

agree on an open set of Ω , they agree on all Ω .

We start with the following theorem

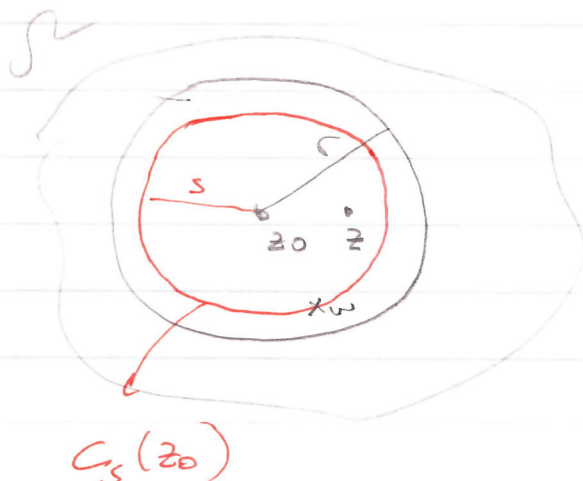
Thm 4.4. Suppose f is holomorphic in an open set Ω . Let $z_0 \in \Omega$, $r > 0$ so that $\overline{D_r(z_0)} \subset \Omega$. Then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } z \in D_r(z_0)$$

and the coefficients a_n are given by the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{\Gamma_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

Proof. Let z_0, r be as above so that $\overline{D_r(z_0)} \subset \Omega$. Fix $s \in (0, r)$ and let $C_s(z_0)$ be the circle of radius s with center z_0 . Then $\gamma = C_s(z_0) \subset \Omega$



and by CIF

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$$

$$\forall z \in D_s(z_0)$$

The trick is to write

$$\frac{1}{w-z} = \frac{1}{(w-z_0) - (z-z_0)}$$

$$= \frac{1}{(w-z_0)} \left(\frac{1}{1 - \frac{(z-z_0)}{(w-z_0)}} \right)$$

Since we are integrating on γ , for $w \in \gamma$

$$\left| \frac{z-z_0}{w-z_0} \right| = \frac{|z-z_0|}{s} < 1 \quad (z \in D_s(z_0))$$

is fixed

Hence $\frac{1}{1 - \left(\frac{z-z_0}{w-z_0}\right)} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n$ for $w \in \gamma$
 $z \in D_s(z_0)$

Convergence is uniform since the bound $\frac{|z-z_0|}{s}$ for $\left(\frac{z-z_0}{w-z_0}\right)$ is indep.

of $w \in \gamma$. Hence we can interchange the series and the integral.

Hence we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} f(w) \sum_{n=0}^{\infty} (z-z_0)^n (w-z_0)^{-n-1} dw$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \underbrace{\int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} dw}_{:= a_n}$$

where
$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

Hence we have that in $D_s(z_0)$, f is the sum of the power series
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

We've seen that power series in their disc of convergence are differentiable with derivatives given by termwise differentiation (Thm 2.6)

Hence in $D_s(z_0)$ we have

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-z_0)^n \quad \forall z \in D_s(z_0) \end{aligned}$$

Being a power series $f'(z)$ is also holomorphic in $D_s(z_0)$. Inductively we get f is differentiable n times for $z \in D_s(z_0)$ and evaluating at $z=z_0$

gives
$$\begin{aligned} a_0 &= f(z_0) \\ a_1 &= f'(z_0) \\ &\vdots \\ n! a_n &= f^{(n)}(z_0) \end{aligned}$$

Hence $a_n = \frac{f^{(n)}(z_0)}{n!}$ is independent of s .

and $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ conv. for all

(70)

$z \in D_S(z_0) \quad \forall S \in (0, r)$. Hence the radius of convergence of

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is at least r

and we are done. \square

Note that the proof also gives

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw = \frac{f^{(n)}(z_0)}{n!}$$

In fact we have

Cor 4.2 If $f \in \mathcal{H}(\Omega)$,

f is ∞ -ly differentiable in Ω .

Moreover if $z_0 \in \Omega$ s.t. $r > 0$ $\overline{D_r(z_0)} \subset \Omega$
then for any $z \in D_r(z_0)$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} dw$$

Cauchy
Int. for
derivatives

Proof The fact that f' is differentiable follows because $\forall z_0 \in \Omega, \exists r > 0$ s.t. $D_r(z_0) \subset \Omega$. By Thm 4.4 f has a power series expansion there, hence holom

in $D_r(z_0)$ for some $r < R$. Since power series are only diff in their disc of convergence we have, f is diff n times at z_0 . Since z_0 was arbitrary f is only diff in Ω .

To prove the IF. Note $n=0$ is simply the CIF in Thm 4.1. We'll use induction.

suppose

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^n} dw$$

for any $z \in D_r(z_0)$

Then for h small, so that $z+h, z$ are away from $C_r(z_0)$

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{h} \left[\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n} \right] dw$$

use $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$

w/ $a = \frac{1}{w-z-h}$ $b = \frac{1}{w-z}$ and take the limit as $h \rightarrow 0$

Note $(a-b) \xrightarrow{h} \frac{1}{(w-z)^2}$ and $(a^{n-1} + \dots + b^{n-1}) \xrightarrow{h} \frac{n}{(w-z)^{n-1}}$ as $h \rightarrow 0$
to get $\lim_h \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}$

$$= \frac{n-1}{2\pi i} \int_{C_r(z_0)} f(w) \left[\frac{1}{(w-z)^2} \cdot \frac{n}{(w-z)^{n-1}} \right] dw$$

$$= \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} dw$$

□

Remark Thm 4.4 says that a holomorphic function f can be locally developed as a power series around each point of the definition domain Ω .

Explicitly for each $z_0 \in \Omega$, $\exists D_r(z_0) \subset \Omega$

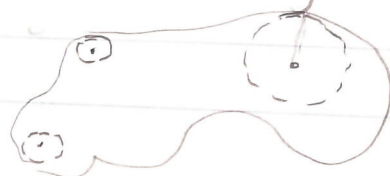
and a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

which converges $\forall z \in D_r(z_0)$ and represents the function f in $D_r(z_0)$

Due to this power series development we have that holomorphic functions are exactly the functions which we everywhere representable as a power series (with a positive radius of convergence) (recall every power series represents a holom. func in their disc of convergence).

And this is why we have the words "holomorphic" / "analytic" used interchangeably in various sources.

Note that the power series $\sum a_n (z - z_0)^n$ might not represent $f(z)$ in all of Ω but represents it at least in a disc whose radius is the distance from the point to the boundary of Ω .



Cor 4.3 (Cauchy inequalities) with the assumptions as in Cor 4.2 we have

$$|f^{(n)}(z_0)| \leq \frac{n! \sup_{|w-z_0|=r} |f(w)|}{r^n} = n! \frac{\|f\|_{C_r(z_0)}}{r^n}$$

where $\|f\|_{C_r(z_0)}$ is the sup of f on $C_r(z_0)$.

~ Proof This follows from

$$|a_n| = \left| \frac{f^{(n)}(z_0)}{n!} \right| = \left| \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw \right|$$

$$|n! a_n| = |f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} r e^{i\theta} d\theta \right|$$

$$\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + re^{i\theta})|}{r^n} d\theta$$

$$\leq \frac{n!}{2\pi} \frac{\|f\|_C}{r^n}$$

An immediate corollary of these results is the remarkable Liouville's thm:

Thm (Liouville's theorem) (Cor 4.5).

If $f \in \mathcal{H}(\mathbb{C})$ and if f is bounded
then f is constant.

Proof Since \mathbb{C} is connected, it is
enough to show $f' = 0$ (Cor. 34. Chap. 1)

let $z_0 \in \mathbb{C}$, $r > 0$ then $D_r(z_0) \subset \mathbb{C} \forall r$
and since f is holom on all \mathbb{C} , we have using Cauchy
inequalities

$$|f'(z_0)| \leq \frac{\|f\|_{D_r(z_0)}}{r}$$

By assumption f is bounded; $\exists M \geq 0$ s.t
 $\forall z \in \mathbb{C} \quad |f| < M$

$$\text{Hence } |f'(z_0)| < \frac{M}{r} \quad \forall r$$

letting $r \rightarrow \infty$ we get $f'(z_0) = 0$

since z_0 was arbitrary $f'(z) = 0 \forall z \in \mathbb{C}$
and hence f is constant

Cor (Fundamental Thm of algebra)

Every poly $P(z) = a_n z^n + \dots + a_0$
of degree $n \geq 1$ has precisely n roots
in \mathbb{C} . If the roots are w_1, \dots, w_n
(with possible repetitions) then

$$P(z) = a_n (z - w_1) \dots (z - w_n)$$

Proof. We first show that $P(z)$ has a root in \mathbb{C} .

Suppose not then the function $Q(z) = \frac{1}{P(z)}$ is in $\mathcal{H}(\mathbb{C})$

If $Q(z)$ is bounded then it would be a constant by Liouville's thm. which then contradicts that $P(z)$ is not constant, i.e. $\deg P \geq 1$ with $P(z) = a_n z^n + \dots + a_0$ where $a_n \neq 0$, $a_i \in \mathbb{C}$.

Claim $Q(z)$ is bounded. For $z \neq 0$ we

$$\begin{aligned} \text{have } |P(z)| &= |a_n z^n + \dots + a_0| \\ &\geq |a_n| |z|^n - \sum_{i=0}^{n-1} |a_i| |z|^i \\ &\geq |z|^n \left(|a_n| - \frac{|a_0|}{|z|^n} - \dots - \frac{|a_{n-1}|}{|z|} \right) \end{aligned}$$

Hence $|P(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$

Hence $|Q(z)| \rightarrow 0$ as $|z| \rightarrow \infty$

Hence $\exists r > 0$ s.t. $|Q(z)| \leq 1$ when $|z| \geq r$ ($Q(z)$ is a continuous function)

but $Q(z)$ is continuous hence bounded on the compact set $|z| \leq r$
 say $|Q(z)| < m$ for some $m \in \mathbb{R}$

Choose $M = \max\{m, 1\}$ then

$$|Q(z)| \leq M \quad \forall z \in \mathbb{C}$$

Hence $Q(z)$ is constant by Liouville's thm. \downarrow

Hence P has a root, say $w_1 \in \mathbb{C}$. Then writing $z = (z - w_1) + w_1$ we have

$$P(z) = a_n \left((z - w_1) + w_1 \right)^n + \dots + a_0$$

$$= b_n (z - w_1)^n + \dots + b_1 (z - w_1) + b_0$$

with new coeffs b_1, \dots, b_{n-1}
 $b_n = a_n$

using $(a+b)^n$

$$= \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Since $P(w_1) = 0$, we get $b_0 = 0$

$$\text{Hence } P(z) = (z - w_1) [b_n (z - w_1)^{n-1} + \dots + b_1]$$

$$= (z - w_1) \tilde{P}(z)$$

where $\tilde{P}(z)$ is a poly of degree $n-1$
 By induction on the degree of the poly we get the result \square