8-10-24

Quick summary

Chapter 1 Thm 3.2 If a continuous function f has a primitive F in Ω, γ is a curve in Ω that begins at w_1 and ends at w_2 then

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1)$$

In particular if γ is closed then $\int_\gamma f(z)\,dz = 0$

Goursat's thm

Thm 1.1' Chapter 2 If f is continuous in an open set Ω and analytic in $\Omega \setminus \{z_0\}$ for some $z_0 \in \Omega$ then

$$\int_R f(z)\,dz = 0$$ for every closed rectangle $R \subset \Omega$ with $\partial R = R$

This is a first example of a general result we want to prove namely that

$$\boxed{\text{If } f \text{ is holom in an open set } \Omega,\ \gamma \subset \Omega \text{ closed curve whose interior is also contained in } \Omega \text{ then } \int_\gamma f(z)\,dz = 0.}$$

Last week we've seen that the simple case of Goursat's thm is enough to show that

Thm 2.2 (Cauchy's thm for a disc)
Suppose D is an open disc in $\mathbb{C}$
f holom on D (or more generally continuous in D and holom in $D \setminus \{z_0\}$ for some $z_0 \in D$) then f has a primitive in D and hence

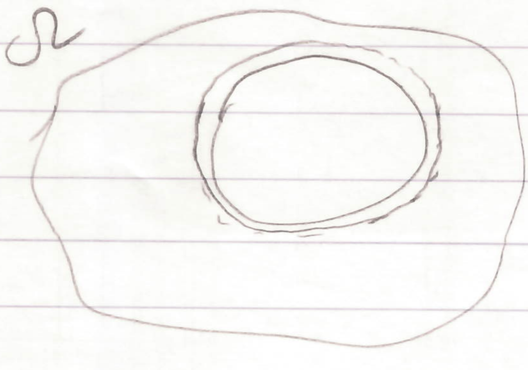
$$\int_\gamma f(z)\,dz = 0$$ for every closed piecewise smooth path in D.

A corollary is

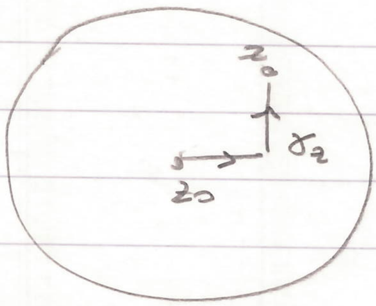
Cor 2.3 Suppose f is holom in an open set containing a circle C and its interior. Then

$$\int_C f(z)\,dz = 0$$

Proof Let D be the disc w/ boundary C
Then $\exists$ a slightly larger disc D' which contains D so that f is holom in D'
We can apply Cauchy's thm for D' to get $\int_C f(z)\,dz = 0$



Rmk Recall in showing f has a primitive in D we used that, for $z \in D_{z_0}(r)$ there is a polygonal path $\gamma_z \in D$



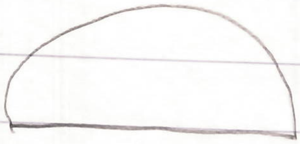
and defined $F(z) := \int_{\gamma_z} f(w)\,dw$

and showed that $F'(z) = f(z)$

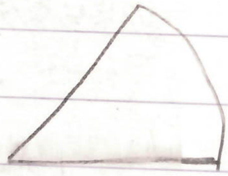
Corollary 2.3 is in fact valid whenever we can define the interior of a contour without ambiguity and construct polygonal paths in an open nbhd of the contour and its interior. S&S call these contours <u>toy contours</u>. For the circle, whose interior is a disc, the geometry made it simple to define the path going horizontally and then vertically from z_0 to z.

Circles, triangles, rectangles, polygons are examples of toy contours. In that case we can copy the arguments as in the circle.

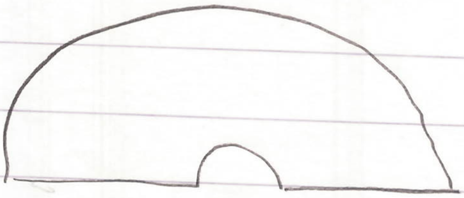
Other toy contours are



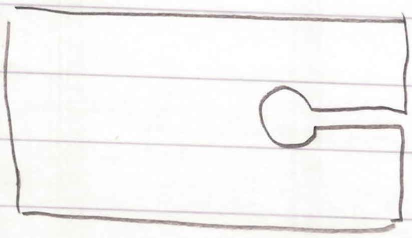
semicircle



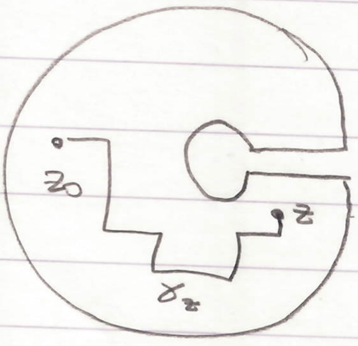
a sector



indented semicircle



rectangular keyhole

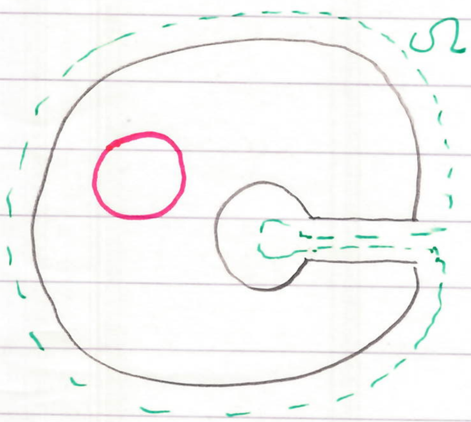


keyhole contour

These can be understood better in terms of closed curves in a "simply connected" region in which f is holomorphic.

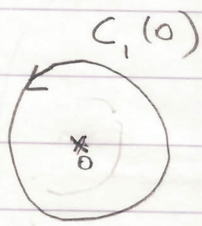
A region in the complex plane is simply connected if any two pair of curves in Ω with same end points are "homotopic" roughly this means one curve can be deformed into the other by a continuous transformation without ever leaving Ω.

We'll see this in chapter 3



Rmk. Note Cauchy's thm doesn't say anything about integral of f in arbitrary open sets and arbitrary closed curves.

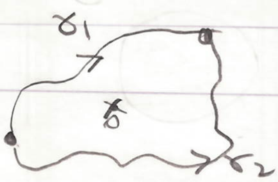
Indeed let $f(z) = \frac{1}{z}$ which is holom in $\mathbb{C} \setminus \{0\} = \Omega$. Unit circle $C_1(0)$ is contained in Ω but



$$\int_{C_1(0)} \frac{1}{z}\, dz = 2\pi i \neq 0.$$

$C_1(0)$ cannot be contained in a disc which is contained in Ω

Note $\mathbb{C} \setminus \{0\}$ is not simply connected

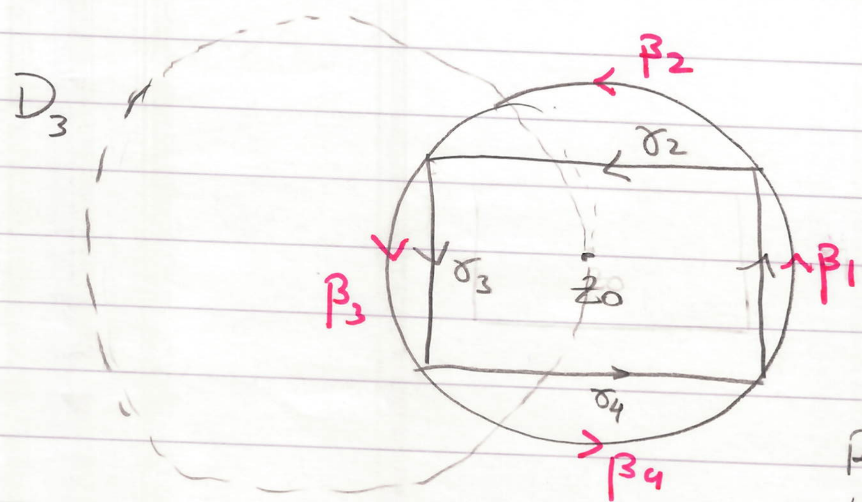


Note γ_1 cannot be deformed to γ_2 continuously in $\mathbb{C} \setminus \{0\}$

Some applications of Cauchy's thm on a disc.

Example Using Cauchy's thm on a disc we can show

$$\int_R \frac{1}{z-z_0}\,dz = 2\pi i$$ for any rectangle with center at z_0.



Let $C_r(z_0)$ be the circle that circumscribes the rectangle R

$R = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$C_r(z_0) = \beta_1 + \beta_2 + \beta_3 + \beta_4$

For each k, $1 \le k \le 4$, choose an open disc D_k so that the trajectory of the closed path $\gamma_k - \beta_k$ is in D_k and $f(z) = \frac{1}{z-z_0}$ is holom in D_k

Now apply Cauchy's thm to $\frac{1}{z-z_0}$ in disc D_k $\gamma_k - \beta_k \in D_k$. Then

$$\int_{\gamma_k - \beta_k} f(z)\,dz = 0 \implies \int_{\gamma_k} \frac{1}{z-z_0}\,dz = \int_{\beta_k} \frac{1}{z-z_0}\,dz$$

But then
$$\int_{\gamma_1+\dots+\gamma_4} \frac{1}{z-z_0}\,dz = \int_{\beta_1+\dots+\beta_4} \frac{1}{z-z_0}\,dz,$$

ie
$$\int_R \frac{1}{z-z_0}\,dz = \int_{C_r(z_0)} \frac{1}{z-z_0}\,dz = 2\pi i$$

$\boxtimes$

<u>Example</u> Fresnel integrals

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{\sqrt{2\pi}}{4}$$

<u>w.t.s</u> $\displaystyle \lim_{r\to\infty}\int_0^r \cos(x^2)\,dx = \lim_{r\to\infty}\int_0^r \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$

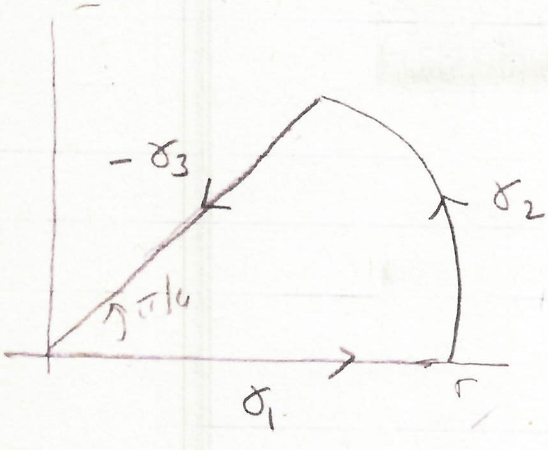
Since $\cos(x^2) + i\sin(x^2) = e^{ix^2}$

<u>w.t.s</u> $\displaystyle \lim_{r\to\infty}\int_0^r e^{ix^2}\,dx = (1+i)\frac{\sqrt{2\pi}}{4}.$

Which naturally leads us to the function

$f(z) = e^{iz^2}$ which is holom everywhere.

less obvious is the path $\gamma = \gamma_1 + \gamma_2 - \gamma_3$



$\gamma_1(t) = t \quad 0 \le t \le r$

$\gamma_2(t) = re^{it} \quad 0 \le t \le \pi/4$

$\gamma_3(t) = te^{i\pi/4} \quad 0 \le t \le r$

$$\int_\gamma e^{iz^2}\,dz = 0$$ by Cauchy's thm in a Disc

Hence
$$\int_0^r e^{it^2}\,dt = -\int_{\gamma_2} e^{iz^2}\,dz + \int_{\gamma_3} e^{iz^2}\,dz$$

$$= -\int_{\gamma_2} e^{iz^2}\,dz + e^{i\pi/4}\int_0^r e^{-t^2}\,dt$$

Claim: $$\left| \int_{\gamma_2} e^{iz^2}\,dz \right| \le \frac{\pi(1 - e^{-r^2})}{4r}$$

Proof (Exercise) (Sheet 4, Question ?

Hence as $r \to \infty$

$$\lim_{r\to\infty} \int_0^r e^{it^2}\,dt \to \frac{(1+i)\sqrt{2}}{2} \int_0^\infty e^{-t^2}\,dt$$

Now the result follows from

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$ which can be evaluated

using the trick

$$\int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \frac{\pi}{4}.$$

$\blacksquare$

Cauchy's Integral Formulas

Thm 4.1 (Chap 2) <u>Cauchy Integral Formula</u> Suppose f is holom in an open set Ω that contains the closure of a disc D. If C denotes the boundary circle of the disc with positive orientation (ccw) Then

$$\boxed{f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw \quad \text{for any } z \in D}$$

<u>Rmk</u> Note CIF says that values of f on D is determined by its boundary values on the circle

Proof let $z \in D_r(z_0)$. $\overline{D_r(z_0)} \subset \Omega$. $\exists\ \varepsilon > 0$ s.t. $D_{r+\varepsilon}(z_0) \subset \Omega$. For $w \in D_{r+\varepsilon}(z_0)$ we define

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & w \neq z \\ f'(z) & w = z \end{cases}$$

Then $g : D_{r+\varepsilon}(z_0) \longrightarrow \mathbb{C}$ is continuous

and away from z is holomorphic

By Cauchy's thm, Thm 2.2' applied to g we have

$$\int_{C_r(z_0)} g(w)\, dw = 0$$

ie $$\int_{C_r(z_0)} \frac{f(w) - f(z)}{w - z}\, dw = 0$$ (Note on $C_r(z_0)$ $w \neq z$ since $z \in D_r(z_0)$)

Hence $$\int_{C_r(z_0)} \frac{f(w)}{w - z}\, dw = f(z) \int_{C_r(z_0)} \frac{dw}{w - z}$$

To finish we claim $$\int_{C_r(z_0)} \frac{dw}{w - z} = 2\pi i$$

<u>Claim</u> $\displaystyle\int_{C_r(z_0)} \frac{dw}{w-z} = 2\pi i.$

> <u>Rmk</u> This was already in one of the exercises. Here we give another proof

<u>Proof</u> $C_r(z_0)$ has parametrization : $\gamma(t) = z_0 + re^{it}$ $0 \le t \le 2\pi$

$\gamma : [0, 2\pi] \to \mathbb{C}$.

It also has the following parametrization

$$\tilde{\gamma}(s) = z + \rho(s) e^{is}$$

where $\rho : [0, 2\pi] \to \mathbb{R}$ $\rho(s) := |\gamma(t(s)) - z|$

Clearly ρ is smooth

$$\tilde{\gamma}'(s) = \rho'(s) e^{is} + i\rho(s) e^{is}$$

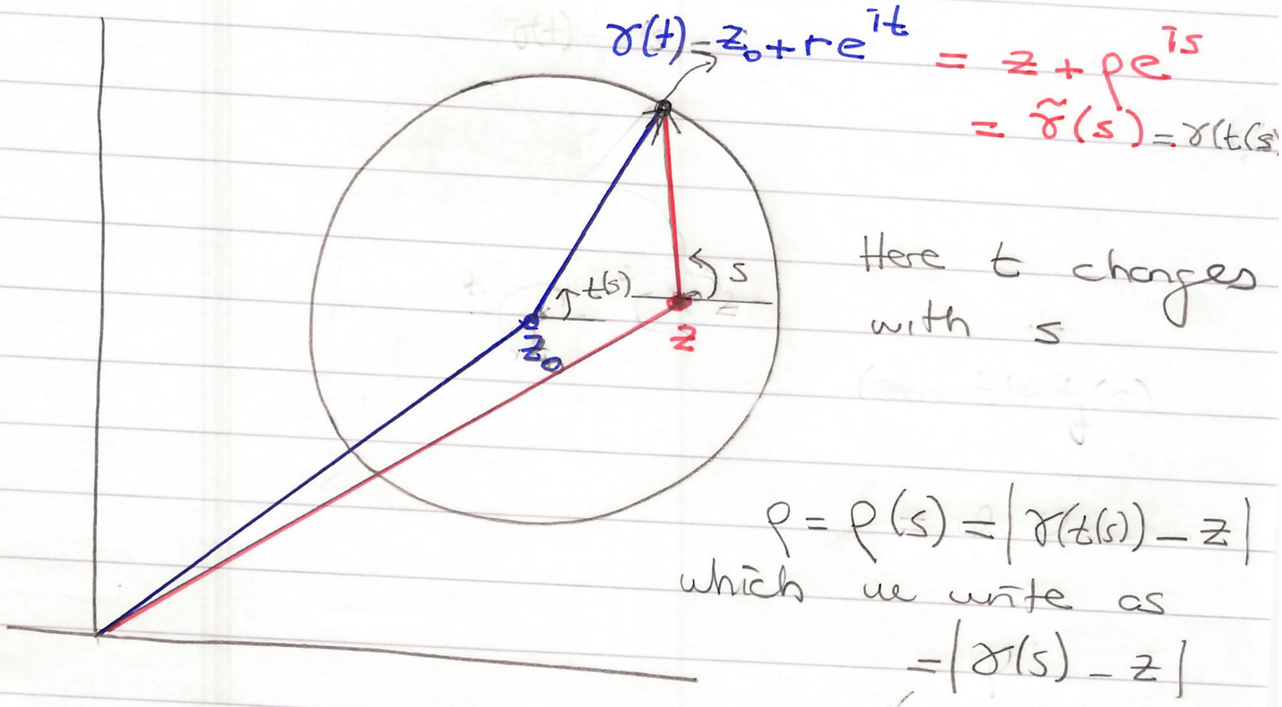
$$\int_{C_r(z_0)} \frac{dw}{w-z} = \int_0^{2\pi} \frac{\rho'(s)e^{is} + i\rho(s)e^{is}}{\rho(s)e^{is}} \, ds$$

$$= \underbrace{\int_0^{2\pi} \frac{\rho'(s)}{\rho(s)} ds}_{} + \underbrace{i \int_0^{2\pi} ds}_{2\pi i}$$

real integral. $\ln|\rho(s)| \Big|_{s=0}^{s=2\pi}$

$= 0$

since $\rho(2\pi) = \rho(0)$



Here t changes with s

$\rho = \rho(s) = |\gamma(t(s)) - z|$ which we write as $= |\gamma(s) - z|$ by abuse of notation

$\tilde{\gamma}(s) = z + \rho(s)e^{is}$ is the new parametrization

$\sigma : [0, 2\pi] \longrightarrow [0, 2\pi]$ is the change of variables
$\quad s \longrightarrow t(s)$

$\tilde{\gamma} = \gamma \circ \sigma$

Before we give important theoretical applications of Cauchy's theorem and Cauchy integral formula we'll look at one more example of contour shifting which helps us to evaluate certain integrals.

Example We'll show that $e^{-\pi x^2}$ is its own Fourier transform:

For a function $f: \mathbb{R} \to \mathbb{C}$ which is Riemann integrable on every $[a, b]$ and $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges, its Fourier transform

$$\boxed{\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx}$$

is well defined for all $\xi \in \mathbb{R}$

we want to show that if $f(x) = e^{-\pi x^2}$ then

$$\hat{f}(\xi) = e^{-\pi \xi^2}$$

ie

w.t.s.

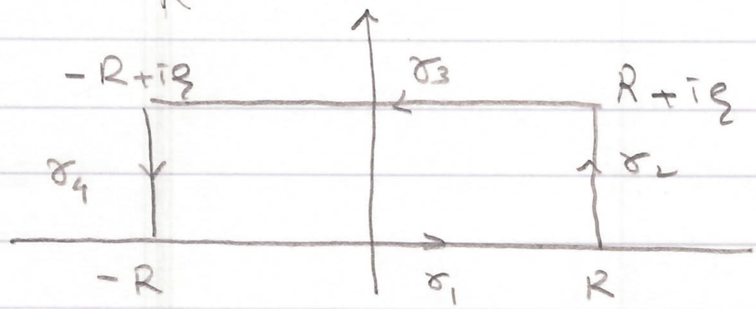
$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi}\, dx$$

If $\xi = 0$ this gives $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

which we know from Analysis

We first suppose $\xi > 0$ and let

$f(z) = e^{-\pi z^2}$ then $f(z)$ is entire and in particular holomorphic in the piecewise smooth contour $\gamma_R = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$



Hence using Cauchy's thm

$$\int_{\gamma_R} f(z)\,dz = 0$$

Note on γ_1 : $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} e^{-\pi x^2}\,dx$

on γ_3 $\displaystyle\int_{\gamma_3} f(z)\,dz = \int_{R}^{-R} e^{-\pi(x+i\xi)^2}\,dx$

$$= -\int_{-R}^{R} e^{-\pi(x^2 + 2\pi i x\xi)} \cdot e^{+\pi\xi^2}\,dx$$

$$= -e^{+\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} \cdot e^{-2\pi i x\xi}\,dx$$

As $R \to \infty$ the first integral over $\gamma_1 = 1$
the integral over γ_3 gives

$$e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x\xi}\,dx$$

On the vertical side on the right

$$\int_{\gamma_2} f(z)\,dz = \int_0^{\xi} f(R+iy)\, i\, dy$$

$$= \int_0^{\xi} e^{-\pi(R^2 + 2iRy - y^2)}\, i\, dy$$

For fixed ξ, the integral can be bounded with

$$\left| \int_{\gamma_2} f(z)\,dz \right| \leq \xi \sup_{0 \leq y \leq \xi} \left| e^{-\pi R^2} \cdot e^{-\pi i R y} \cdot e^{\pi y^2} \right|$$

$$\leq C\, e^{-\pi R^2}$$

A similar bound holds for the γ_4

Hence as $R \to \infty$ both integrals go to 0 (since ξ is fixed)

And we obtain that $\lim_{R \to \infty} \int_{\gamma_R} f(z)\,dz = 0$

$$= \lim_{R \to \infty} \left[\int_{\gamma_1} + \cdots \int_{\gamma_4} f(z)\,dz \right]$$

$$= 1 + 0 - e^{\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x}\, dx + 0$$

$$\Rightarrow \boxed{e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi i x \xi}}$$

$\square$

Next we see that Cauchy's theorem and CIF will imply fundamental properties of holomorphic functions.

Namely we'll see that they're enough to prove

① If $\Omega \subset \mathbb{C}$ open and $f \in \mathcal{H}(\Omega)$ then $f' \in \mathcal{H}(\Omega)$. Hence f is ∞ly differentiable

And if $z_0 \in \Omega$ and $r > 0$ s.t $D_r(z_0) \subset \Omega$ then f has a power series expansion at z_0

$$f(z) = \sum a_n (z - z_0)^n \quad \forall z \in D_r(z_0)$$

ie f is analytic in $D_r(z_0)$

② If f is entire, ie $f: \mathbb{C} \to \mathbb{C}$ holom everywhere and bdd then f is constant

③ Fund. thm of algebra holds ie any polynomial $p(z) \in \mathbb{C}[z]$ of degree n, has n roots in $\mathbb{C}$ (counted w/ multiplicity)

④ If f, g are holom in Ω and $f(z) = g(z)$ $\forall z$ in some sequence of distinct points with a limit point in Ω then $f(z) = g(z)$ $\forall z \in \Omega$. in particular if f, g