Remark If f has a pole of order 1 at z_0

then $\text{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$.

Since if f has a simple pole at z_0

then $f(z) = \dfrac{a_{-1}}{z - z_0} + g(z)$, $g \in \mathcal{H}(D_r(z_0))$

Hence $(z - z_0) f(z) = a_{-1} + (z - z_0) g(z)$

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1} + \lim_{z \to z_0} (z - z_0) g(z)$$

$$= a_{-1}$$

Conversely if $\lim_{z \to z_0} (z - z_0) f(z)$ exists and

is non-zero, then $(z - z_0) f(z)$ is bounded in some nbhd of z_0

z_0 is a pole of order 1, by our defn of a pole

If the limit exists but is equal to zero then it means f has a removable singularity at z_0.

More generally we have :

Thm 1.4 If f has a pole of order n at z_0 then $\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{(n-1)} \left((z-z_0)^n f(z)\right)$

Proof let $f(z) = P_f(z) + G(z) \quad z \in D_r^*(z_0)$

w/ $P(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}$

and $G(z) \in \mathcal{H}(D_r(z_0))$

Then $(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) +$

$+ \cdots + a_{-1}(z-z_0)^{n-1} + G(z)(z-z_0)^n$

differentiating $(n-1)$ times gives

$$\left(\frac{d}{dz}\right)^{(n-1)} \left((z-z_0)^n f(z)\right) = (n-1)!\, a_{-1} + \underbrace{\frac{d^{(n-1)}}{dz}\left(G(z)(z-z_0)^n\right)}_{\downarrow \text{ product rule.}}$$

$$\lim_{z \to z_0} \left(\frac{d}{dz}\right)^{(n-1)} (z-z_0)^n f(z) = (n-1)!\, a_{-1} + \lim_{z \to z_0} (z-z_0) \tilde{G}(z)$$

hence : $\operatorname{Res}_{z_0} f = a_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{(n-1)}}{dz} \left((z-z_0)^n f(z)\right)$

Example ① $\text{Res}_i \left(\frac{1}{z^2+1}\right)$

$$= \lim_{z \to i} (z-i) \frac{1}{z^2+1} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$

② The function $f = \frac{1}{(z^2+1)^2}$ has double poles at $z = \pm i$

$$\text{Res}_i f = \lim_{z \to i} \frac{d}{dz} \frac{1}{(z+i)^2} = \lim_{z \to i} \frac{-2}{(z+i)^3} = \frac{1}{4i}$$

$$\text{Res}_{-i} f = \lim_{z \to -i} \frac{d}{dz} \left(\frac{1}{(z-i)^2}\right) = \lim_{z \to -i} \frac{-2}{(z-i)^3} = \frac{-2}{(-2i)^3} = \frac{1}{4i^3} = \frac{-1}{4i}$$

Remark The following is a useful tool to calculate residues

Lemma If f, g are holom at z_0, $f(z_0) \neq 0$ and $g(z)$ has a simple zero at z_0 then $\frac{f}{g}$ has a simple pole at z_0

and $$\text{Res}_{z_0} \left(\frac{f(z)}{g(z)}\right) = \frac{f(z_0)}{g'(z_0)}$$

Proof. It is clear that if g has simple zero then $g(z) = (z-z_0)\tilde{g}(z)$ where $\tilde{g}(z_0) \neq 0$

and $$\frac{f(z)}{g(z)} = (z-z_0)^{-1} \underbrace{\frac{f(z)}{\tilde{g}(z)}}_{\text{holom in } D_r(z_0)}$$ and non-zero in some $D_r(z_0)$

So $\dfrac{f(z)}{g(z)}$ has a simple pole at z_0.

We then apply thm 1.4 to f/g

$$\operatorname{Res}_{z_0}\left(\frac{f}{g}\right) = \lim_{z\to z_0} (z-z_0)\frac{f(z)}{g(z)}$$

$$\underset{g(z_0)=0}{=} \lim_{z\to z_0} \frac{f(z)(z-z_0)}{g(z)-g(z_0)}$$

$$= f(z_0)\lim_{z\to z_0}\frac{z-z_0}{g(z)-g(z_0)} = \frac{f(z_0)}{g'(z_0)}.$$

Rk Note if $f(z_0)=0$ then f/g has a removable sing. at z_0. ☑

Example ① $\operatorname{Res}_i\left(\dfrac{1}{z^2+1}\right) = \operatorname{Res}_i\left(\dfrac{1}{g(z)}\right) = \dfrac{1}{g'(i)} = \dfrac{1}{2i}$

where $g(z)=z^2+1$ has simple zero at $z=i$

② $\operatorname{Res}_i\left(\dfrac{z^3}{z^2+1}\right) = ?$ We can either use partial fraction expansion

$$\frac{z^3}{z^2+1} = z - \frac{z}{z^2+1} = z - \frac{1}{2}\frac{1}{z-i} - \frac{1}{2}\frac{1}{z+i}$$

and get $\operatorname{Res}_i \dfrac{z^3}{z^2+1} = \dfrac{-1}{2}$

or use the above Lemma

$$\operatorname{Res}_i\left(\frac{z^3}{z^2+1}\right) = \frac{f(i)}{g'(i)} = \frac{i^3}{2i} = -1/2 \quad \text{with } f=z^3,\ g=z^2+1$$

Remark Note if $f(z) = P(z) + G(z)$ $z \in D_r^*$

where $P(z) =$ principal part at pole z_0

$G(z)$ - Holomorphic func.

Let C be any circle centered at z_0 and contained $D_r(z_0)$

then
$$\int_C P(z)\,dz = \int_C \left(\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}\right) dz$$

$$= 2\pi i\, a_{-1}$$

Since
$$\int_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

By Cauchy's thm we also know that if $C \subset D_r(z_0)$ then $\int_C G(z)\,dz = 0$

Hence we have

$$\int_C f\,dz = 2\pi i\, a_{-1}$$

In fact we have the general formula

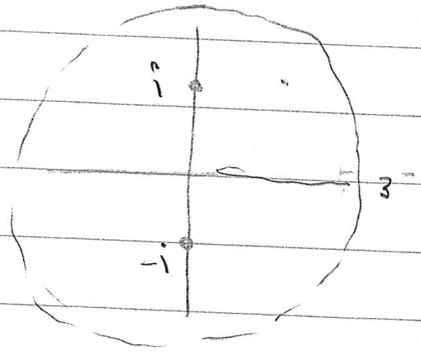
Thm (2.1) **The Residue Formula** Let $\Omega \subset \mathbb{C}$ open, $F = \{z_0, \dots, z_n\}$ a finite set in Ω. Suppose $f \in \mathcal{H}(\Omega \setminus F)$ holomorphic except for poles at $z_0, z_1, \dots, z_n \in F$. Let γ be any circle contained in Ω with counter clockwise orientation and such that $\gamma \cap F = \phi$. Let D be the open disc bounded by γ. Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in F \cap D} \operatorname{res}_{z_i} f}$$

Before we give the proof we'll look at simple examples which uses this formula to calculate integrals.

Example (1) Let γ be the circle $|z| = 3$

$$\int_\gamma \frac{dz}{(z^2+1)^2}$$



$$= 2\pi i \operatorname{Res}_i \left(\frac{1}{(z^2+1)^2}\right) + 2\pi i \operatorname{Res}_{-i} \left(\frac{1}{(z^2+1)^2}\right)$$

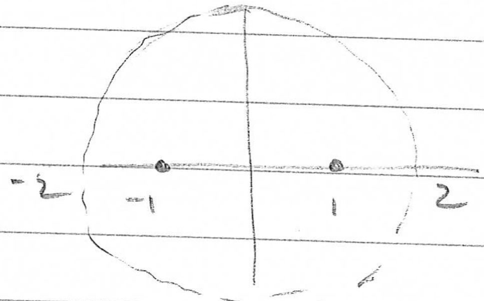
$$= 2\pi i \left[\frac{1}{4i} + \left(\frac{-1}{4i}\right)\right] = 0$$

(2) $$\int_{|z|=3} \frac{z^3}{z^2+1}\,dz = 2\pi i\left(\operatorname{Res}_i \frac{z^3}{z^2+1} + \operatorname{Res}_{-i} \frac{z^3}{z^2+1}\right)$$

$$= 2\pi i\left(\frac{-1}{2} + \frac{-1}{2}\right) = -2\pi i$$

(3) $$\int_{|z-1|=\frac{1}{2}} \frac{dz}{(z^2+1)^2} = 0$$ since there is no pole of $\frac{1}{(z^2+1)^2}$ inside the circle $|z-1| = \frac{1}{2}$

(4) $$\int_{|z|=2} \frac{e^z\,dz}{z^2-1}$$



$$= 2\pi i\left(\operatorname{Res}_1\left(\frac{e^z}{z^2-1}\right) + \operatorname{Res}_{-1}\left(\frac{e^z}{z^2-1}\right)\right)$$

$$\operatorname{Res}_1 \frac{e^z}{z^2-1} = \frac{f(1)}{g'(1)}$$ where $f(z) = e^z$, $g(z) = z^2-1$ using the lemma.

$$= \frac{e}{2}$$

$$\operatorname{Res}_{-1} \frac{e^z}{z^2-1} = \frac{f(-1)}{g'(-1)} = \frac{e^{-1}}{-2}$$

Hence $$\int_{|z|} \frac{e^z}{z^2-1}\,dz = \pi i(e - e^{-1})$$