III. Meromorphic functions and Residue Formula

Goal: To extend Cauchy's theorem and C.I.F from holomorphic functions to functions which might have singularities.

Recall: Cauchy's thm, $\int_\gamma f\,dz = 0$

for any γ closed curve, $f \in \mathcal{H}(\Omega)$
γ and its interior is contained in Ω.

CIF: $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w - z}$

$\forall\, z \in D$, a disc, $\partial D = C$ and f is holom in D.

To this end we'll first look at isolated singularities of a function f.

We'll see there are 3 prototypes: $\frac{\sin z}{z}$, $\frac{1}{z}$, $e^{1/z}$

$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$ shows that $z=0$ is a "removable" singularity

One can view the RHS as an analy cont. to all $\mathbb{C}$ of LHS $= \frac{\sin z}{z}$

$\left|\frac{1}{z}\right| \to \infty$ as $z \to 0$

where as $|e^{1/z}|$ oscillates. For example
if $z \to 0$ on positive real numbers
then $|e^{1/x}| \to \infty$ as $x \to 0$, $(x>0)$
if $z \to 0$ on negative real numbers
then $|e^{1/x}| \to 0$ as $x \to 0$ $(x<0)$
$\lim_{z\to 0} |e^{1/z}|$ fails to exist either in strict sense or as an ∞ limit
These 3 examples of singularities
are what we call removable, a pole
and an essential singularity resp.

We'll prove a generalization of Cauchy's
thm to functions that are holom
except for finitely many isolated points
This will lead us to the

<u>Residue formula:</u> If f is holom in an open set Ω
containing a circle C and its interior
except for finitely many points $z_1, \dots, z_N$
inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \underbrace{\mathrm{Res}_{z_k} f}_{a_{-1}}$$

where we'll also see that f in a nbhd of z_0
has the form

$$f(z) = \dots + \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0} + G(z),$$ where $G \in \mathcal{H}(D_r(z_0))$

This thm, like Cauchy's thm can be used to evaluate many real integrals and complex line integrals.

It will also lead to many theoretical results just like Cauchy's thm did

- Argument principle = which allows us to count the number of zeroes (and poles) of holomorphic (meromorphic) functions inside closed curves.

Rouché's thm. A holom. function can be perturbed slightly without changing the number of its zeros.
f, g holom in an open set containg a circle C and its interior. If $|f(z)| > |g(z)|$ $\forall\ z \in C$ then f and $f+g$ have the same # of zeroes inside C.

Open mapping thm: f is holom, non-constant in an open connected region Ω then f is an open map i.e. image of an open set is open

Maximum modulus principle If f is non-constant

on Ω open connected, compact closure $\overline{\Omega}$
if f is continuous on $\overline{\Omega}$ then

$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\overline{\Omega}-\Omega}|f(z)|$$

Another way to say this

$$\underbrace{\max_{z\in\overline{\Omega}}}|f(z)| = \max_{z\in\overline{\Omega}-\Omega}|f(z)|$$

exists because
$\overline{\Omega}$ is compact

We start with definitions of singularities.

Defn Let $z_0 \in \mathbb{C}$. z_0 is called a (possible) isolated singularity of f if $\exists\ r>0$ such that f is holomorphic in the punctured disc $D_r(z_0)\setminus\{z_0\} =: D_r^*(z_0)$)

(or if $\exists$ U open, $z_0 \in U \subset \Omega$ s.t $f \in \mathcal{H}(U-\{z_0\})$

eg. $f = \mathbb{C}\setminus\{0\} \to \mathbb{C}$
$z \to z$

$z=0$ is an isolated singularity of f because f is not defined there, but f can in fact be extended to all $\mathbb{C}$ by defining $f(0)=0$. In this case

$z=0$ is a "removable singularity"

On the other hand $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$
$z \longmapsto 1/z$

has a singularity at $z=0$, which cannot be removed.

Defn An isolated singularity z_0 of a function $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ is called removable if f is holomorphically extendable to all Ω. ie $\exists\ F : \Omega \to \mathbb{C}$ holom. s.t $F(z) = f(z)$ $\forall\ z \in \Omega \setminus \{z_0\}$.

We have the following thm of Riemann sometimes called Riemann continuation thm.

Thm Let $z_0 \in \mathbb{C}$. Then the following assertions for a function $f \in \mathcal{H}(\Omega\setminus\{z_0\})$ are equivalent (Ω open, $\neq \emptyset$)

(i) f is holomorphically extendable to Ω

(ii) f is continuously extendable to Ω

(iii) f is bounded in a nbhd of z_0
i.e. $\exists\ r>0$ s.t f is bounded in $D_r^*(z_0)$

(iv) $\lim\limits_{z \to z_0} (z-z_0) f(z) = 0$.

Proof Exercise 5.5 There $\Omega = \mathbb{C}$, $z_0 = 0$ but the proof is verbatim the same. Apply it to $\tilde{f}(z) = f(z) - z_0$.

As a consequence of Riemann's continuation thm we have

Riemann's thm on removable singularities

Thm 3.1 Suppose f is holom in an open set Ω except possibly at a pt $z_0 \in \Omega$.
If f is bounded in $D_r(z_0) \setminus \{z_0\}$ for some $D_r(z_0) \subset \Omega$ then z_0 is a removable singularity of f.
ie $\exists\, F : \Omega \to \mathbb{C}$ holom on Ω s.t $F(z) = f(z)$, $\forall z \in \Omega \setminus \{z_0\}$.

Proof Note this is just (iii) $\Rightarrow$ (i) of the Riemann's continuation thm which shows (iii) $\Leftrightarrow$ (i).

Cor Let f be a function with an isolated singularity at z_0. Then the singularity is removable $\Leftrightarrow$ f is bdd in a punctured disc around z_0 $\Leftrightarrow$ $\lim_{z \to z_0} |f(z)|$ exists.

Rmk The condition $\lim_{z \to z_0} f(z)$ exists could be substituted for the existence of $\lim_{z \to z_0} |f(z)|$ in the above corollary.
The point is that the nature of the singularity is already apparent in the behaviour of the magnitude of f, a real quantity, as z approaches z_0.

Example ① let $f(z) = \frac{\sin z}{z}$, $z \neq 0$

Then f has a removable singularity at $z=0$

We can see this either using Riemann construction thm part (iv)

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \sin z = 0$$

or $$\lim \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1$$

extension of L'hospital's to $\mathbb{C}$.

Note this imply $f(z) = \frac{\sin z}{z}$ is bdd in a nbhd of 0.

[$\lim \frac{\sin z}{z} = 1 \Rightarrow$ For $\varepsilon = 1/2$ $\exists$ $r>0$ s.t $\forall$ $|z|<r$, $z \neq 0$,

$$\left| \frac{\sin z}{z} - 1 \right| < 1/2$$

$\Rightarrow \frac{\sin z}{z}$ is bdd in $D_r^*(0)$

or Using $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ $\forall z \in \mathbb{C}$ we get

$F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$ $\forall z$, which gives a holom ext of $\sin z / z$ to all $\mathbb{C}$.

② $f(z) = \frac{z}{e^z - 1}$ has removable sing. at $z=0$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{e^z - 1} = \lim_{z \to 0} \frac{1}{e^z} = 1$$

If f does not have a removable singularity at z_0 and $f \in \mathcal{H}(\Omega \setminus \{z_0\})$

then f is not bounded near z_0.

We can ask whether its unboundedness is similar to the unboundedness of

$\frac{1}{(z-z_0)^n}$. ie we can ask whether

$(z-z_0)^n f(z)$ is bounded near z_0 for n sufficiently large.

If such an $n \in \mathbb{N}$ exists, then

z_0 is called <u>a pole of f</u> and

the natural number $m := \min\{n \in \mathbb{N} \mid (z-z_0)^n f(z)$ is bdd near $z_0\}$
≥ 1

is called the <u>order of the pole of f at z_0</u>

Poles of first order are called <u>simple poles</u>

<u>eg</u>. $f(z) = (z-z_0)^{-m}$ has a pole of order m at $z = z_0$.

We will see soon that poles arise from reciprocals of holomorphic functions with zeroes

Before we make this more precise let's recall that zeroes of holom. functions are isolated and we have the following theorem for their behaviour near a zero

Thm (1.1 III) Suppose f is holomorphic in a connected open set Ω, and has a zero at a point $z_0 \in \Omega$. And f does not vanish identically on Ω. Then $\exists\ r > 0$ s.t $D_r(z_0) \subset \Omega$ and a unique non-vanishing holom. func $g \in \mathcal{H}(D_r(z_0))$ and a unique positive integer n (s.t

$$f(z) = (z - z_0)^n g(z) \quad \forall z \in D_r(z_0)$$

$$n = \min\{n \mid f^{(n)}(z_0) \neq 0\}$$

The analogous thm for the poles is the following.

Thm (1.2)' For $m \in \mathbb{N}$, $m \geq 1$ the following statements about $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ are equivalent

(I) f has a pole of order m at z_0 (i.e $(z-z_0)^m f(z)$ is bounded near z_0 and m is the smallest such integer.

(II) $\exists\, r>0$, $g \in \mathcal{H}(D_r(z_0))$ s.t $g(z_0) \neq 0$.

and $f(z) = (z-z_0)^{-m} g(z) \quad \forall z \in D_r^*(z_0)$

(III) $\exists\, r>0$ s.t $D_r(z_0) \subset \Omega$ and $h \in \mathcal{H}(D_r(z_0))$ s.t $h(z) \neq 0$ $\forall z \in D_r^*(z_0)$, h has a zero of order m at z_0 and such that

$$f(z) = \frac{1}{h(z)} \quad \forall z \in D_r^*(z_0)$$

Proof (I) $\Rightarrow$ (II) f has a pole of order m at z_0 means that $(z-z_0)^m f(z)$ is bounded near z_0, and m is minimal. The thm of Riemann on removable singularity says that $\exists\, g \in \mathcal{H}(D_r(z_0))$ s.t $g(z) = (z-z_0)^m f(z)$, whenever $z \neq z_0$

If $g(z_0)$ were zero then it would imply ~~by~~ the previous thm

that $g(z) = (z - z_0)\, h(z)$ where h is holomorphic in $D_r(z_0)$

consequently this would give that

$h(z) = (z - z_0)^{m-1} f(z)$ is bounded near z_0. This would contradict the minimality of m. Hence $g(z_0) \neq 0$

and we get $f(z) = (z - z_0)^{-m} g(z)$

for $z \in D_r^*(z_0)$, and $g(z_0) \neq 0$

(ii) $\Rightarrow$ (iii) Suppose $\exists\, g \in \mathcal{H}(D_s(z_0))$ s.t $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^{-m} g(z) \quad \forall z \in D \setminus \{z_0\}$$

Since $g(z_0) \neq 0$, g cont, $\exists\, r > 0$ s.t $g(z) \neq 0 \quad \forall z \in D_r(z_0)$

Then let $h := \dfrac{(z - z_0)^m}{g(z)} \quad \forall z \in D_r(z_0)$

Then $h(z) \neq 0 \quad \forall z \in D_r^*(z_0)$ and

$h \in \mathcal{H}(D_r(z_0))$

and $\dfrac{1}{h(z)} = g(z)(z-z_0)^{-m} = f(z)$

$\forall\, z \in D_r^*(z_0)$

Note h has a zero of order m
since $h(z) = (z-z_0)^m (1/g)$
and $\dfrac{1}{g(z)} \neq 0 \quad \forall\, z \in D_r(z_0)$.

(iii) $\Rightarrow$ (i) Suppose $\exists\, r>0$ s.t

$D_r(z_0) \subset \Omega$ and $h \in \mathcal{H}(D_r(z_0))$
s.t $h(z) \neq 0 \quad \forall\, z \in D_r^*(z_0)$

$h(z)$ has a zero of order m at z_0
and
$f(z) = \dfrac{1}{h(z)} \quad \forall\, z \in D_r^*(z_0)$.

Since h has a zero of order m
at z_0, $\exists\ g_1 \in \mathcal{H}(D_r(z_0))$ st

$h(z) = (z-z_0)^m g(z)$ and $\exists\, s>0$

s.t $g(z) \neq 0 \quad \forall\, z \in D_s(z_0) \subset D_r(z_0)$

Since g is holom and non vanishing
$1/g$ is holom in $D_s(z_0)$

But then

$$f(z) = \frac{1}{h(z)} = (z - z_0)^{-m} \frac{1}{g(z)} \quad \forall\, z \in D_\delta^*(z_0)$$

would imply that $(z - z_0)^m f(z) = \frac{1}{g(z)}$
is holom on $D_\delta^*(z_0)$ and has
the holom extension $\frac{1}{g(z)}$ in $D_\delta(z_0)$

($1/g$ is holom on $D_\delta(z_0)$ since $g \neq 0$ on $D_\delta(z_0)$)

By Riemann's extendability thm

$(z - z_0)^m f(z)$ is bounded in
a nbhd of z_0.

Moreover $(z - z_0)^{m-1} f(z) = \left(\frac{1}{g(z)}\right)\left(\frac{1}{z - z_0}\right)$
is not bounded since
$\frac{1}{g(z_0)} \neq 0$ and $\frac{1}{z - z_0} \to \infty$ as $z \to z_0$.

Hence m is minimal and f has
a pole of order m at z_0

$\blacksquare$

<u>Example</u> ① $f(z) = \dfrac{1}{e^z - 1}$ has a pole of order 1 at $z = 0$

since $\dfrac{1}{f} = e^z - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$= z + \dfrac{z^2}{2!} + \dots$

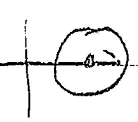
$= z\left(1 + \dfrac{z^2}{2!} + \dots\right)$

has a zero of order 1 at $z = 0$.

(Note $\dfrac{1}{e^z-1}$ has simple poles at $z = 2\pi i n$, $n \in \mathbb{Z}$)

② $f(z) = \dfrac{z}{z^2 - 1}$ has poles of order 1 at $z = \pm 1$

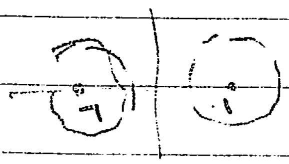
Since $f(z) = (z-1)^{-1}\left(\dfrac{z}{z+1}\right)$



and $h(z) = \dfrac{z}{z+1}$ is holom and non-vanishing $\forall z \in D_{1/2}(1)$

Similarly $f(z) = (z+1)^{-1}\left(\dfrac{z}{z-1}\right)$

and $\bar{h}(z) = \dfrac{z}{z-1}$ is holom and non vanishing in $D_{1/2}(-1)$



The next theorem is the analog of the power series expansion of a holomorphic function.

Recall if $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$ s.t $D_r(z_0) \subset \Omega$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0) \quad \forall z \in D_r(z_0)$$

For functions with poles we have

Thm (1.3 III) If f has a pole of order n at z_0 then

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

where G is holomorphic in a nbhd of z_0.

Proof: f has a pole of order n at z_0

$\Rightarrow$ we can write $f(z) = (z - z_0)^{-n} g(z)$

$\forall z \in D_r^*(z_0)$ and $g \in \mathcal{H}(D_r(z_0))$ and $g(z_0) \neq 0$

We expand $g(z)$ in a power series

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k \quad , z \in D_r(z_0)$$

Then for $z \in D_r^*(z_0)$

$$f(z) = \frac{1}{(z-z_0)^n} \Big[\overbrace{g(z_0)}^{\neq 0} + \frac{g'(z_0)}{1}(z-z_0) + \cdots$$
$$+ \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots \Big]$$

$$= \frac{g(z_0)}{(z-z_0)^n} + \frac{g'(z_0)}{(z-z_0)^{n-1}} + \cdots + \frac{g^{(n-1)}(z_0)/(n-1)!}{(z-z_0)}$$
$$+ \underbrace{\sum_{k=n}^{\infty} \frac{g^{(k)}(z_0)(z-z_0)^{k-n}}{k!}}_{:= G(z)}$$

$$= \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

☒

Remark: $f(z) = \sum_{k=-n}^{\infty} a_k (z-a)^k$ is a special case of a Laurent series

Defn The number a_{-1} (ie coef. of $(z-z_0)^{-1}$) is called the residue of f at the pole z_0, denoted by $\boxed{\text{res}_{z_0} f = a_{-1}}$

The function $\sum_{j=1}^{n} \frac{a_{-j}}{(z-z_0)^j}$ is called The principal part of f at the pole z_0