As a corollary of Homotopy thm we have that

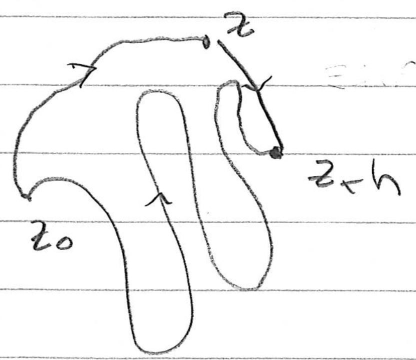
Thm 5.2. Any holomorphic function on a simply connected domain Ω has a primitive. In particular $\int f dz = 0$ for every closed curve $\gamma \in \Omega$. Any 2 primitives differ by a constant

Proof. Fix $z_0 \in \Omega$. For $z \in \Omega$, Ω connected

Define $$F(z) = \int_\gamma f(w)\,dw$$

where γ is any curve connecting z_0 to z. The definition is well defined since Ω is simply connected. Hence if $\tilde{\gamma}$ is another curve connecting z_0 to z, then $\gamma \sim \tilde{\gamma}$ and by Homotopy thm

$$\int_\gamma f(w)\,dw = \int_{\tilde{\gamma}} f(w)\,dw.$$



Choose h small so that the line segment joining z to $z+h$ is in Ω

Then $F(z+h) - F(z) = \int_z^{z+h} f(w)\,dw$.

Arguing as in the proof of Thm 2.1, II or using continuity of f as below we get $\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z)$

which shows F is a primitive of f in Ω ∎.

$$F(z+h) - F(z) = \int_{[z,z+h]} \left(f(w) - f(z) + f(z)\right) dw$$

$$= f(z) \underbrace{\int_{[z,z+h]} dw}_{h.} + \int_{[z,z+h]} f(w) - f(z)\,dw$$

$$\left| \int_{[z,z+h]} f(w) - f(z)\,dw \right| \le \left(\sup_{w\in[z,z+h]} |f(w) - f(z)| \right) h$$

Hence $\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \le \sup_{w\in[z,z+h]} |f(w) - f(z)|$

But f is continuous. Hence $\sup_{w\in[z,z+h]} |f(w) - f(z)| \to 0$ as $h \to 0$

The complex logarithm

For a non-zero complex number $z \in \mathbb{C}$ we want to define a "logarithm", ie a complex number w s.t $z = e^w$.
We want the "logarithm" to be the inverse of the exponential.

ie $w = \log z$ if $e^w = z$
if $z = re^{i\theta}$ then
We can set $\log z = \log r + i\theta$

where $\log r$ is the usual logarithm $\log : \mathbb{R}^+ \longrightarrow \mathbb{R}$ of the positive real number r.
The problem is that this is not single valued since θ is only unique up to an integer multiple of $2\pi i$. For $z=1$ $e^0 = 1$ but also for $w = 2\pi i k$, $e^{2\pi i k} = 1$
We want a holomorphic function $\ell : \Omega \to \mathbb{C}$ which satisfy $\exp \circ \ell = \mathrm{id}$ throughout its domain of definition.

Defn: Let $\Omega \subset \mathbb{C}$ be open. A branch of the logarithm, $\log_\Omega$, on Ω is a holomorphic function s.t $\exp(\log_\Omega(z)) = z$ $\forall z \in \Omega$.

Remark ① Since $\exp z \neq 0 \ \forall z \in \mathbb{C}$ such $\log_{\Omega}$ function can exist only if $0 \notin \Omega$.

② If $\Omega = \mathbb{C} - \{0\}$. Even though

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\} \quad \text{is surjective}$$

there is no holomorphic choice of logarithm in $\mathbb{C} - \{0\}$. Indeed if there were $f \in \mathcal{H}(\mathbb{C} - \{0\})$ s.t

$$\exp(f(z)) = z \quad \forall z \in \mathbb{C} - \{0\}$$

then differentiating both sides would give

$$f'(z)\exp(f(z)) = 1 \quad \forall z \in \Omega. \quad \text{Hence}$$

$$f'(z) = \frac{1}{z} \quad \forall z \in \mathbb{C} - \{0\}. \text{ ie } \frac{1}{z} \text{ has a primitive } f$$

but then we would get $\int_{\gamma} \frac{1}{z} \, dz = 0$

for $\gamma = C_1(0)$ which we know is $2\pi i$, not zero.

3' If Ω is open and connected and $l = \log_{\Omega} : \Omega \to \mathbb{C}$ is a logarithm on Ω. Then $\tilde{l} : \Omega \to \mathbb{C}$ is also a logarithm on Ω if and only if

$\tilde{l} = l + 2\pi i n$ for some $n \in \mathbb{Z}$.

Indeed if $\tilde{l}$ is a logarithm function then

$\exp(\tilde{l}(z)) = z$ and $\exp(l(z)) = z$

Hence $\exp(\tilde{l}(z) - l(z)) = 1 \quad \forall z \in \Omega$

$\tilde{l}(z) - l(z) \in 2\pi i \mathbb{Z} \quad \forall z \in \Omega.$

ie $\dfrac{\tilde{l} - l}{2\pi i}$ is a cont., integer valued function on Ω which is connected

Hence its image under $\dfrac{\tilde{l} - l}{2\pi i}$ is connected and a subset of $\mathbb{Z}$ hence it is a single point n.

Conversly if $\tilde{l} = l + 2\pi i n$ then

$\exp(\tilde{l}(z)) = \exp(l(z)).\exp(2\pi i n) =$
$= \exp(l(z)) = z$

We have for a simply connected domain $\Omega \subset \mathbb{C} - \{0\}$ the following

Thm 6-1 Let $\Omega \subset \mathbb{C} - \{0\}$ be a simply connected set. Then there exists a branch of the logarithm on Ω

ie a function $F : \Omega \to \mathbb{C}$ s.t

F is holom on Ω and $\exp(F(z)) = z$ $\forall z \in \Omega$.

Proof Since $0 \notin \Omega$, $\frac{1}{z} \in \mathcal{H}(\Omega)$ and since Ω is simply connected

It has a primitive on Ω.

let $f(z)$ be a primitive of $1/z$

let $G(z) := z \exp(-f(z))$

$$G'(z) = -\underbrace{f'(z)}_{\frac{1}{z}} z \exp(-f(z)) + \exp(-f(z))$$

$$= -\exp(-f(z)) + \exp(-f(z)) = 0$$

Ω connected hence $G(z) = \text{constant} = a \Rightarrow z e^{-f(z)}$

Since $\exp \neq 0, z \neq 0, a \neq 0$, and $\exists$ b s.t $a = \exp(b)$, and $\exp(f(z)) = \frac{z}{a}$

let $F(z) = f(z) + b$

then $\exp(F(z)) = \underbrace{\exp f(z)}_{\frac{z}{a}} \cdot \underbrace{\exp(b)}_{a} = z$

and $F(z)$ is a branch of the log on Ω.

Defn let $\Omega := \mathbb{C}^- = \mathbb{C} - (-\infty, 0]$
the principal branch of the logarithm is the unique $\log_\Omega \in \mathcal{H}(\Omega)$ s.t

$\log(1) = 0$. Sometimes $\log_{\mathbb{C}^-}$ is also denoted by Log

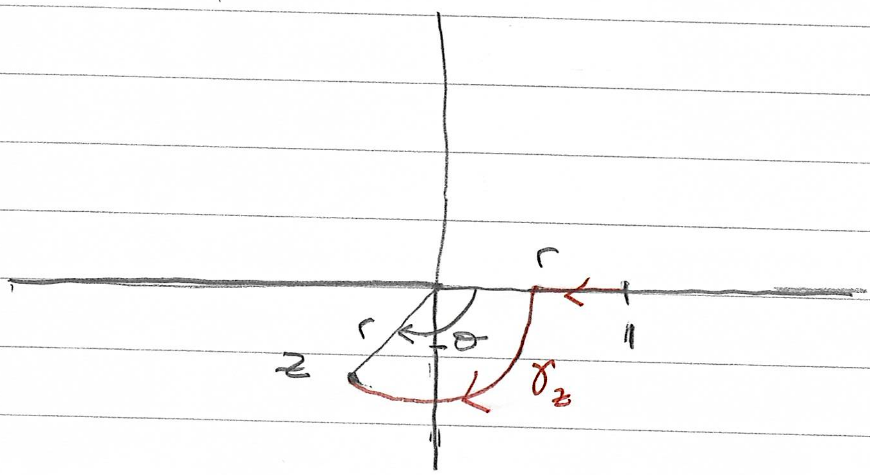
Proposition If $z = re^{i\theta} \in \mathbb{C}^-$ with $r > 0$
$-\pi < \theta < \pi$, then the principal branch of logarithm is given by the formula

$$\mathrm{Log}\, z := \log_{\mathbb{C}^-} z = \log r + i\theta.$$

Proof. let $\log z := \int_{\gamma_z} \frac{dw}{w}$

be a primitive of $\frac{1}{z}$ where we take the path γ_z which starts at 1 and ends at z.

Note $\int_{\gamma_1} \frac{dw}{w} = 0$. hence $\log 1 = 0$.



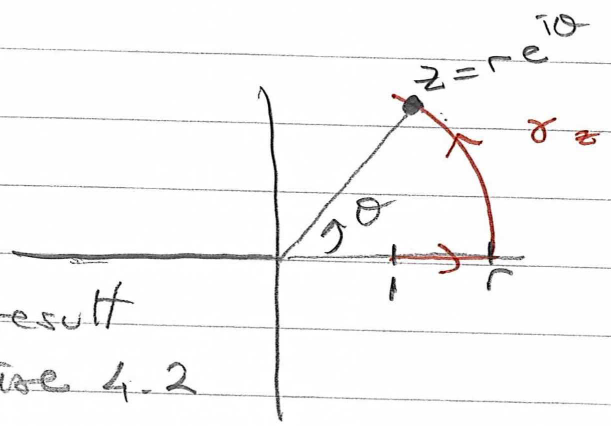
If $z = re^{i\theta}$ w/ $r < 1$ take the path

γ_z which goes on the real line from 1 to r then on the circular arc to z.

$$\log z = \underbrace{-\int_r^1 \frac{dx}{x}}_{\text{on the segment}} + \underbrace{\int_0^{-\theta} \frac{-i\cancel{r}\cancel{e^{-it}}}{\cancel{re^{-it}}}\,dt}_{\text{on the arc } z = re^{-it}} \qquad 0 < t < -\theta$$

$= \log r + i\theta$

If $r > 1$ take the path



Similar calculation gives the result (which was also on exercise 4.2 in the sheet 4)

∎

Remark (1) The identity $\log z + \log w = \log wz$ does not hold for all $z, w, wz \in \mathbb{C}^-$

If $w = re^{i\alpha}$, $z = se^{i\beta}$ $wz = rse^{i\theta}$

with $\theta, \alpha, \beta \in (-\pi, \pi)$

then $\exists$ $\gamma \in \{-2\pi, 0, 2\pi\}$ s.t.

$$\theta = \alpha + \beta + \gamma$$

Then $\log wz = \log rs + i\theta$

$$= \log r + \log s + i(\alpha + \beta + \gamma)$$

$$= \log r + i\alpha + \log s + i\beta + i\gamma$$

$$= \log w + \log z + i\gamma$$

In particular $\log wz = \log w + \log z$

$$\Leftrightarrow \gamma = 0 \Leftrightarrow \alpha + \beta \in (-\pi, \pi).$$

Since the condition is met whenever $\mathrm{Re}\, w > 0$ $\mathrm{Re}\, z > 0$ we have

$\log wz = \log z + \log w$ $\forall w, z \in \mathbb{C}^-$ with $\mathrm{Re}\, z > 0$, $\mathrm{Re}\, w > 0$.

Remark 2 For the principal branch of log one has the Taylor expansion

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n, \quad |z-1|<1$$

To see this note the derivative of LHS is $\frac{1}{z}$, and RHS's derivative is

$$\sum_{n=1}^{\infty} (-1)^{n-1} (z-1)^{n-1} = \sum_{n=0}^{\infty} (1-z)^n = \frac{1}{1-(1-z)} = \frac{1}{z}$$

for $|z-1|<1$

Hence RHS $= \log z$, LHS $= \sum \frac{(-1)^{n-1}}{n}(z-1)^n$ differ by a constant. Looking at $z=1$ gives both sides are equal to zero. Hence

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n \quad |z-1|<1.$$

Remark (3)

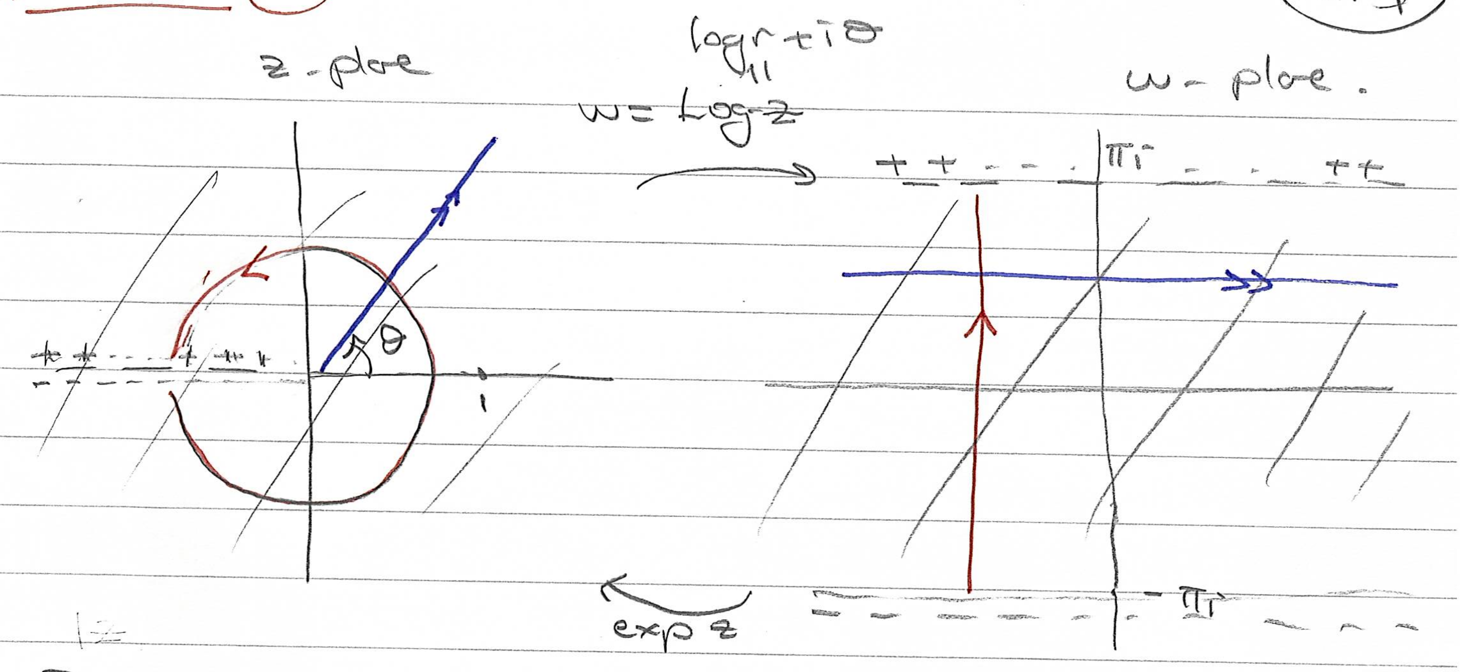


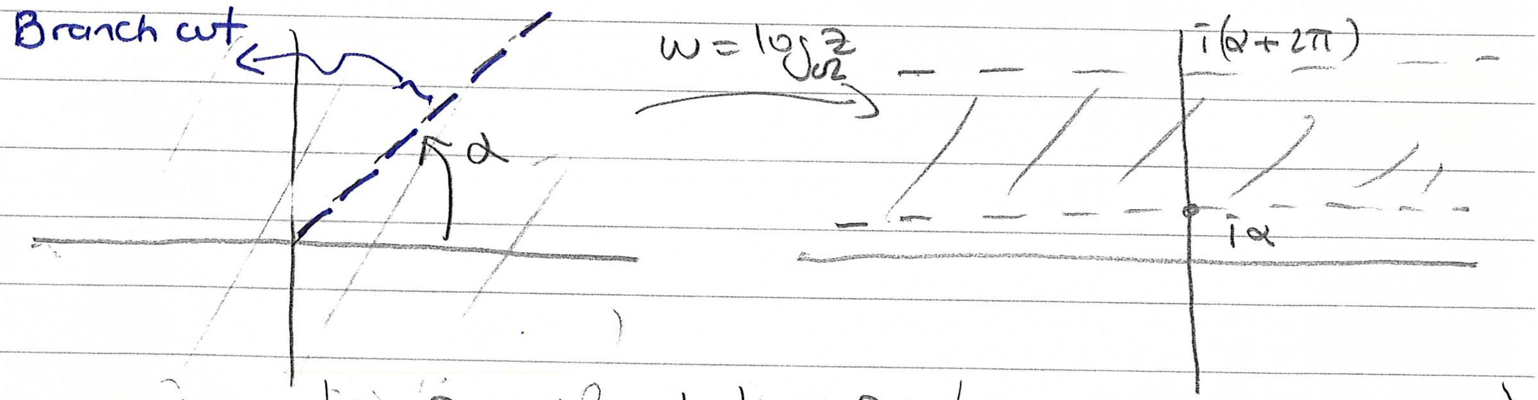
Image of a punctured circle
$\{|z| = r \mid -\pi < \arg z < \pi\}$ is the vertical interval $\{\operatorname{Re} w = \log|z|, -\pi < \operatorname{Im} w < \pi\}$

(if $r < 1$ then $\operatorname{Re} w < 0$
if $R > 1$ $\operatorname{Re} w > 0$)

Image of $\{z \mid \operatorname{Arg} z = \theta\}$, a ray from 0 to ∞ is the

horizontal line $\{w \mid \operatorname{Im} w = \theta\}$

(4) We can define a holom. 'branch' of logarithm for any $\Omega := \mathbb{C} \setminus (\{z \mid \operatorname{Arg} z = \alpha\} \cup \{0\})$



$w = \log z = \log|r| + i\theta, \ (r > 0, \ \alpha < \theta < \alpha + 2\pi)$

Remark 5 Let $\Omega \subset \mathbb{C}^*$ be simply connected and $\log_\Omega : \Omega \to \mathbb{C}$ a branch of logarithm

Let $\alpha \in \mathbb{C}$, $z \in \Omega$, we define

$$\boxed{z^\alpha := \exp(\alpha \log_\Omega z)}.$$

Note this definition depends on the choice of $\log_\Omega$. If we choose $\log_\Omega + 2\pi i k$ instead then

$$\exp(\alpha(\log_\Omega z + 2\pi i k)) = z^\alpha e^{2\pi i k \alpha}$$

If we choose the principal branch of log with $\log 1 = 0$, $\alpha = \frac{1}{m}$

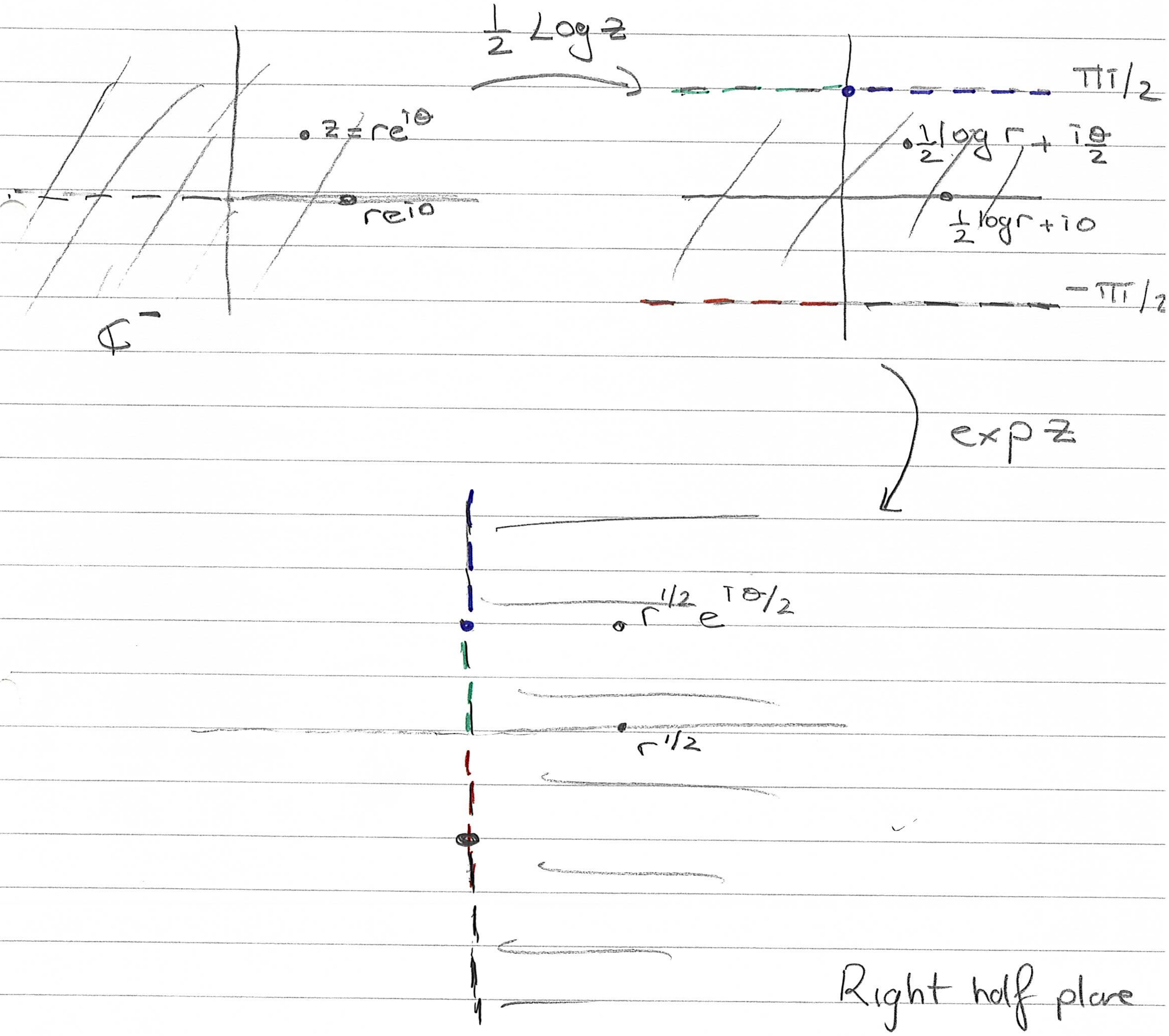
then $z^{1/m} = e^{\frac{1}{m}\log z}$ satisfy

$$(z^{1/m})^m = \exp\left(\frac{1}{m}\log z\right) \cdots \exp\left(\frac{1}{m}\log z\right)$$

$$= \exp\left(m \frac{1}{m} \log z\right) = \exp(\log z) = z.$$

Example: let $\operatorname{Log} z$ be the principal branch of log on $\mathbb{C}^-$

$$z^{1/2} = \exp\left(\tfrac{1}{2}\operatorname{Log} z\right)$$



Note for $z \in \mathbb{R}^+$, $z^{1/2}$ is the usual positive square root.

If we choose $\log_{\mathbb{C}} z = \log r + i(\theta + 2k\pi)$

then

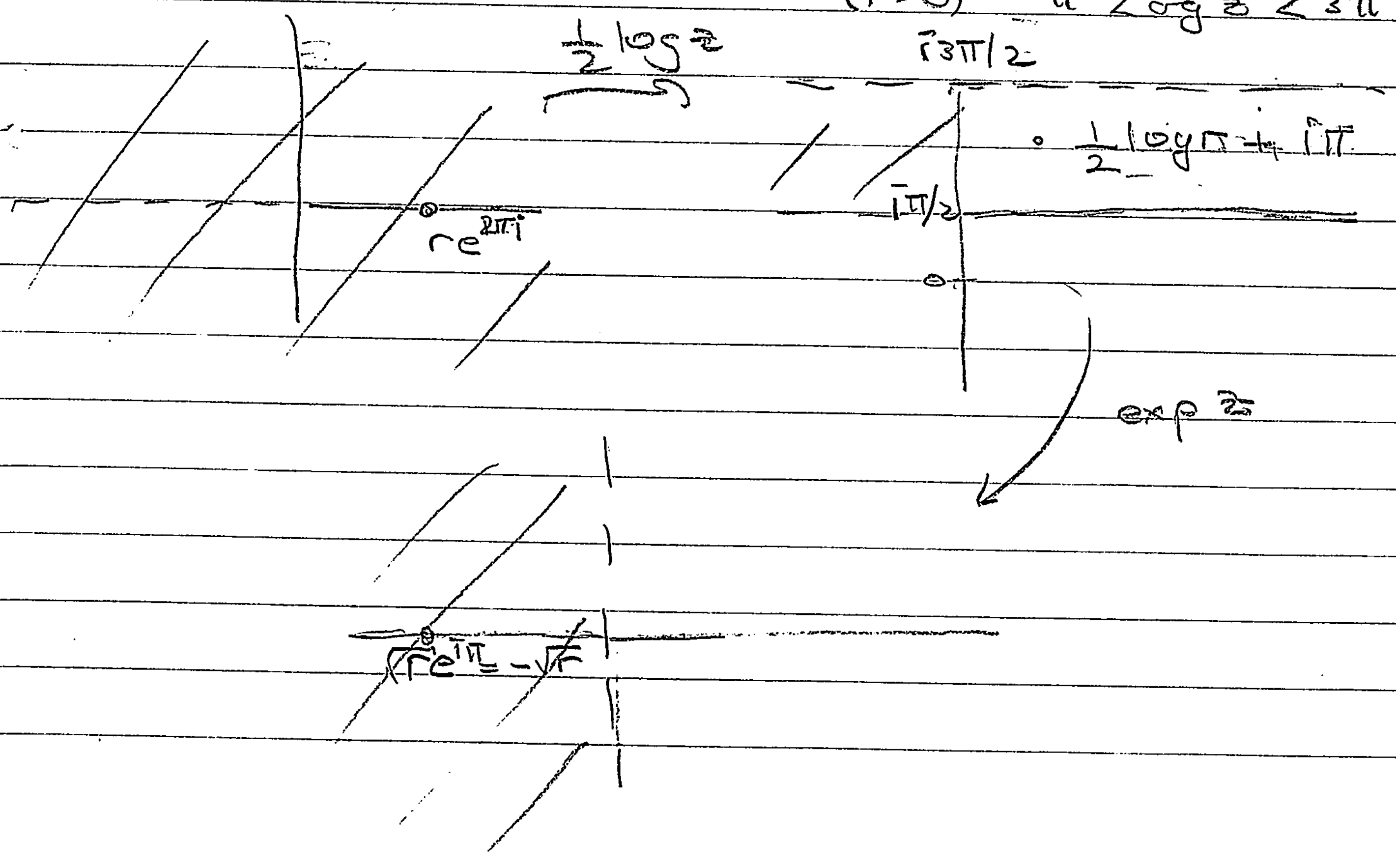
$$z^{1/2} = \exp\left(\tfrac{1}{2}\log z\right) = r^{1/2} e^{i\frac{\theta + 2k\pi}{2}}$$

$$= r^{1/2} e^{\frac{i\theta}{2}} \cdot e^{ik\pi}$$

$$= r^{1/2} e^{\frac{i\theta}{2}} (-1)^k = [z^{1/2}](-1)^k$$

∞ly many branches of logarithm yield precisely 2 branches of the square root.
(Where we wrote $[z^{1/2}]$ for the principal branch of the square root.

For the choice $\log_{\mathbb{C}} z = \log r + i(\theta + 2\pi)$
we have $(r>0, \ \pi < \arg z < 3\pi)$



Finally we have that if $f \in \mathcal{H}(\Omega)$ on a simply connected domain Ω and f is non-vanishing in all of Ω, then f has a logarithm in Ω. ie $\exists$ a holom g on Ω s.t

$$f(z) = e^{g(z)}$$

The function $g(z)$ is called a logarithm of f and is denoted by $\log f(z)$.

Thm (6.2) If $f \in \mathcal{H}(\Omega)$, non-vanishing in all of Ω, a simply connected domain. Then $\exists$ a holom function $g : \Omega \to \mathbb{C}$, called logarithm of f, such that

$$f(z) = e^{g(z)}$$

Proof ~~Exercise~~ Define g as a primitive of $\frac{f'}{f}$

Cor If $f \in \mathcal{H}(\Omega)$, non-vanishing in all of Ω simply connected

Then f has a square root in Ω

ie $\exists$ $h : \Omega \to \mathbb{C}$ holom such that

$$h^2(z) = f(z)$$

Pf Let $h(z) = \exp\left(\frac{1}{2}\log f\right) = \exp\left(\frac{1}{2}g(z)\right)$

then $h^2 = \exp g(z) = f(z)$ from Thm 6.2

Before we move to conformal maps in the next section,

I mention that there are various ways to look at simply connected domains. This is taken up in the book in the Appendix B.

We've seen that if Ω is simply connected (i.e. any 2 curves in Ω w/ same end points are homotopic) then

$$\int_\gamma f(z)\,dz = 0 \quad \forall\ \gamma \text{ closed curve in } \Omega \text{ and } f \in \mathcal{H}(\Omega).$$

An open connected region Ω is called holomorphically simply connected if

$\forall\ \gamma \in \Omega$ closed, $f \in \mathcal{H}(\Omega)$

$$\int_\gamma f(z)\,dz = 0$$

Clearly Ω simply connected $\Rightarrow$ Ω holom. simply connected

in fact the converse is also true. i.e. we have

Thm A region Ω is holomorphically simply connected $\Leftrightarrow$ Ω is simply connected

The other direction

holom simply connected $\Longrightarrow$ simply connected

uses Riemann mapping thm (which we will see soon)

For bounded domains we also have

Thm If Ω is a bounded region in $\mathbb{C}$ then Ω is simply connected $\Leftrightarrow$ $\mathbb{C} \setminus \Omega$ is connected

(Thm 1.2 in Appendix B of Stein & Shakarchi)

The proof $\underset{\text{bdd}}{\Omega \text{ simply connected}} \Rightarrow \mathbb{C}\setminus\Omega$ connected

uses the notion of winding numbers which we will talk about briefly next since it also leads to the natural generalization of the residue thm.

Rmk. In the above thm Ω bdd is important since the infinite strip is simply connected unbounded, its complement has 2 components

However if the complement is taken in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the conclusion holds if Ω is bdd or not.