The Rouche's thm also leads us to two other important thms.

Thm 4.4 (Open mapping thm) Let Ω be open connected, $f \in \mathcal{H}(\Omega)$, f is not a constant, then f is open.

(A map is open if it sends open sets to open sets).

Proof. Let $z_0 \in U \subset \Omega$, $f(z_0) = w_0$.

We want to show that a nbhd of w_0 is also contained in $f(U)$.
ie if w is near w_0, w.t.s $\exists\, z \in U$ s.t $w = f(z)$ ie $w \in f(U)$.

Let $r > 0$ s.t $\overline{D_r(z_0)} \subset U$ and s.t

$f(z) - w_0 \neq 0$ $\forall z \in \overline{D_r^*(z_0)}$. This we can do since zeroes of $\tilde{f}(z) := f(z) - w_0$ are isolated.
In particular $f(z) - w_0 \neq 0$ on the circle $C_r(z_0)$.
Since $C_r(z_0)$ is compact, and $f(z) - w_0 \neq 0$ on $C_r(z_0)$, we can find $\delta > 0$

$$|f(z) - w_0| \geq \delta \quad \text{for } z \text{ on the circle } C_r(z_0) \text{ ie if } |z - z_0| = r.$$

Now let $w \in \mathbb{C}$ s.t $|w - w_0| < \delta$
ie $w \in D_\delta(w_0)$

let

$$F(z) := f(z) - w = \underbrace{(f(z) - w_0)}_{:= \tilde{f}} + \underbrace{(w_0 - w)}_{\tilde{g}}$$

w.t.s. that $F(z)$ has a zero insite the circle $C_r(z_0)$

This will show that $\exists z \in D_r(z_0)$ s.t $f(z) = w$, Hence $w \in f(D_r(z_0))$

Now we apply Rouche's thm to $\tilde{f}, \tilde{g}$ on the circle $|z - z_0| = r$ we have

$$|\tilde{f}| \geq \delta, \quad |\tilde{g}| < \delta$$

Hence on $|z - z_0| = r$, $|\tilde{f}| > |\tilde{g}|$

Since $\tilde{f} = f(z) - w_0$ has a zero inside $D_r(z_0)$ (namely z_0)

$F = \tilde{f} + \tilde{g} = f(z) - w$ also has a zero inside $D_r(z_0)$.

Hence $\exists\, z \in D_r(z_0)$ s.t $w = f(z)$

ie $w \in f(D_r(z_0))$ as wanted $\blacksquare$.

Rmk. This thm for example says that if $f \in \mathcal{H}(D_r(0))$, f not constant then it is not possible that $f(z) \in \mathbb{R}$ for all z since any sset of $\mathbb{R}$ is not open in $\mathbb{C}$.

Thm 4.5 Maximum modulus principle
(Cor 4.6)

Let $\Omega \subset \mathbb{C}$ open connected. $f \in \mathcal{H}(\Omega)$ not constant. Then there is no $z_0 \in \Omega$ s.t

$$|f(z)| \leq f(z_0) \quad \forall z \in \Omega$$

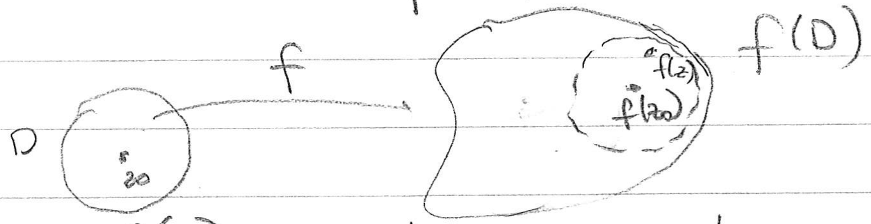
ie f cannot attain a maximum in Ω.

In particular if $\overline{\Omega}$ is bounded and f is continuous on $\overline{\Omega}$ (holom on Ω) then

$$\underbrace{\max_{z \in \overline{\Omega}} |f(z)|}_{} = \max_{z \in \overline{\Omega} - \Omega} |f(z)|$$

exists because $\overline{\Omega}$ compact

Proof: Suppose $f \in \mathcal{H}(\Omega)$ non constant and Suppose f attained a maximum at $z_0 \in \Omega$. Since by open mapping thm f is an open map, if $D = D_r(z_0) \subset \Omega$, then $f(D)$ is open and contains $f(z_0)$



Hence $f(D)$ contains a disc around $f(z_0)$ but this means there are points $z \in D$ s.t $|f(z_0)| < |f(z)|$

which contradicts that $|f(z)|$ attains its maximum at z_0.

If $\overline{\Omega}$ is bounded and f is nonconstant and continuous on $\overline{\Omega}$ then

$|f(z)|$ attains its maximum on $\overline{\Omega}$ since it is a continuous function on a compact set. By the first part this point where it attains its maximum cannot be inside Ω. Hence it has to be on the boundary $\overline{\Omega} \setminus \Omega$.

If f is constant the statement is trivially true.

Remark The assumption $\overline{\Omega}$ is bdd (ie compact) is crucial

Let $\Omega = \{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2} \}$. open connected but $\overline{\Omega}$ is not bounded.

Let $f(z) = \exp(e^z)$, $\left(f|_{\partial\Omega}\right)(z) = \exp(e^{x \pm i\pi/2})$
$= \exp(\pm i e^x)$

Hence $|f(z)| = 1$ for $z \in \partial\Omega$.

but $f(x) = \exp(e^x) \to \infty$ for $x \in \mathbb{R}$ as $x \to \infty$.

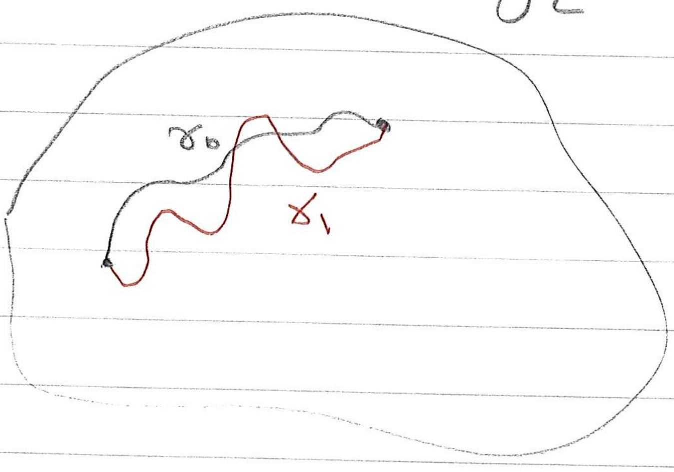
Homotopy and simply connected domains

The key to understand the general form of Cauchy's formula is the idea that if $f : \Omega \to \mathbb{C}$ holomorphic

and if we "continuously deform" γ_0 to γ_1 while staying in Ω and keeping the end points fixed then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz \qquad \begin{array}{l} \gamma_0 : [a, b] \to \Omega \\ \gamma_1 : [a, b] \to \Omega \end{array}$$

Ω



Such curves are called homotopic with fixed end points. This means for each $s \in [0, 1]$ $\exists$ a curve $\gamma_s \subset \Omega$

parametrized by $\gamma_s(t)$, $\gamma_s : [a, b] \to \Omega$

s.t. $\gamma_s(a) = \gamma_0(a) = \gamma_1(a)$

$\gamma_s(b) = \gamma_0(b) = \gamma_1(b)$

and at $s=0$ $\quad \gamma_s(t)\big|_{s=0} = \gamma_0(t)$

and $s=1$

$\gamma_s(t)\big|_{s=1} = \gamma_1(t)$

This should be done continuously.

Defn (Homotopy) Let $\Omega \subset \mathbb{C}$ open
let $\gamma_0 : [a,b] \to \Omega$, $\gamma_1 : [a,b] \to \Omega$
be 2 curves s.t $\gamma_0(a) = \gamma_1(a)$
$\gamma_0(b) = \gamma_1(b)$

We say γ_0 is homotopic to γ_1 in Ω
with endpoints fixed
if $\exists$ $H : [a,b] \times [0,1] \to \Omega$
continuous $(t,s) \longmapsto H(t,s)$
s.t

① $H(t,0) = \gamma_0(t)$ $\forall t \in [a,b]$
$H(t,1) = \gamma_1(t)$

② $H(t,s) := \gamma_s(t)$ is continuous $\forall s \in [0,1]$
$\forall t \in [a,b]$

and $H(a,s) = \gamma_0(a) = \gamma_1(a)$ $\forall s$
$H(b,s) = \gamma_0(b) = \gamma_1(b)$

ie $\gamma_s(t)$ has the same ends points as γ_0, γ_1

Similarly if $\gamma_0 : [a,b] \to \Omega$, $\gamma_1 : [a,b] \to \Omega$
are 2 closed curves, we say
γ_0 is homotopic to γ_1 in Ω if
$\exists$ $H : [a,b] \times [0,1] \to \Omega$ s.t
① $H(t,0) = \gamma_0(t)$ $\forall t \in [a,b]$
$H(t,1) = \gamma_1(t)$

(2) $\gamma_s(t) := H(t,s)$ is a smooth curve $\gamma_s : [a,b] \to \Omega$ in Ω and

$$H(a,s) = H(b,s) \quad \forall\ s \in [0,1]$$

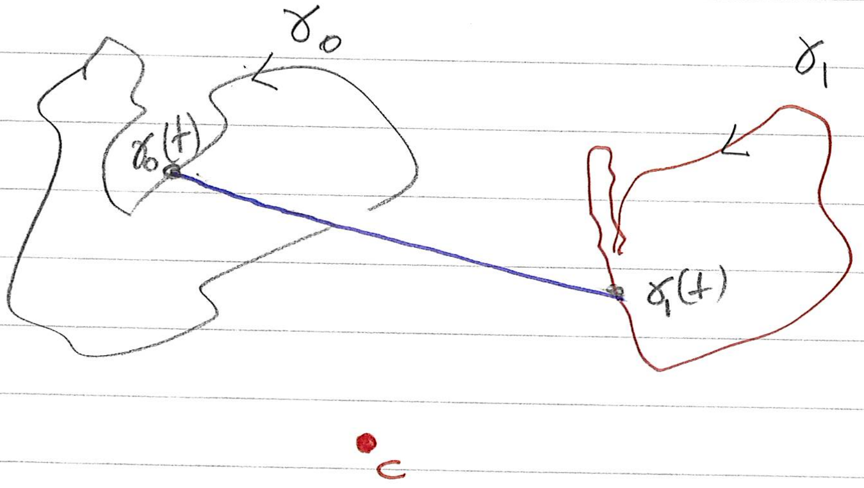
ie

$\gamma_s(t)$ is a closed curve in Ω, $\forall s \in [0,1]$

Examples. (1) If $\Omega = \mathbb{C}$ then

any 2 closed curves γ_0, γ_1 are homotopic in particular every closed curve homotopic to the constant curve

$$\gamma : [a,b] \to \mathbb{C} \quad \text{for every } c \in \mathbb{C}.$$
$$t \to c$$



let $H : [a,b] \times [0,1] \to \mathbb{C}$

$$(t,s) \longmapsto (1-s)\gamma_0(t) + s\,\gamma_1(t)$$

H is a combination of continuous funcs hence continuous

$H(t, 0) = \gamma_0(t)$
$H(t, 1) = \gamma_1(t)$

$H(a, s) = (1-s)\gamma_0(a) + s\gamma_1(a)$
$H(b, s) = (1-s)\gamma_0(b) + s(\gamma_1(b))$

Since $\gamma_0(a) = \gamma_0(b)$ and $\gamma_1(a) = \gamma_1(b)$
$H(a, s) = H(b, s)$ $\forall s$
Hence $\gamma_s(t) : [a, b] \to \mathbb{C}$ are all closed curves.

Note geometrically H is defined using the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ for each fixed t. Hence s varies over the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ for fixed t.

For the constant curve $\sigma = c$, we can take the homotopy between σ, γ as

$$H : [a, b] \times [0, 1] \longrightarrow \mathbb{C}$$
$$(t, s) \longmapsto c(1-s) + s\,\sigma(t)$$

Note the same defn we used for closed curves γ_0, γ_1 also gives a homotopy w/ fixed points if $\gamma_0 : [a, b] \to \Omega$, $\gamma_1 : [a, b] \to \Omega$ are 2 curves w/ fixed end points
ie $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$

(2) Note the same formula for the homotopy works for any Ω which is convex.
Ie if we have 2 curves $\gamma_0(t)$ $\gamma_1(t)$ either closed or with fixed end points in a convex set Ω. Then since for a convex set the line segment between any 2 points is also in the set, the function defined by

$$H : [0,b] \times [0,1] \longrightarrow \Omega$$
$$(t, s) \longmapsto (1-s)\gamma_0(t) + s\gamma_1(t)$$

gives a homotopy in Ω.

In particular this works for Ω a disc.

(3) An example of 2 curves which are <u>not</u> homotopic in Ω is if we take $\Omega := \mathbb{C} - \{0\} = \mathbb{C}^*$

$$\gamma_0(t) : [0, \pi] \longrightarrow \Omega$$
$$t \longrightarrow e^{it}$$

$$\gamma_1(t) : [0, \pi] \longrightarrow \Omega$$
$$t \longrightarrow e^{-it}$$

Note $\displaystyle\int_{\gamma_0 - \gamma_1} \frac{1}{z}\,dz = 2\pi i$

$$= \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\,dt = 2\pi i$$

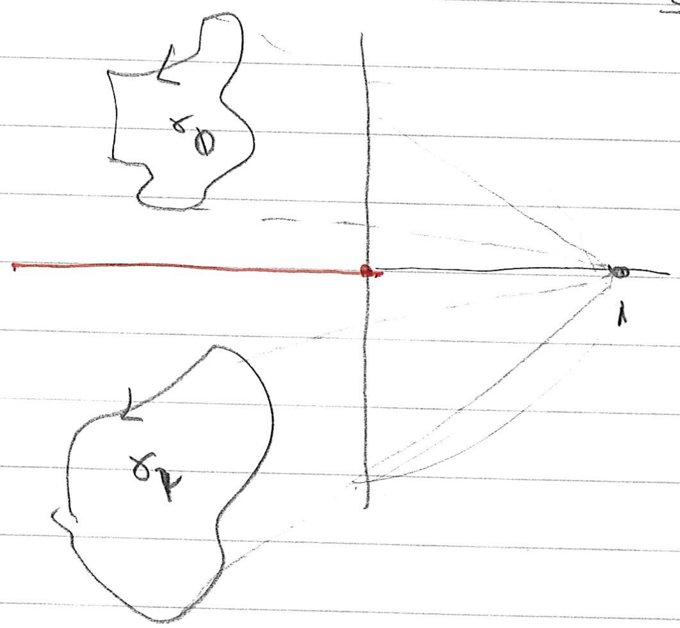
$$\int_{\gamma_1} \frac{1}{z}\,dz = 2\pi i + \int_{\gamma_1} \frac{1}{z}\,dz$$

Then γ_0, γ_1 are not homotopic in Ω

We'll see a simple proof of this when we see the homotopy version of Cauchy's thm.

Intuitively to deform γ_0 to γ_1 we have to go through 0 which is not in Ω

(4) $\Omega = \mathbb{C} - (-\infty, 0]$. Ω is not convex, so we cannot use the previous formula. But we can still deform γ_0 to γ_1



The idea is to choose any point the real line say 1 and the constant curve

$$\Gamma : [a,b] \to \Omega$$
$$t \to 1.$$

we deform γ_0 to 1 then 1 to γ_2

$$H(t,s) := \begin{cases} 1 + (1-2s)(\gamma_0(t) - 1) & 0 \le s \le \frac{1}{2} \\ 1 + (2s-1)(\gamma_1(t) - 1) & \frac{1}{2} < s \le 1 \end{cases}$$

H is continous, the only point to check is $s = \frac{1}{2}$

$$H(t, \tfrac{1}{2}) = 1 = \lim_{s \to \frac{1}{2}} 1 + (2s-1)(\gamma_1(t) - 1) = 1$$

To see the image of $H(t,s)$ is contained in Ω $\forall$ $a \le t \le b$, $0 \le s \le 1$ check

for exemple that if $a \le t \le b$, $0 \le s \le \frac{1}{2}$

then if $H(t,s) \notin \Omega$ for some t, s means $H(t,s) \in (-\infty, 0]$

$H(t,s)$ is a real number which is non-positive

ie $1 + (1-2s)(\gamma_0(t) - 1) \le 0$

$\Leftrightarrow (1-2s)(\gamma_0(t) - 1) \le -1$

$\Leftrightarrow \gamma_0(t) \le \dfrac{-1}{1-2s} + 1 = 1 + \dfrac{1}{2s-1} = \dfrac{2s}{2s-1}$

but $0 \le s \le \frac{1}{2} \Rightarrow 2s \ge 0$, $2s - 1 \le 0 \Rightarrow 2s/_{2s-1} \le 0$

Hence $\gamma_0(t) \le 0$ but $\gamma_0(t) \in \Omega$ so this cannot happen

$\frac{1}{2} < s \leq 1$ is similar

<u>Remark</u>. If γ_0 is homotopic to γ_1 in Ω (either closed or w/ fixed end points) we write $\gamma_0 \underset{\Omega}{\sim} \gamma_1$

and simply write $\gamma_0 \sim \gamma_1$ if Ω is fixed and clear.

Then "$\sim$" is an equivalence relation

If $\gamma_0 \sim \gamma_1$ with $H(t, s)$ then

$\gamma_1 \sim \gamma_0$ with $\tilde{H}(t, s) = H(t, 1-s)$

If $\gamma_0 \sim \gamma_1$ with F, $\gamma_1 \sim \gamma_2$ with G

define
$$H(t,s) := \begin{cases} F(t, 2s) & 0 \leq s \leq \frac{1}{2} \\ G(t, 2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

then H gives a homotopy between γ_0 and γ_2.

The Homotopy thm

We can now state the homotopy thm.

Thm 5.1 (Chap 3) let $\Omega \subset \mathbb{C}$ open

γ_0, γ_1 two curves $\gamma_i : [a,b] \to \Omega$

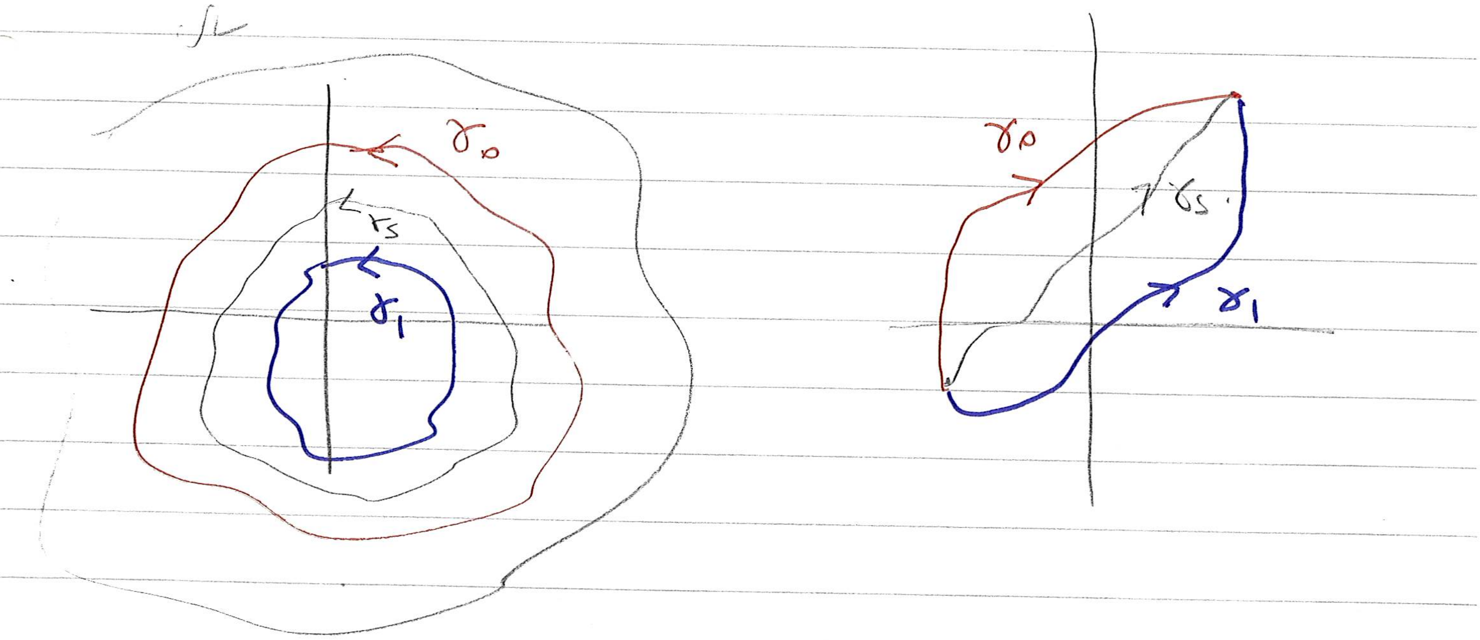
s.t (1) either γ_0, γ_1 are closed and homotopic

or

(2) γ_0, γ_1 have the same end points and are homotopic with fixed end points.

Then for $f \in \mathcal{H}(\Omega)$ we that

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz .$$

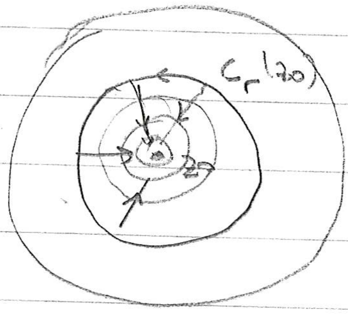


Example ① let $\Omega = D_R(z_0)$, $R>0$.
let

$\gamma_0 = C_r(z_0)$, $0<r<R$. Then γ_0 can be deformed into the point z_0 (by dilation which can be thought as the constant curve $\gamma_1(t) = z_0$ $\forall t \in [a,b]$.

So $$\int_{C_r(z_0)} f\,dz = \underbrace{\int_{C_0(z_0)} f\,dz}$$

$= 0$ since $\gamma_1'(t) = 0$ $\forall t$.



$H(t,s) : [0,2\pi]\times[0,1] \to D$
$(t,s) \to (1-s)e^{it} + s z_0$

In fact in D any closed curve γ is homotopic to a constant curve. Hence $\int_\gamma f = 0$.

② let $\Omega = \mathbb{C} - \{0\}$

$\gamma_0(t) : [0,\pi] \to \Omega$, $t \to e^{it}$ $\gamma_1 : [0,\pi] \to \Omega$, $t \to e^{-it}$.

They are <u>NOT</u> homotopic (with fixed end points since if they <u>were</u> then we would get that for $\frac{1}{z} \in \mathcal{H}(\Omega)$

$$\int_{\gamma_0} \frac{1}{z}\,dz = \int_{\gamma_1} \frac{1}{z}\,dz$$

$$\Rightarrow \int_{\gamma_0} f\,dz - \int_{\gamma_1} f\,dz = 0$$

$$\int_{C_1(0)} f(z)\,dz \quad . \quad \text{But} \quad \int_{C_1(0)} f(z)\,dz = 2\pi i \neq$$

We now look at the proof of the homotopy thm. We'll look at the case of closed curves. (The book does fixed end points)

Proof of Homotopy thm: (1) Simpler version:

If we also assume that the homotopy $H(t,s)$ also has continuous 2nd partial derivatives and hence

$$\frac{\partial^2 H}{\partial t\,\partial s} = \frac{\partial^2 H}{\partial s\,\partial t} \quad \forall (t,s) \in [a,b]\times[0,1]$$

then we can give a simpler proof.

For this first recall from real analysis

$h : [a,b]\times[0,1] \to \mathbb{R}$ be a function

Suppose $\frac{\partial h}{\partial s}$ exists and continuous in $[a,b]\times[0,1]$. If we define

$$G : [0,1] \longrightarrow \mathbb{R}$$
$$s \longrightarrow G(s) := \int_a^b h(t,s)\,dt$$

Then G is differentiable and

$$G'(s) = \int_a^b \frac{\partial h}{\partial s}(t,s)\,dt$$

Applying this to real and imaginary parts we define

$$I(s) := \int_a^b \underbrace{f(H(t,s)).\frac{\partial H}{\partial t}(t,s)}_{h(t,s)}\,dt$$

$$= \int_a^b f(\gamma_s)\,\gamma_s'(t)\,dt = \int_{\gamma_s} f(z)\,dz$$

Note $I(0) = \int_{\gamma_0} f(z)\,dz$ $\quad I(1) = \int_{\gamma_1} f(z)\,dz$

We want to show $I(s)$ is constant.

$$I'(s) = \int_a^b \frac{\partial}{\partial s}\left(f(H(t,s))\frac{\partial H}{\partial t}(t,s)\right)dt$$

$$= \int_a^b \left(\frac{\partial}{\partial s}\left[(f \circ H)\frac{\partial H}{\partial t}\right]\right)dt$$

$$I'(s) = \int_a^b \Big[f'(H(t,s)) \frac{\partial H}{\partial s}(t,s) \frac{\partial H}{\partial t}(t,s) +$$

$$f(H(t,s)) \frac{\partial^2 H}{\partial s \, \partial t}(t,s) \Big] \, dt$$

But note what is inside the parenthesis [...] is also equal to

$$\frac{\partial}{\partial t} \left[f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \right]$$

Hence $I'(s) = \displaystyle\int_a^b \frac{\partial}{\partial t} \left[f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \right] dt$

$$= f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \Big|_{t=a}^{t=b}$$

$$= f(H(b,s) \frac{\partial H}{\partial s}(b,s) -$$

$$f(H(a,s)) \frac{\partial H}{\partial s}(a,s) = 0.$$

Since $H(t,s)$ is homotopy of closed curves $\gamma_s(a) = H(a,s) = H(b,s) = \gamma_s(b)$ for every $s \in [0,1]$, we also have that

$$\frac{\partial H(a,s)}{\partial s} = \frac{\partial H}{\partial s}(b,s) \quad \forall s$$