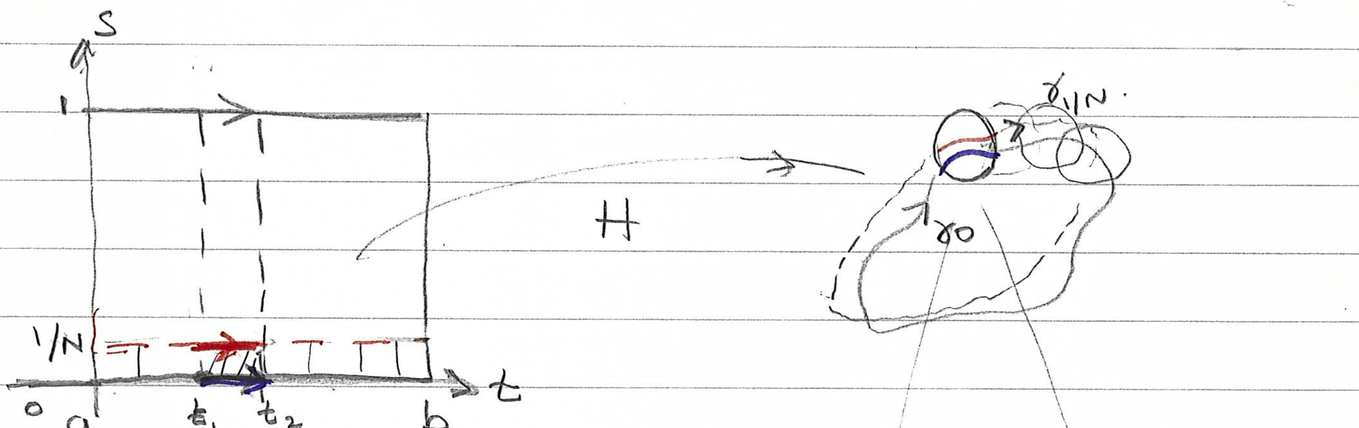
Hence $I'(s) = 0 \quad \forall s \in [0,1]$
and $I(s)$ is constant, in particular

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$$

$\boxtimes$

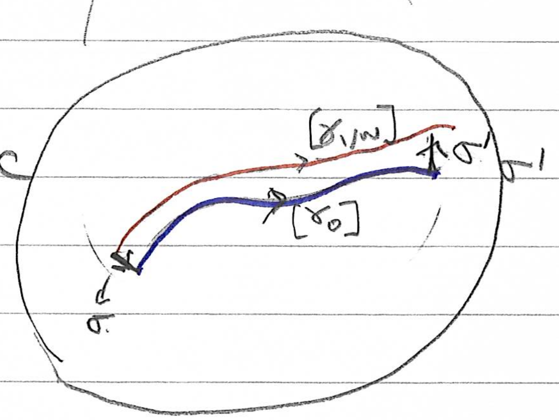
For the general proof the idea is the following: if we make a small deformation of one of the curves $\gamma_s(t)$, say $\gamma_0(t)$ to $\gamma_{1/N}(t)$ so that if we look at a small piece around a point of $\gamma_0(t)$ say $t_1 < t < t_2$ then we can show these are contained in a small disc in Ω.



$$H(t,0) = \gamma_0(t)$$

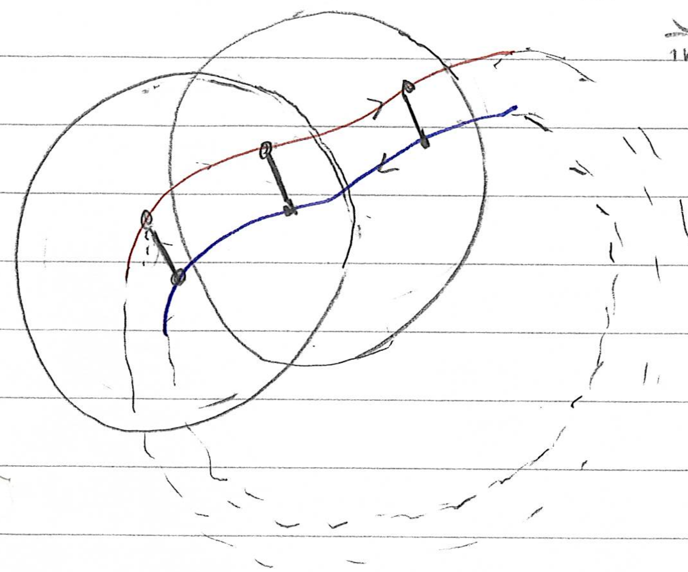
$$H(t,1) = \gamma_1(t)$$

We can apply Cauchy's thm in a disc to get



$$\int_{[\gamma_0]} f(z)\,dz = \int_{[\gamma_{1/N}]+\sigma+\sigma'} f(z)\,dz$$

Now we move over the whole curve $\gamma_0, \gamma_{1/N}$ using small discs contained in Ω to get



$$\int_{\gamma_0} f\,dz = \int_{\gamma_{1/N}} f\,dz.$$

Now we make this idea more precise:

For this we use 2 facts

1. If $K = \text{Image } H = H([a,b]\times[0,1])$ then K is compact (Sine H is continous and $[a,b]\times[0,1]$ is compact)

This gives the following

Lemma $\exists\ \varepsilon > 0$ s.t $\forall z \in K$ the disc $D_\varepsilon(z)$ is contained in Ω.

Proof: Assume on the contrary no such ε exists. Then $\forall n \geq 1$, $\exists\ z_n \in K$ s.t $D_{1/n}(z_n)$ is not contained in Ω

i.e $\exists\ w_n \in \mathbb{C} - \Omega$ such that $|z_n - w_n| \leq \frac{1}{n}$

(z_n) is a sequence in K, K is compact

Hence $\exists$ a conv. subseq. $(z_{n_k})_{k\geq 1}$ s.t

$\lim z_{n_k} = z$. Since K is closed, $z \in K$.

But now, since $|w_n - z_n| \leq \frac{1}{n}$

$|w_{n_k} - z_{n_k}| \leq \frac{1}{n_k}$ Hence $w_{n_k} \to z$ as well

$w_{n_k} \in \mathbb{C} - \Omega$ which is also closed since Ω is open.

Hence $z \in \mathbb{C} - \Omega$. But this is a contradiction since $z \in K \subset \Omega$.

$\boxtimes$

(2) This lemma, together with the fact that H is uniformly continuous on the compact set $[a,b] \times [0,1]$ will now allow us to find the small discs that are contained in Ω, that we need to apply Cauchy's thm on the disc.

This is because we can divide the rectangle $[a,b] \times [0,1]$ into small rectangles s.t the image of these small rectangles are contained in small discs of radius ε.

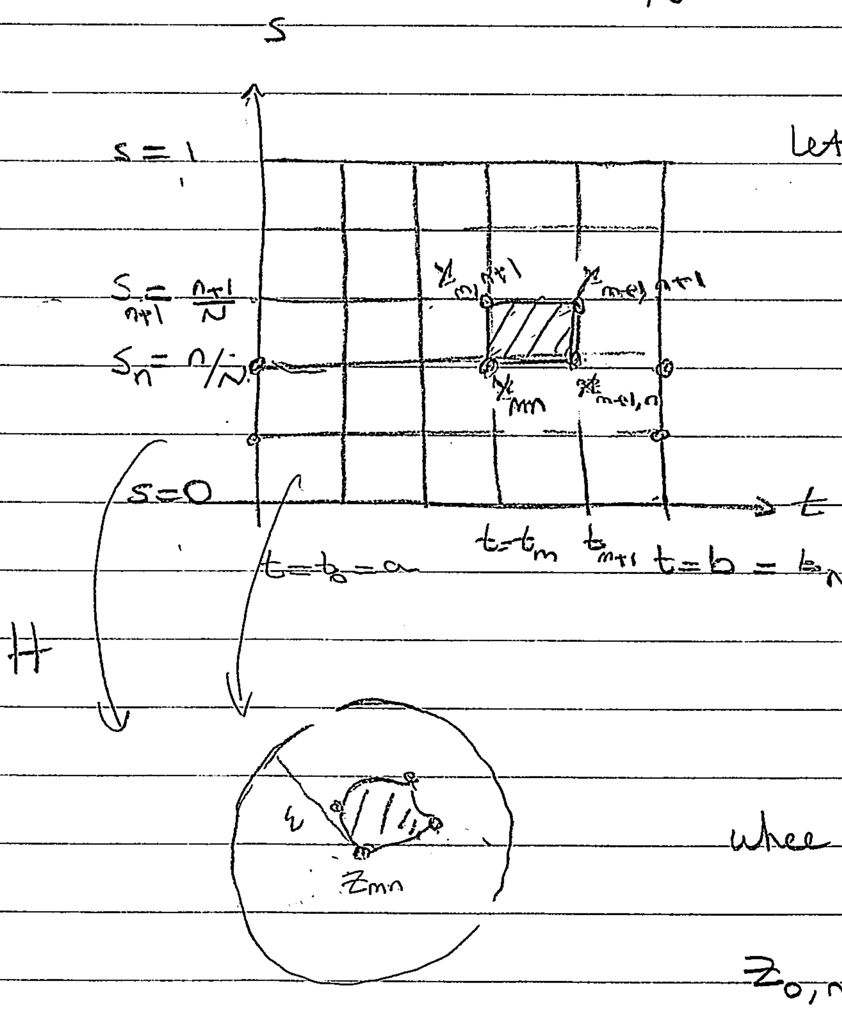
More precisely, since H is unif cont. on $[a,b]\times[0,1]$, $\exists N>0$ s.t

$$\left| H(t,s) - \underbrace{H(t_m, s_n)}_{:= z_{mn}} \right| < \varepsilon \qquad (*)$$

whenever $\left|(t,s) - (t_m, s_n)\right| < \dfrac{2}{N}$

where $t_m = a + \dfrac{b-a}{N}\cdot m \qquad 0 \le m \le N$

$s_n = \dfrac{n}{N} \qquad , \qquad 0 \le n \le N$



Let $Q_{mn} = [t_m, t_{m+1}] \times [s_n, s_{n+1}]$

Since the diameter of Q_{mn} is $\dfrac{\sqrt{2}}{N}$.

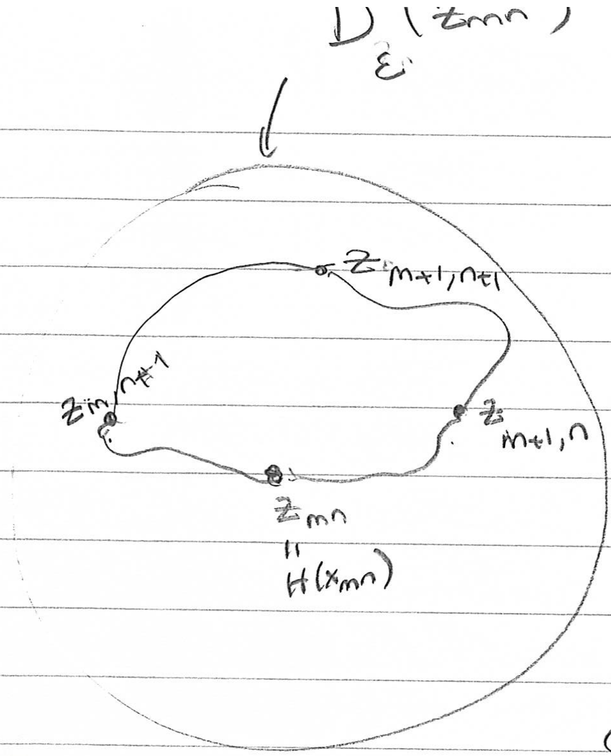
it follows from $(*)$ that

$$H(Q_{mn}) \subset D_\varepsilon(z_{mn}) \subset \Omega$$

where $z_{mn} := H(\underbrace{t_m, s_n}_{:= \gamma_{mn}})$

$z_{0,n} = H[a, n/N]$, i.e. $z_{0,n} = z_{N,n}$

$z \quad H[b, n/N]$



We use now induction on n, $0 \le n \le N$ to show that

$$\int_{\gamma_{n/N}} f(z)\,dz = \int_{\gamma_0} f(z)\,dz$$

Clearly true for $n = 0$.

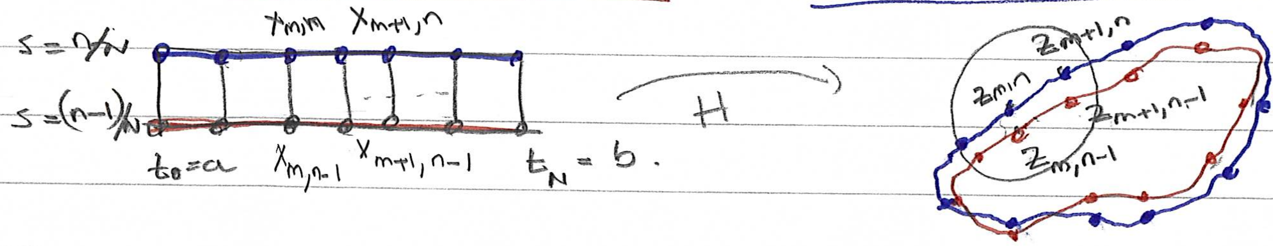
Assume $n \ge 1$ and it holds that

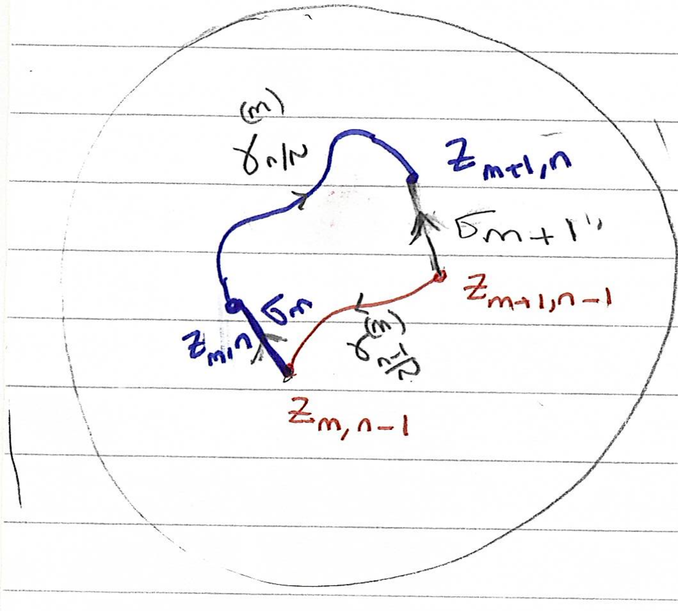
$$\int_{\gamma_{\frac{n-1}{N}}} f(z)\,dz = \int_{\gamma_0} f(z)\,dz$$

It is then enough to show that

$$\int_{\gamma_{\frac{n-1}{N}}} f(z)\,dz = \int_{\gamma_{n/N}} f(z)\,dz$$

$$\boxed{\gamma_{\frac{n-1}{N}} = H\left(t, \frac{n-1}{N}\right)} \qquad \boxed{\gamma_{n/N} = H\left(t, \frac{n}{N}\right)}$$





let $\boxed{\gamma^{(m)}_{\frac{n-1}{N}} = \gamma_{\frac{n-1}{N}}\Big|_{[t_m,\, t_{m+1}]}}$

and $\boxed{\gamma^{(m)}_{\frac{n}{N}} = \gamma_{\frac{n}{N}}\Big|_{[t_m,\, t_{m+1}]}}$

Let σ_m = line segment between $z_{m,n-1}$ and $z_{m,n}$

σ_{m+1} = line seg. between $z_{m+1,n-1}$, $z_{m+1,n}$

$\sigma_0 = [z_{0,n-1}, z_{0,n}]$

$\sigma_N = [z_{N,n-1}, z_{N,n}]$

Cauchy's thm on the disc $D_\varepsilon(z_{m,n-1})$ gives

$$\int_{\gamma^{(m)}_{\frac{n}{N}}} f - \int_{\sigma_{m+1}} f - \int_{\gamma^{(m)}_{\frac{n-1}{N}}} f + \int_{\sigma_m} f = 0.$$

Summing over m,

$$\int_{\gamma_{\frac{n-1}{N}}} f = \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{\frac{n-1}{N}}} f\,dz = \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{\frac{n}{N}}} f\,dz$$

$$+ \sum_{m=0}^{N-1} \left(\int_{\sigma_m} f - \int_{\sigma_{m+1}} f \right)$$

$$= \int_{\gamma_{\frac{n}{N}}} f + \int_{\sigma_0} f - \int_{\sigma_N} f = \int_{\gamma_{\frac{n}{N}}} f \, dz$$

$$H(a, \tfrac{n-1}{N}) = H(b, (n-1)/N)$$

Now $\sigma_0 = \sigma_N$ since $\gamma_{\frac{n-1}{N}}(a) = \gamma_{\frac{n-1}{N}}(b)$

and similarly $\gamma_{\frac{n}{N}}(a) = \gamma_{\frac{n}{N}}(b)$

ie the curves $\gamma_s(t)$ are closed.

∎

We've seen in $\mathbb{C}$, $\mathbb{C} - (-\infty, 0]$ or any convex set any 2 closed curves are homotopic or any 2 curves with the same end points are homotopic. This motivates the

Definition An open set $\Omega \subset \mathbb{C}$ is called simply connected if it is connected and if any 2 curves with the same endpoints are homotopic.

$\mathbb{C}$, $\mathbb{C} - (-\infty, 0]$, $D_r(z_0)$ are simply connected

$\mathbb{C} - \{0\}$ is not simply connected