}|\gamma'(t)|\,dt$$

Proof ① Follows from the linearity of the Riemann integral

② If $\gamma : [a,b] \to \mathbb{C}$, then $\gamma^- : [a,b] \to \mathbb{C}$
$t \to \gamma(a+b-t)$

$$(\gamma^-)'(t) = -\gamma'(a+b-t)$$

$$\int_{-\gamma} f(z)\,dz = -\int_a^b f(\gamma(a+b-t))\,\gamma'(a+b-t)\,dt$$

$u = b+a-t$
$du = -dt$

$$= \int_b^a f(\gamma(u))\,\gamma'(u)\,du = -\int_a^b f(\gamma(u))\,\gamma'(u)\,du$$

$$= -\int_\gamma f\,dz$$

③ Exercise

④
$$\left|\int_\gamma f(z)\,dz\right| = \left|\sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} f(\gamma(t))\,\gamma'(t)\,dt\right|$$

$$\leq \sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq \sup_{t\in[a,b]} |f(\gamma(t))| \sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} |\gamma'(t)|\,dt$$

$\square$