Complex Analysis

Summary of the lecture Complex Analysis by
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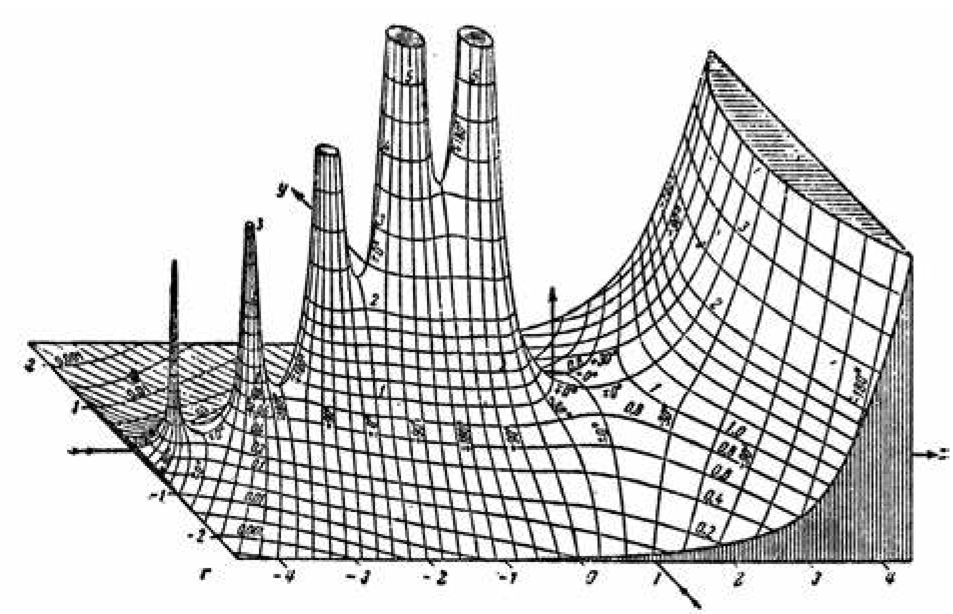

ETH Zürich - December 23, 2024

Preface

This work follows quite accurately the course taught by **Professor O.Imamoglu** at **ETH Zürich** in the years 2023 and 2024, based once again on the book by Stein and Shakarchi [SS10], properly cited during the proceedings. The material has then undergone some changes in the notation and also some slight changes in the reformulation of a few concept. It does not contain nevertheless any change in the content itself, which is a good thing, given the title on the title-page.

In case the reader shall notice any imprecision, mistake, typo or similar, we kindly encourage the reader to report them by sending an e-mail to:

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specifying "Complex Analysis - " in the object, immediately followed by the topic under discussion. Please note that "relatively long" times of response ought to be expected.

Two quick remarks about the notation:

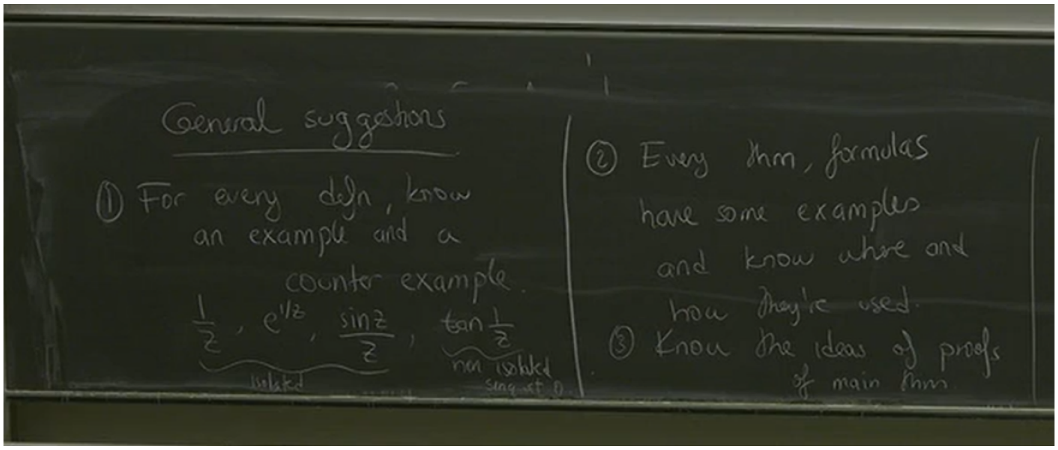
(i) It is possible that the reader might still see some parts of this work being written in red. These are my comments or remarks about the content: they might of interest (expecially to understand the notation), but are not necessarily content covered by the Professor in class. This also applies to the Appendix B.

(ii) To avoid misinterpretation of notation, when possible, hence when a line ends with a mathematical symbol with nothing following (except a displayed mathematical formula), the last punctuation symbol of the above mentioned line is omitted.

The webpage of the lecture is available here:

Funktionentheorie/Complex Analysis Autumn 2024

We shall conclude by showing the very last blackboard of the course, to keep in mind while reading.



I wish the reader an interesting view.

Sincerely,
Andreas Compagnoni

Contents

Chapter 0

Introduction

Our goal this semester is to study functions

$$f : \mathbb{C} \to \mathbb{C}$$

defined on the complex plane $\mathbb{C}$, or on an open set Ω of $\mathbb{C}$.

We will see that the study of **Complex Function Theory** is not simply the study of functions on $\mathbb{R}^2$: in fact, the theory of functions of one real variable is in many ways more complicated than the theory of functions of a complex variable.

To give on idea of what I mean let's try to compare and contrast:

1. It is not too difficult to find a function of a real variable that is in $D^n(\mathbb{R})$ but not in $D^\infty(\mathbb{R})$. Consider

   $$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & , x \in \mathbb{R} \setminus \{0\} \\ 0 & , x \in \{0\} \end{cases}$$

   The derivative of f exists for every $x \in \mathbb{R}$, including $x = 0$, with $f'(0) = 0$. Hence, f is differentiable, but its derivative is not continuous, therefore it's not differentiable twice.

   By integrating f as many times as one likes, one can obtain a function h, that is differentiable that many times, but not infinitely differentiable.

   In contrast: we will see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable once, then it is differentiable infinitely many times.

2. There are functions $f : \mathbb{R} \to \mathbb{R}$ that are infinitely many times differentiable, whose Taylor series does not represent f, i.e. f is not analytic. E.g.

   $$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) = \begin{cases} \exp\left(\frac{-1}{x^2}\right) & , x \in \mathbb{R} \setminus \{0\} \\ 0 & , x \in \{0\} \end{cases}$$

Then f is infinitely differentiable. Unfortunately, at $x = 0$ all derivatives are zero. Hence, its Taylor series is identically zero and cannot represent f

In contrast: if $f : \mathbb{C} \to \mathbb{C}$ is a function of a complex variable, which is differentiable, then f is analytic, i.e. it can be represented by a power series (differentiable = analytic).

3. There are plenty of $C^\infty(\mathbb{R})$ functions of a real variable that are bounded, e.g. $\sin(x), \cos(x)$

   In contrast: we will see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable and bounded, then it is constant (Liouville's Theorem 2.8)

4. For two functions of a real variable f, g, they both can "agree" (be equal) on an open set without being equal.

   In contrast: if $f, g : \mathbb{C} \to \mathbb{C}$ are two differentiable functions which coincide on an arbitrarily small disc (or even a convergent sequence $(z_n)_{n\in\mathbb{N}}$), then $f = g$ (Analytic continuation principle 2.10)

Remark 0.1. *The power of Complex Function Theory comes from this "robustness" or rigidity. It is a field in which, in some sense, Analysis, Geometry and Algebra come together.*
This, we will see, allows one to prove Theorems that a priori seem to have nothing to do with complex numbers.

Example 0.1. *1. The integral*

$$\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt = \frac{\sqrt{2\pi}}{4}$$

2. Let $\pi(x) := \#\{p \in \mathbb{P} : p \leq x\}$ with $\mathbb{P}$ denoting the set of prime numbers. Then

$$\pi(x) \approx_{x\to\infty} \frac{x}{\log(x)}$$

Result know as the Prime Number Theorem A.5, as

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1$$

3. If $f \in \mathbb{C}[X]$ is a non-zero polynomial, then f has a zero in $\mathbb{C}$ (Fundamental Theorem of Algebra 2.5, not valid in $\mathbb{R}$ for instance).

4. Let $r_4(n) := \#\{(m_1, ..., m_4) \in \mathbb{Z}^4 : \sum_{l=1}^4 m_l^2 = n\}$, then $r_4(n) = 8 \sum_{\substack{d\mid n \\ 4\nmid d}} d$

Before we start with the definition of differentiability of a function of a complex variable, we recall the definitions and basic properties of complex numbers.

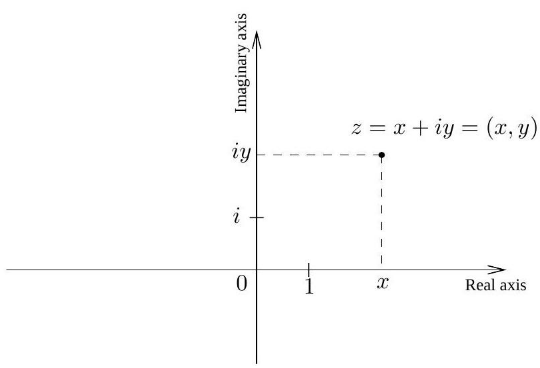
Chapter 1

Preliminaries to Complex Analysis

1.1 The complex numbers and the complex plane

Definition 1.1. The set of complex numbers is

$$\mathbb{C} := \{x + iy : x, y \in \mathbb{R} \text{ and } i^2 = -1\}$$



We consider $\mathbb{R} \subseteq \mathbb{C}$ using the following fact:

$$\exists \iota_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}, \quad r \mapsto r + i \cdot 0$$

Definition 1.2. For $z = x + iy \in \mathbb{C}$ we define

$$Re(z) := x$$

as the **real part of** z and

$$Im(z) := y$$

as the **imaginary part of** z. Moreover, we define the **complex conjugate of** z as

$$\bar{z} := x - iy$$

Proposition 1.1. For $z \in \mathbb{C}$ it holds that:

(i) $Re(z) = \frac{z+\bar{z}}{2}$

(ii) $Im(z) = \frac{z-\bar{z}}{2i}$

(iii) $z \in \mathbb{R} \iff z = \bar{z}$

(iv) $z \in i\mathbb{R} \iff z = -\bar{z}$

Proof. **(i)** $Re(z) = x = \frac{2x}{2} = \frac{x+iy+x-iy}{2} = \frac{z+\bar{z}}{2}$

(ii) $Im(z) = y = \frac{2iy}{2i} = \frac{x+iy-x+iy}{2i} = \frac{z-\bar{z}}{2i}$

(iii) We prove both directions of the equivalence:

$\implies$: Assume $z \in \mathbb{R}$, then $z = x + i0 = x - i0 = \bar{z}$

$\impliedby$: Assume $z = \bar{z}$, then $z - \bar{z} = 0 = x + iy - (x - iy) = 2iy$. This means that $y = 0$ and hence that $z \in \mathbb{R}$

(iv) We prove once again both directions of the equivalence:

$\implies$: Assume $z \in i\mathbb{R}$, then $z = 0 + iy = 0 - (-iy) = -\bar{z}$

$\impliedby$: Assume $z = -\bar{z}$, then $z + \bar{z} = 0 = 2x$. This means that $x = 0$ and consequently that $z \in i\mathbb{R}$

□

Definition 1.3. For $z \in \mathbb{C}$, if $z = Re(z)$, then z is said to be **(purely) real**, whereas if $z = Im(z)$, then z is said to be **purely imaginary**.

Algebraic Structure of $\mathbb{C}$

Complex numbers can also be represented as ordered pairs of real numbers in $\mathbb{R}^2$, so for $z \in \mathbb{C}$ we have that

$$z \cong (x, y)$$

and where for another complex number $w \in \mathbb{C}$ we have that $z = w$ with $w \cong (u, v) \iff x = u$ and $y = v$. Moreover, we defined the following operations:

Addition in $\mathbb{C}$: if $z = x + iy$ and $w = u + iv$, then

$$z + w := (x + u) + i(y + v)$$

or as pairs in $\mathbb{R}^2$

$$z + w \cong (x + u, y + v)$$

Multiplication in $\mathbb{C}$: if $z = x + iy$ and $w = u + iv$, then

$$z \cdot w := (x + iy) \cdot (u + iv) = xu + i(xv + yu) + i^2 yv = (xu - yv) + i(xv + yu)$$

or as pairs in $\mathbb{R}^2$

$$z \cdot w \cong (xu - yv, xv + yu)$$

Note that $i \cong (0, 1)$ and $(0, 1) \cdot (0, 1) = (-1, 0) \cong -1$, as $i^2 = -1$

$\mathbb{R}^2$ with these two operations $+, \cdot$ becomes a field, i.e. $(\mathbb{R}, +, \cdot)$ satisfies the following:

- $(\mathbb{R}^2, +)$ is an abelian group with additive identity $0 \cong (0, 0)$
- $\left(\mathbb{R}^2 \setminus \{(0, 0)\}, \cdot\right)$ is an abelian group with multiplicative identity $1 \cong (1, 0)$
- The (commutative) distributive law holds:

$$\forall z_1, z_2, z_3 \in \mathbb{C} : z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 = (z_2 + z_3)z_1$$

Therefore, complex numbers form a 2-dimensional commutative algebra over $\mathbb{R}$: in this case "$\cong$" can be substituted with with "$=$". Interchanging hence the meaning of $\mathbb{R}^2$ between field structure and the simple usual real vector space, we can prevent any abuse of notation, leaving then any other accentuation of the difference between these structures only to possibly increase the clarity in the proceedings.

Definition 1.4. The real number

$$|z| := \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \in \mathbb{R}$$

is called the **norm**, **modulus** or **absolute value of** $z \in \mathbb{C}$

Additive inverse of $z \in \mathbb{C}$:

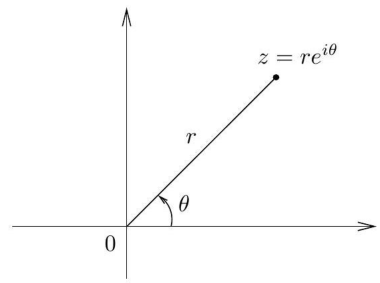
$$-z \cong (-x, -y)$$

Multiplicative inverse of $z \in \mathbb{C}$:

$$z^{-1} = \frac{\bar{z}}{|z|^2} \cong \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

Polar coordinates representation of complex numbers

We also have the following polar coordinates representation of complex numbers:



$$z = x + iy = r\big(\cos(\theta) + i\sin(\theta)\big) = re^{i\theta}$$

where $r \geq 0$, $\theta \in \mathbb{R}$ with $|z| = r$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$
The polar representation is not unique unless $z \neq 0$ and we restrict $\theta \in (-\pi, \pi]$ (or any other interval of length 2π).

Definition 1.5. The number $\theta \in \mathbb{R}$ is called **the argument of** $z \in \mathbb{C}$, which is defined uniquely up to a multiple of 2π and is denoted by

$$arg(z) := \left\{\theta \in \mathbb{R} : z \in |z|e^{i\theta}\right\}$$

From this we define then

Definition 1.6. The argument of $z \in \mathbb{C}$ chosen in the interval $(-\pi, \pi]$ is called **the principal argument of** $z \in \mathbb{C}$ and denoted by $\text{Arg}(z) \in \arg(z)$

It holds that: $\text{Arg}(i) = \frac{\pi}{2}$ and $\forall c \in (0, +\infty) : \text{Arg}(-c) = \pi$

Remark 1.1. *No assignment of argument is made to* $0 \in \mathbb{C}$*; therefore we often consider* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$*, also known as the group of units of* $\mathbb{C}$

Proposition 1.2. Let $z = x + iy \in \mathbb{C}^*$, then

$$\operatorname{Arg}(z) = \begin{cases} \arcsin\left(\frac{y}{|z|}\right) & , x \in \mathbb{R}_{\geq 0} \\ \pi - \arcsin\left(\frac{y}{|z|}\right) & , x \in \mathbb{R}_{<0}, y \in \mathbb{R}_{\geq 0} \\ -\pi - \arcsin\left(\frac{y}{|z|}\right) & , x, y \in \mathbb{R}_{<0} \end{cases}$$

Proof. Here is enough to observe that for $t \in [-1, 1]$, arcsin is the unique number $u \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(u) = t$. By using the various mirroring in the unit circle (indeed we divided by $|z|$), we obtain the result. □

Proposition 1.3. For $z \in \mathbb{C}$, it holds that

$$\arg(z^{-1}) = -\arg(z)$$
$$\arg(zw) = \arg(z) + \arg(w)$$

Proof. The proofs of these equalities are direct

$$\begin{aligned} \arg(z^{-1}) &= \left\{\theta \in \mathbb{R} : z^{-1} = |z^{-1}|e^{i\theta}\right\} = \left\{\theta \in \mathbb{R} : \frac{\bar{z}}{|z|^2} = |z^{-1}|e^{i\theta}\right\} = \\ &= \left\{\theta \in \mathbb{R} : \frac{\bar{z}}{|z|^2} = \frac{1}{|z|}e^{i\theta}\right\} = \left\{\theta \in \mathbb{R} : \bar{z} = |z|e^{i\theta}\right\} = \\ &= \left\{\theta \in \mathbb{R} : z = |z|e^{-i\theta}\right\} = \left\{-\theta \in \mathbb{R} : z = |z|e^{i\theta}\right\} = \\ &= -\left\{\theta \in \mathbb{R} : z = |z|e^{i\theta}\right\} = -\arg(z) \end{aligned}$$

where θ is defined with respect to z^{-1} and where we used that $|z^{-1}| = \frac{1}{|z|}$ as consequence of a simple computation. Moreover, we have that

$$\begin{aligned} \arg(zw) &= \left\{\theta \in \mathbb{R} : zw = |zw|e^{i\theta}, \theta = \theta_z + \theta_w\right\} = \\ &= \left\{\theta_z + \theta_w \in \mathbb{R} : zw = |zw|e^{i(\theta_z + \theta_w)}\right\} = \\ &= \left\{\theta_z + \theta_w \in \mathbb{R} : zw = |z|e^{i\theta_z}|w|e^{i\theta_w}\right\} = \\ &= \arg(z) + \arg(w) \end{aligned}$$

□

Remark 1.2. *Despite these results, it is not always the case that for $z \in \mathbb{C}$ we have*

$$\operatorname{Arg}(z^{-1}) = -\operatorname{Arg}(z)$$
$$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$$

as shown in the following example.

Example 1.1. *It holds that* $\text{Arg}\left(\frac{-1}{2}\right) = \pi \neq -\text{Arg}(-2)$ *and also that* $\pi = \text{Arg}(-1) = \text{Arg}\left((-i)(-i)\right) \neq \text{Arg}(-i) + \text{Arg}(-i) = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi$, *hence the properties mentioned above do not hold for* Arg

Matrix representation of complex numbers

This representation is obtained by associating a 2×2 matrix to a complex number written in standard form; some properties of the operations between matrices translate directly into the ones in $\mathbb{C}$, as exposed below.

For $z = a + ib \in \mathbb{C}$, let $Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ while for $w = c + di$, let $W = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$, then

$$ZW = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix}$$

On the other hand, we have $zw = (ac - bd) + i(bc + ad)$. The multiplication in $\mathbb{C}$ hence corresponds to the multiplication of the respective matrices in $\mathbb{R}^{2\times 2}$

We can represent any $z \in \mathbb{C}$ with the matrix

$$Z = \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix} = Re(z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Im(z) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

where $i \in \mathbb{C}$ corresponds to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
In polar form $z = re^{i\theta}$, the corresponding matrix is

$$Z = r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Topological results

Proposition 1.4. The following properties hold for the complex norm $|\cdot|$:

(i) $\forall z \in \mathbb{C} : |z| = 0 \iff z = 0$

(ii) $\forall z_1, z_2 \in \mathbb{C} : \big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

(iii) $\forall z_1, z_2 \in \mathbb{C} : |z_1 z_2| = |z_1||z_2|$

(iv) $\forall z \in \mathbb{C} : |\bar{z}| = |z|$

(v) $\forall z \in \mathbb{C} : |Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$

Proof. **(i)** It is very convenient to prove both implications directly:

$\Longleftarrow$: This direction is trivial, consider $z = 0$, then $|z| = \sqrt{0^2 + 0^2} = 0$

$\Longrightarrow$: $0 = \sqrt{x^2 + y^2} = |z| \Longleftrightarrow x^2 + y^2 = 0$ with both $x^2, y^2 \geq 0$. This implies that $x, y = 0$ and with it that $z = 0$

(ii) First, we prove the triangular inequality:

$$\begin{aligned}\forall z_1, z_2 \in \mathbb{C} : |z_1 - z_2| &= \left(|z_1|^2 + |z_2|^2 + 2(x_1x_2 + y_1y_2)\right)^{\frac{1}{2}} \leq \\ &\leq \left(|z_1|^2 + |z_2|^2 + 2(|z_1||z_2|\right)^{\frac{1}{2}} = \\ &= \left((|z_1| + |z_2|)^2\right)^{\frac{1}{2}} = \\ &= |z_1| + |z_2|\end{aligned}$$

in which we used the **Cauchy-Schwarz Inequality** in $\mathbb{C}$. Applying then this initial result, we obtain

$$\begin{aligned}|z_1 - z_2 + z_2| &\leq |z_1 - z_2| + |z_2| \\ \Longleftrightarrow |z_1| - |z_2| &\leq |z_1 - z_2|\end{aligned}$$

and since this holds for all $z_1, z_2 \in \mathbb{C}$, then by swapping z_1 and z_2 we obtain the same inequality in absolute value, namely

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

(iii) Let $z_1, z_2 \in \mathbb{C}$, then $|z_1z_2| = \sqrt{z_1z_2\bar{z}_1\bar{z}_2} = \sqrt{z_1\bar{z}_1z_2\bar{z}_2} = \sqrt{z_1\bar{z}_1}\sqrt{z_2\bar{z}_2} = |z_1||z_2|$

(iv) For $z \in \mathbb{C}$ we have $|\bar{z}| = |x - iy| = \sqrt{x^2 + y^2} = |x + iy| = |z|$

(v) For $z \in \mathbb{C}$ we have

$$\begin{aligned}|Re(z)|^2 &= x^2 \leq x^2 + y^2 = |z|^2 \\ |Im(z)|^2 &= y^2 \leq x^2 + y^2 = |z|^2\end{aligned}$$

Applying the $\sqrt{\cdot}$ function, which is a monotonically increasing function, we obtain the wished result.

□

Proposition 1.5. If $z = re^{i\theta} \in \mathbb{C}^*$ and $w = se^{i\nu} \in \mathbb{C}^*$, then

$$zw = rse^{i(\theta+\nu)} \in \mathbb{C}^*$$

Proof. This result follows from simple multiplications rules:

$$zw = re^{i\theta}se^{i\nu} = rse^{i(\theta+\nu)}$$

Since both r, s are non-zero then also their product is not, hence $zw \in \mathbb{C}^*$ □

Next, we recall some definitions that we need from Topology and Analysis.

Definition 1.7. We denote **the open disc of radius** $r > 0$ **centred at** z with $D_r(z)$ or $D(z, r)$ and **the closed disc of radius** $r \geq 0$ **centred at** z with $\overline{D_r}(z)$ or $\overline{D}(z, r)$. They are both defined as follows

$$D_r(z) := \{w \in \mathbb{C} : |w - z| < r\}$$
$$\overline{D_r}(z) := \{w \in \mathbb{C} : |w - z| \leq r\}$$

The **boundary of** $D_r(z)$ is the circle

$$C_r(z) := \partial D_r(z) := \overline{D_r}(z) \setminus D_r(z) = \{w \in \mathbb{C} : |w - z| = r\}$$

Remark 1.3. *If $r > 0$, then $\forall z \in \mathbb{C} : \overline{D_r}(z) = \overline{D_r(z)}$, hence in this case the closed disc is equal to the closure of the open disc.*

Definition 1.8 (Open set)**.** A subset $U \subseteq \mathbb{C}$ is **open**, if

$$\forall z \in U \exists r > 0 : D_r(z) \subseteq U$$

The set of all such open subsets of $\mathbb{C}$ is called **the standard topology of open sets of** $\mathbb{C}$ and denoted by $\mathcal{O}_{\mathbb{C}}$

E.g. $\emptyset, \mathbb{C}, D_r(z)$ and $\mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\}$

Definition 1.9 (Closed set)**.** A subset $U \subseteq \mathbb{C}$ is **closed**, if $\mathbb{C} \setminus U$ is open.

E.g. $\emptyset, \mathbb{C}, \overline{D_r}(z), C_r(z)$ and $\mathbb{R}$

Proposition 1.6. The following equivalence holds:

$$U \text{ is closed } \iff \left(\forall (z_n)_{n\in\mathbb{N}^*} \in U^{\mathbb{N}^*} : \lim_{n\to\infty} z_n = z \Longrightarrow z \in U\right)$$

Definition 1.10. A subset $K \subseteq \mathbb{C}$ is **compact**, if it is closed and bounded, i.e. if it is closed and if $\exists M > 0 \forall z \in K : |z| < M$

Proposition 1.7. $K \subseteq \mathbb{C}$ is compact, if and only if every sequence $(z_n)_{n\in\mathbb{N}^*} \in U^{\mathbb{N}^*}$ has a subsequence that converges to a point in U

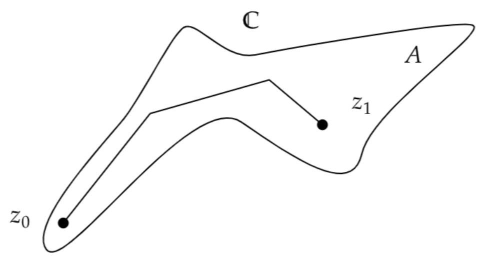
E.g. $\emptyset, \overline{D_r}(z), C_r(z)$ and $[a, b] \times [c, d]$

Definition 1.11. A subset $A \subseteq \mathbb{C}$ is called **disconnected**, if

$$\exists U, V \in \mathcal{O}_{\mathbb{C}} : (U \cap V = \emptyset) \text{ and } (A \cap U \neq \emptyset) \text{ and } (A \cap V \neq \emptyset) \text{ and } (A \subseteq U \cup V)$$

A subset $A \subseteq \mathbb{C}$ is called **connected**, if it is not disconnected.
Moreover, a connected open non-empty set $\emptyset \neq U \subseteq \mathbb{C}$ is called a **region** or **domain**.

We mention here that in any euclidean space: a connected open set is automatically open and path-connected and vice versa. Any two distinct points z_0, z_1 in an open connected set $A \subseteq \mathbb{C}$ can be connected by a polygonal path lying in A



E.g. $\emptyset, \mathbb{C}, D_r(z), \overline{D_r}(z), C_r(z)$ and $\mathbb{R}$ are connected, whereas $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R} \cup D_1(2i)$ are disconnected.

Definition 1.12 (Convergence of a complex sequence)**.** A sequence $(z_n)_{n\in\mathbb{N}^*} = (x_n + iy_n)_{n\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}^*}$ **converges to** $z = x + iy$ **in** $\mathbb{C}$, if one of the following equivalent conditions holds:

(i) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in $\mathbb{R}$

(ii) $\lim_{n\to\infty} |z_n - z| = 0$ in $\mathbb{R}$

(iii) $\forall \varepsilon > 0 \exists N \in \mathbb{N}^* \forall m, n \geq N : |z_m - z_n| < \varepsilon$, i.e. $(z_n)_{n\in\mathbb{N}^*}$ is a complex Cauchy-sequence.

Definition 1.13 (Limits)**.** Let $U \subseteq \mathbb{C}$ be an open subset and $f \in \mathbb{C}^U$ any function. For $z_0 \in U$ and $w_0 \in \mathbb{C}$ we have

$$\lim_{\substack{z\to z_0\\ z\in U}} f(z) = w_0$$

if one of the following equivalent conditions holds:

(i) $\forall \varepsilon > 0 \exists \delta > 0 \forall z \in U : |z - z_0| < \delta \Longrightarrow |f(z) - w_0| < \varepsilon$

(ii) If $(z_n)_{n\in\mathbb{N}^*} \in U^{\mathbb{N}^*}$ is a sequence with $\lim_{n\to\infty} z_n = z_0$, then $\lim_{n\to\infty} f(z_n) = w_0$

Definition 1.14 (Continuity of a function)**.** A function $f \in \mathbb{C}^U$ **is continuous on** U, if and only if

$$\forall z_0 \in U : \lim_{z\to z_0} f(z) = f(z_0)$$

that is, if and only if

$$\forall (z_n)_{n\in\mathbb{N}^*} \in U^{\mathbb{N}^*} : \lim_{n\to\infty} z_n = z_0 \Longrightarrow \lim_{n\to\infty} f(z_n) = f(z_0)$$

The set of all continuous functions on an set U is denoted by $C^0(U)$ or $C^0(U;\mathbb{C})$

1.2 Holomorphic Functions

1.2.1 Definition and basic properties

This is a central notion for the rest of the class.

Definition 1.15 (Holomorphic function)**.** Let $\Omega \subseteq \mathbb{C}$ an open set and $f \in \mathbb{C}^\Omega$ a complex valued function on Ω, then

- f is called **holomorphic at** $z_0 \in \Omega$, if

$$\exists f'(z_0) := \lim_{\substack{z\to z_0\\ z\neq z_0}} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{f(z_0 + h) - f(z_0)}{h} \in \mathbb{C}$$

Here $h \in \mathbb{C}$, $z_0 + h \in \Omega$ (so that the quotient is well defined).
If the limit exists, we denote it with $f'(z_0)$ and we call it **the derivative of f at z_0**

- f is called **holomorphic on** Ω, if $\forall z_0 \in \Omega : f$ is holomorphic at $z_0 \in \Omega$
- If f is holomorphic on all of $\mathbb{C}$, then it is called **entire**.

Remark 1.4. *Regular or complex differentiable are other words used for holomorphic.*

Example 1.2. *Let $f \in \mathbb{C}^{\mathbb{C}}, f(z) = z$. Then f is entire, since*

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h\to 0} \frac{z_0 + h - z_0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence, it holds that $\forall z \in \mathbb{C} : f'(z) = 1$

Definition 1.16. The set of all holomorphic functions on Ω is defined as follows

$$\mathscr{H}(\Omega) := \{f \in \mathbb{C}^{\Omega} : f \text{ is holomorphic on } \Omega\}$$

As in the case of real variables, we have

Proposition 1.8. [SS10, Proposition I.2.2]

(i) The set $\mathscr{H}(\Omega)$ is a $\mathbb{C}$-vector space. More precisely, if $f, g \in \mathscr{H}(\Omega)$, then

$$\forall \alpha, \beta \in \mathbb{C} : \alpha f + \beta g \in \mathscr{H}(\Omega)$$

and

$$\forall \alpha, \beta \in \mathbb{C} : (\alpha f + \beta g)' = \alpha f' + \beta g'$$

(The zero-function $0 \in \mathbb{C}^{\Omega}$ is the zero element of the vector space).

(ii) If $f, g \in \mathscr{H}(\Omega)$, then $fg \in \mathscr{H}(\Omega)$ and

$$(fg)' = f'g + fg'$$

(iii) If $g(z_0) \neq 0$, then $\frac{f}{g}$ is holomorphic at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}$$

Moreover, if $g \in \mathscr{H}(\Omega)$ and $\forall z \in \mathbb{C} : g(z) \neq 0$, then $\frac{f}{g} \in \mathscr{H}(\Omega)$

(iv) If $f \in U^\Omega$ and $g \in \mathbb{C}^U$ are both holomorphic, then $g \circ f \in \mathbb{C}^\Omega$ is holomorphic and

$$\forall z \in \Omega : (g \circ f)'(z) = g'\big(f(z)\big)f'(z)$$

Proof. The claims of this Propositions are very similar to the case of Real Analysis; therefore here we will only show the fourth.

(iv) Let $z_0 \in \Omega$ and $w_0 := f(z_0) \in U$. Consider $F \in \mathbb{C}^\Omega$ and $G \in \mathbb{C}^U$ defined by

$$F(z) := \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & , z \neq z_0 \\ f'(z_0) & , z = z_0 \end{cases}$$
$$G(z) := \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} & , w \neq w_0 \\ g'(w_0) & , w = w_0 \end{cases}$$

Since

$$\lim_{z \to z_0} F(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = F(z_0)$$

we have that F is continuous at z_0
Similarly, G is continuous at w_0. Hence, since f is differentiable at z_0 and hence continuous at z_0, $G \circ f$ is continuous at z_0

For $z \in \Omega \setminus \{z_0\}$ we have

$$\begin{aligned} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} &= \frac{g\big(f(z)\big) - g\big(f(z_0)\big)}{z - z_0} = \\ &= \left\{ \begin{array}{ll} \frac{g\left(f(z)\right)-g(w_0)}{f(z)-w_0} \frac{f(z)-f(z_0)}{z-z_0} & , f(z) \neq w_0 \\ 0 & , f(z) = w_0 \end{array} \right\} = \\ &= G\big(f(z)\big)F(z) \end{aligned}$$

Note that if $f(z) = w_0$, then $F(z) = \frac{w_0 - f(z)}{z - z_0} = 0$

Hence, we finally obtain

$$\begin{aligned} (g \circ f)'(z_0) &= \lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = \\ &= \lim_{z \to z_0} G\big(f(z)\big)F(z) = \\ &= G\big(f(z_0)\big)F(z_0) = \\ &= G(w_0)F(z_0) = \\ &= g'(w_0)f'(z_0) = \\ &= g'\big(f(z_0)\big)f'(z_0) \end{aligned}$$

since $G \circ f$ and F are continuous at z_0

□

Remark 1.5. *Note that if $f \in \mathbb{C}^\Omega$ is complex differentiable at $z_0 \in \Omega$, then there exists a complex number $c \in \mathbb{C}$ such that for all $z \in \mathbb{C}$*

$$f(z) = f(z_0) + c(z - z_0) + E(z, z_0)$$

with $E \in \mathbb{C}^\Omega$ satisfying

$$\lim_{z \to z_0} \left| \frac{E(z, z_0)}{z - z_0} \right| = 0$$

Here we have that $c = f'(z_0)$

Example 1.3. *1. Example 1.2 and Proposition 1.8 applied repeatedly show that any polynomial $p \in \mathbb{C}[X]$ is complex differentiable at every point $z \in \mathbb{C}$*

For $p(z) = z^n$ with $n \in \mathbb{N}$ we have that in $z_0 \in \mathbb{C}$:

$$p'(z_0) = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0) \sum_{k=0}^{n-1} z^{n-1-k} z_0^k}{z - z_0} = \lim_{z \to z_0} \sum_{k=0}^{n-1} z^{n-1-k} z_0^k = n z_0^{n-1}$$

2. *Important non-example: Let $f(z) = \bar{z}$, then*

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \bar{z}_0}{h} = \frac{\bar{z}_0 + \bar{h} - \bar{z}_0}{h} = \frac{\bar{h}}{h}$$

For $h = t$ and $t \in \mathbb{R}$ this limit is 1
For $h = it$ and $t \in \mathbb{R}$ this limit is -1

Hence $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ does not exists for any $z_0 \in \mathbb{C}$ and consequently $f(z) = \bar{z}$ is not holomorphic at any point in $\mathbb{C}$. This procedure can be used more generally to disprove the existence of a limit, as done here.

Note that $f(z) = \bar{z}$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x, -y) \end{aligned}$$

Hence it is a linear function and is differentiable with

$$DF(x_0, y_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Recall: A function

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto \big(u(x, y), v(x, y)\big) \end{aligned}$$

is differentiable at a point $P_0 = (x_0, y_0)$ if it exists a linear map $dF : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{\substack{P \to P_0 \\ P \neq P_0}} \frac{\big\|F(P) - F(P_0) - dF(P - P_0)\big\|}{\|P - P_0\|} = 0$$

or equivalently

$$F(P) - F(P_0) = dF(P - P_0) + \Psi(P - P_0)\|P - P_0\|$$

with $\|\Psi(P - P_0)\| \xrightarrow{P \to P_0} 0$

The linear map $dF : \mathbb{R}^2 \to \mathbb{R}^2$ is unique and called **the differential of** F **at** P_0. In the standard basis of $\mathbb{R}^2$, dF is represented by the **Jacobian Matrix of** F, namely DF:

$$DF(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Recall: We can view $\mathbb{C}$ as a 1-dimensional vector space over $\mathbb{C}$ with basis $B_{\mathbb{C}} = \{1\}$ or as a 2-dimensional real vector space with basis $B_{\mathbb{R}} = \{1, i\}$

A map $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear if

$$T(z) = \lambda z$$

for some $\lambda \in \mathbb{C}$. On the other hand, a map $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear if

$$T(z) = T(x + iy) = xT(1) + yT(i) = \lambda z + \mu \bar{z}$$

with

$$\begin{aligned} \lambda &= \frac{1}{2}(T(1) - iT(i)) \\ \mu &= \frac{1}{2}(T(1) + iT(i)) \end{aligned}$$

using that $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$

Hence, every $\mathbb{C}$-linear map $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear, but a $\mathbb{R}$-linear map $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear only if $\mu = 0$, i.e. $T(i) = iT(1)$ (quick and fun to verify).

If $T(1) = a + ib$ and $T(i) = c + id$ for $a, b, c, d \in \mathbb{R}$, then $T(i) = iT(1) \implies b = -c$ and $a = d$

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ with $z = x + iy = \begin{pmatrix} x \\ y \end{pmatrix}$, since every $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$ is given by a 2×2 real matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix}$$

such a map is also $\mathbb{C}$-linear if A is of the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Remark 1.6. *Note that in Example 1.3, the function $f(z) = \bar{z}$ as map $\mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with Jacobian equal to*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

but which is not of the above form.

Our next goal is to see how this Linear Algebra fact about $\mathbb{R}$-linear versus $\mathbb{C}$-linear maps is reflected in the case of a linear function $f : \mathbb{C} \to \mathbb{C}$ and of its complex differentiability.

1.2.2 Cauchy-Riemann Equations

Let $f \in \mathbb{C}^{\mathbb{C}}$ be holomorphic at z_0. If $f(x + iy) = u + iv$, via some linking $\mathbb{R}$-linear isomorphism we can also view f as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, such as

$$\begin{aligned} \tilde{f} &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y) &\mapsto \big(u(x, y), v(x, y)\big) \end{aligned}$$

As specified in 1.1, we can identify the two spaces $\mathbb{R}^2$ and $\mathbb{C}$ as equal, hence $\tilde{f} \cong f \implies \tilde{f} = f$

This said, the derivative $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists independently of how $z \to z_0$

In particular, we can have z tending to z_0 along the line $z = x + iy_0$ by letting $x \to x_0$, hence

$$\begin{aligned} f'(z_0) &= \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} = \\ &= \lim_{x \to x_0} \frac{\tilde{f}(x, y_0) - \tilde{f}(x_0, y_0)}{x - x_0} = \\ &= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} = \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0) \end{aligned}$$

We can conclude from this that the usual partial derivatives $u_x(z_0), v_x(z_0)$ exist and hence also the partial derivative $f_x(z_0) = u_x(z_0) + iv_x(z_0)$ exists and that

$$\boxed{f'(z_0) = u_x(z_0) + iv_x(z_0) = f_x(z_0)}$$

On the other hand approaching $z_0 = x_0 + iy_0$ along $z = x_0 + iy$ with $y \to y_0$ gives

$$\begin{aligned} f'(z_0) &= \lim_{y\to y_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{y-y_0} = \\ &= \lim_{y\to y_0} \frac{\tilde{f}(x_0,y) - \tilde{f}(x_0,y_0)}{y-y_0} = \\ &= \lim_{x\to x_0} \frac{v(x_0,y) - v(x_0,y_0)}{y-y_0} - i \lim_{y\to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{y-y_0} = \\ &= u_y(x_0,y_0) - iv_y(x_0,y_0) = \\ &= v_y(z_0) - iu_y(z_0) \end{aligned}$$

We obtain that the partial derivatives $u_y(z_0), v_y(z_0)$ also exist together with $f_y(z_0) = u_y(z_0) + iv_y(z_0)$ and that

$$\boxed{f'(z_0) = v_y(z_0) - iu_y(z_0) = -if_y(z_0)}$$

By pulling together all previous results we obtain the **Cauchy-Riemann Equations (CR)**

$$\begin{aligned} u_x(z_0) &= v_y(z_0) \\ u_y(z_0) &= -v_x(z_0) \end{aligned}$$

If we introduce two differential operators

$$\begin{aligned} \frac{\partial}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{aligned}$$

What we have shown can be summarised in

Proposition 1.9. [SS10, Proposition I.2.3] If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0$$

and

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

If we write $f(z) = \tilde{f}(x,y)$, then $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto \big(u(x,y), v(x,y)\big)$ is complex differentiable with

$$D\tilde{f}(x_0, y_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

and

$$\det\left(D\tilde{f}(x_0, y_0)\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \left|f'(z_0)\right|^2 = u_x^2(z_0) + u_y^2(z_0) = u_x^2(z_0) + v_x^2(z_0)$$

Proof. Using the Cauchy-Riemann Equations we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}(u_x + iv_x + iu_y - v_y) = \frac{1}{2}\big((u_x - v_y) + i(v_x + u_y)\big) = 0$$

By summing together the Cauchy-Riemann Equations and dividing by 2, one can obtain

$$f'(z_0) = \frac{1}{2}\big(f_x(z_0) - if_y(z_0)\big)$$

These equations also give

$$\frac{\partial f}{\partial z}(z_0) = u_x + iv_x = 2\frac{1}{2}(u_x - iu_y) = 2\frac{\partial u}{\partial z}(z_0)$$

If $z_0 = x_0 + iy \in U$ and $h = h_1 + ih_2 \in \mathbb{C}$, then for f being holomorphic at z_0 means that

$$f(z_0 + h) = f(z_0) + f'(z_0)h + hE(h)$$

with $\lim_{h\to 0} E(h) = 0$. If $f'(z_0) = a + ib$, then

$$f'(z_0)h = (a + ib)(h_1 + ih_2) = ah_1 - bh_2 + i(bh_1 + ah_2)$$

Hence, if we write $\tilde{f}(x, y) = f(z)$ for $\tilde{h} = (h_1, h_2)^T$, we have that

$$\frac{\left|\tilde{f}\big((x_0, y_0) + (h_1, h_2)\big) - \tilde{f}(x_0, y_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right|}{|\tilde{h}|} \xrightarrow{|\tilde{h}|\to 0} 0$$

This means that $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with a $\mathbb{C}$-linear differential

$$D\tilde{f}(x_0, y_0)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Using $a = u_x = v_y$ and $b = v_x = -u_y$ we get

$$D\tilde{f}(x_0, y_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Consequently, computing the determinant results in

$$\det\left(D\tilde{f}(x_0, y_0)\right) = u_x^2 + v_x^2 = \left|f'(z_0)\right|^2$$

□

Remark 1.7. *Recall that we can represent any complex number $z = a + bi$ with a 2×2 real matrix:*

$$a + ib \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

If $f(z) = u(z) + iv(z)$ is complex differentiable at z_0, then $f'(z_0) = u_x(z_0) + iv_x(z_0)$ has matrix representation

$$f'(z_0) \longleftrightarrow \begin{pmatrix} u_x(z_0) & -v_x(z_0) \\ v_x(z_0) & u_x(z_0) \end{pmatrix}$$

On the other hand, the corresponding function

$$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto \big(u(x, y), v(x, y)\big)$$

has a Jacobian matrix

$$D\tilde{f}(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Comparing these two matrices, we get exactly the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}$$

If one remembers the general form of the Jacobian of a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ and the matrix representation of a complex number, then can remember the Cauchy-Riemann equations.

The previous Proposition 1.9 shows that f holomorphic implies that $\frac{\partial f}{\partial \bar{z}} = 0$ using the Cauchy-Riemann Equations, hence that f satisfies the same Cauchy-Riemann Equations. We also have the following partial converse.

Theorem 1.1. [SS10, Theorem I.2.4] Suppose $f = u + iv \in \mathbb{C}^\Omega$ for an open set $\Omega \subseteq \mathbb{C}$. If $u, v \in C^1(\Omega; \mathbb{R})$ and satisfy the Cauchy-Riemann Equations (CR) in Ω, then $f \in \mathscr{H}(\Omega)$ and

$$f'(z) = \frac{df}{dz}(z) = \frac{\partial f}{\partial z}(z)$$

Proof. Let $z_0 = x_0 + iy_0 \in \Omega$ and $h = h_1 + ih_2 \in \mathbb{C}$. Having $u, v \in C^1(\Omega; \mathbb{R})$ implies that

$$u(z_0 + h) - u(z_0) = \partial_x u(z_0) h_1 + \partial_y u(z_0) h_2 + |h| \varepsilon_1(h)$$

with $\varepsilon_1(h) \xrightarrow{h \to 0} 0$. Similarly,

$$v(z_0 + h) - v(z_0) = \partial_x v(z_0) h_1 + \partial_y v(z_0) h_2 + |h| \varepsilon_2(h)$$

with $\varepsilon_2(h) \xrightarrow{h\to 0} 0$. By then summing these two together we obtain

$$\begin{aligned} f(z_0+h)-f(z_0) &= (u+iv)(z_0+h)-(u+iv)z_0 = \\ &= (\partial_x u + i\partial_x v)h_1 + (\partial_y u + i\partial_y v)h_2 + |h|\varepsilon(h) \end{aligned}$$

where $\varepsilon(h) = (\varepsilon_1+\varepsilon_2)(h) \xrightarrow{h\to 0} 0$. Using now the Cauchy-Riemann Equations we can reform the previous expansion in what follows

$$\begin{aligned} f(z_0+h)-f(z_0) &= (\partial_x u - i\partial_y u)h_1 + (\partial_y u + i\partial_x u)h_2 + |h|\varepsilon(h) = \\ &= (\partial_x u - i\partial_y u)(h_1+ih_2) + |h|\varepsilon(h) \end{aligned}$$

Hence $f(z_0+h)-f(z_0) = (\partial_x u - i\partial_y u)h + |h|\varepsilon(h)$ where $\varepsilon(h) \xrightarrow{h\to 0} 0$

This says that

$$\frac{f(z_0+h)-f(z_0)}{h} \xrightarrow{h\to 0} \partial_x u - i\partial_y u$$

Hence $f'(z_0)$ exists and is equal to $\partial_x u - i\partial_y u = 2\partial_z u = \partial_z f$ □

Example 1.4. *Let $f \in \mathbb{C}^{\mathbb{C}}$ such that $z \mapsto f(z) = x^2+y^2+2ixy$, considering that $u(z) = x^2+y^2 \in \mathbb{R}$ and $v(z) = 2xy \in \mathbb{R}$. We analyse the partial derivatives to see whether and where they satisfy the Cauchy-Riemann equations:*

$$\begin{array}{ll} \partial_x u(z) = 2x & \partial_x v(z) = 2y \\ \partial_y u(z) = 2y & \partial_y v(z) = 2x \end{array}$$

It holds that for all $z \in \mathbb{C}$ we have $\partial_x u = 2x = \partial_y v$, while $\partial_y u = 2y = -2y = -\partial_x v$ is only true if $y = 0$. f therefore satisfies the Cauchy-Riemann equations only when $Im(z) = 0$. Hence, f is holomorphic only for points on the real axis. For these points

$$f'(x_0) = \partial_x u(x_0) + i\partial_x v(x_0) = 2x_0$$

Remark 1.8. *Many books distinguish between complex differentiability at a point and holomorphicity at a point as follows:*

- *Complex differentiability at a point is given, if the limit in 1.15 exists at that point.*
- *Holomorphicity at a point is instead given instead when the limit in 1.15 exists in a neighbourhood of that point.*

[SS10] does not make such distinction (and in this course we will not need it).

A quick summary

Let Ω be an open subset of $\mathbb{C}$, then

1. For $f \in \mathbb{C}^\Omega$ with $f(z) = u(z) + iv(z)$ we have that if f is holomorphic on Ω, then u, v satisfy the Cauchy-Riemann Equations $u_x = v_y$ and $u_y = -v_x$. Moreover it holds that
$$f'(z) = u_x(z) - iu_y(z)$$
2. If $u, v \in C^1(\Omega; \mathbb{C})$ and satisfy the Cauchy-Riemann Equations, then $f = u + iv$ is holomorphic.
3. If we write $\tilde{f}(x, y) = f(z)$, for $f \in \mathscr{H}(\Omega)$ and identifying $\mathbb{C}$ with $\mathbb{R}^2$, then $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with
$$D\tilde{f}(x_0, y_0) = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} \text{ and } \det\left(D\tilde{f}(x_0, y_0)\right) = \left|f'(z_0)\right|^2$$

Remark 1.9. *A matrix of the form* $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *defines a linear map*

$$\begin{aligned} L : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

which preserves angles and orientation, i.e. it is a rotation and a dilation. If $a + ib \neq 0$ *with* $a + ib = |a + ib|e^{i\theta}$*, then it is a rotation by the angle* θ *and a dilation by* $|a + ib|$

Our next result gives important examples of holomorphic functions.

1.2.3 Power series

Recall: A (complex-) power series is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$

Theorem 1.2. [SS10, Theorem I.2.5] Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series. Then $\exists R \in [0, +\infty]$ such that

(i) if $|z| < R$, the series converges absolutely.

(ii) if $|z| > R$, the series diverges.

Moreover, with the convention that $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$, R is given by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

in this case R is called **the radius of convergence** and

$$D_R(0) := \{z \in \mathbb{C} : |z| < R\}$$

is called **disc of convergence** (or **region of convergence**).

Proof. Exercise (as in Real Analysis). □

Example 1.5 (Exponential function)**.** *An important example of a power series is the complex exponential function*

$$\boxed{e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}}$$

This series converges absolutely for all $z \in \mathbb{C}$. Also,

$$|e^z| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$$

Hence, e^z convergence uniformly on compact subsets of $\mathbb{C}$

The following Theorem shows that e^z in particular and power series in general give examples of holomorphic functions in their disc of convergence.

Theorem 1.3. [SS10, Theorem I.2.6] The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence, i.e. $f \in \mathscr{H}\big(D_R(0)\big)$, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Moreover, f' has the same radius of convergence as f, i.e. $f' \in \mathscr{H}\big(D_R(0)\big)$

Proof. Let R be the radius of convergence of f, since

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

then it holds that[1]

$$\limsup_{n\to\infty} |na_n|^{\frac{1}{n}} = 1 \cdot \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = R$$

[1]If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two sequences of non-negative real numbers and $b := \lim_{n\to\infty} b_n$ is the limit of the second, then $\limsup_{n\to\infty} a_n b_b = (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$. In general, lim sup is submultiplicative for sequences of non-negative real numbers.

Hence , $\sum_{n=1}^{\infty} na_n z^{n-1}$ has the same radius of convergence. Repeated applications of this same argument show that the sum

$$\sum_{n=k+1}^{\infty} \frac{n!}{(n-k-1)!} a_n z^{n-k}$$

has radius of convergence R for any $k \in \mathbb{N}$

We now want to show that $\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1}\right| \xrightarrow{h\to 0} 0$ in the following way.

Let now $z \in D_R(0) \subseteq \mathbb{C}$ and choose $\delta > 0$ such that $|z| + \delta < R$, e.g. one can take $\delta = \frac{R-|z|}{2}$, with $h \in \mathbb{C}$ such that $|h| < \delta$, then

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1}\right| &= \left|\sum_{n=1}^{\infty}\left(\frac{a_n(z+h)^n - a_n z^n}{h} - na_n z^{n-1}\right)\right| \leq \\
&\leq \sum_{n=1}^{\infty} |a_n| \left|\frac{1}{h}\left(\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{B}in} h^k z^{n-k}\right) - z^n\right) - nz^{n-1}\right| = \\
&= \sum_{n=2}^{\infty} |a_n| \left|\sum_{k=2}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{B}in} h^{k-1} z^{n-k}\right| \leq \\
&\leq \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{B}in} |h|^{k-1}|z|^{n-k} \leq \\
&\overset{(*^1)}{\leq} \sum_{n=2}^{\infty} |a_n| n(n-1) \sum_{k=2}^{n} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{\mathcal{B}in} |h|^{k-2}|z|^{n-k}|h| \leq \\
&\leq \sum_{n=2}^{\infty} |a_n| n(n-1)\big(|z| + |h|\big)^{n-2}|h| \leq \\
&\overset{(*^2)}{\leq} |h| \underbrace{\sum_{n=2}^{\infty} |a_n| n(n-1)\left(\frac{R+|z|}{2}\right)^{n-2}}_{\text{independent of } h}
\end{aligned}$$

using in the $(*^1)$-step that $\forall k \geq 2 : \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{B}in} = \frac{n}{k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\mathcal{B}in} = \frac{n(n-1)}{k(k-1)}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{\mathcal{B}in} \leq n(n-1)\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{\mathcal{B}in}$ and using lastly in the $(*^2)$-step that $|h| < \frac{R-|z|}{2}$

We therefore obtain

$$|h| \sum_{n=2}^{\infty} |a_n| n(n-1)\left(\frac{R+|z|}{2}\right)^{n-2} \xrightarrow{h\to 0} 0$$

and so we finally get that

$$\frac{f(z+h)-f(z)}{h} \xrightarrow{h\to 0} \sum_{n=1}^{\infty} n a_n z^{n-1}$$

which concludes the proof. □

Example 1.6. *1. The exponential function*

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C} \\ z \mapsto \exp(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{n!} \end{aligned}$$

converges on all of $\mathbb{C}$ *and as such is holomorphic on all of* $\mathbb{C}$*, moreover we can easily extract*

$$\exp'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z)$$

2. Trigonometric functions

$$\boxed{\begin{aligned} \sin(z) &:= \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ \cos(z) &:= \frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \end{aligned}}$$

with

$$\sin(i) = \frac{e^{i^2}-e^{-i^2}}{2i} = i\left(\frac{e^2-1}{2e}\right)$$

and

$$\cos(i) = \frac{e^{i^2}+e^{-i^2}}{2} = \frac{e^{-1}+e}{2} = \frac{e^2+1}{2e}$$

For the records, these functions are not bounded.

3. The series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

has convergence radius 1*, i.e. it converges for all* $z \in D_1(0)$*, since* $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

4. The geometric series

$$\sum_{n=0}^{\infty} z^n$$

converges for $z \in D_1(0)$

5. *The series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$$

converges for all $z \in D_1(0)$. Moreover, for $z = 1$ the series converges (Leibniz' criteria), while for $z = -1$ the series diverges (Harmonic series).

1.3 Complex Line Integrals (Integrals along curves)

We start by recalling the main definitions and properties of curves.

Definition 1.17.

- A **parametrised curve in** $\mathbb{C}$ is a continuous function $\gamma : [a,b] \to \mathbb{C}$, i.e. $\gamma \in C^0\big([a,b]\big)$, where $[a,b]$ is a closed interval in $\mathbb{R}$
- A **smooth curve** is a curve

  $$\begin{aligned}\gamma : [a,b] &\to \mathbb{C}\\ t &\mapsto \gamma(t) = x(t) + iy(t)\end{aligned}$$

  such that its derivative

  $$\gamma'(t) = x'(t) + iy'(t)$$

  exists for all $t \in [a,b]$, $\gamma \in C^1\big([a,b]\big)$ and $\gamma'(t) \neq 0$ for all $t \in [a,b]$
  Here, we consider

  $$\gamma'(a) := \lim_{h \searrow 0} \frac{\gamma(a+h) - \gamma(a)}{h}$$

  and

  $$\gamma'(b) := \lim_{h \nearrow 0} \frac{\gamma(b+h) - \gamma(b)}{h}$$

  as the right and left derivatives respectively.
- A **piecewise smooth curve** is a curve $\gamma : [a,b] \to \mathbb{C}$ such that γ is continuous on $[a,b]$, i.e. $\gamma \in C^0\big([a,b]\big)$, and exist points

  $$a = a_0 < a_1 < ... < a_n = b$$

  such that γ is smooth on each interval $[a_k, a_{k+1}]$
- A **closed curve** is a curve $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$
- A curve is **simple** if it's not self intersecting, i.e. $\gamma(t) \neq \gamma(s)$ unless $s = t$ or $s = a$ and $t = b$

- $\tilde{\gamma}$ is called **reparametrisation** of $\gamma : [a,b] \to \mathbb{C}$ if there exists a continuously differentiable bijective function $\sigma \in C^1\big([c,d];[a,b]\big)$ with $\forall t \in [c,d] : \sigma'(t) > 0$ and with $\tilde{\gamma} = \gamma \circ \sigma$ (the condition $\sigma' > 0$ means that the orientation is preserved), i.e. γ and $\tilde{\gamma}$ represent the same geometric object with different parametrisation.

Remark 1.10. *For us in this course the curves will always be piecewise smooth. From now on when we say "a curve" we mean "a piecewise smooth one", even if we forget to write it.*

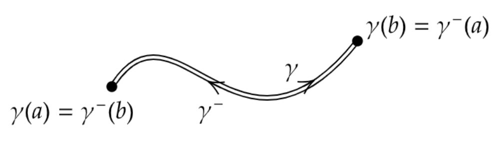
Remark 1.11. *We will often work with a particular parametrisation, since most important notions will be independent of parametrisation (for example path integrals). Because of this independence, we often describe curves by drawing them as geometric objects in the plane.*

There are two elementary methods to modify or combine paths in order to obtain new paths.

Definition 1.18.
- If $\gamma : [a,b] \to \mathbb{C}, t \mapsto \gamma(t)$ is a path, **the reverse path** γ^- is the path

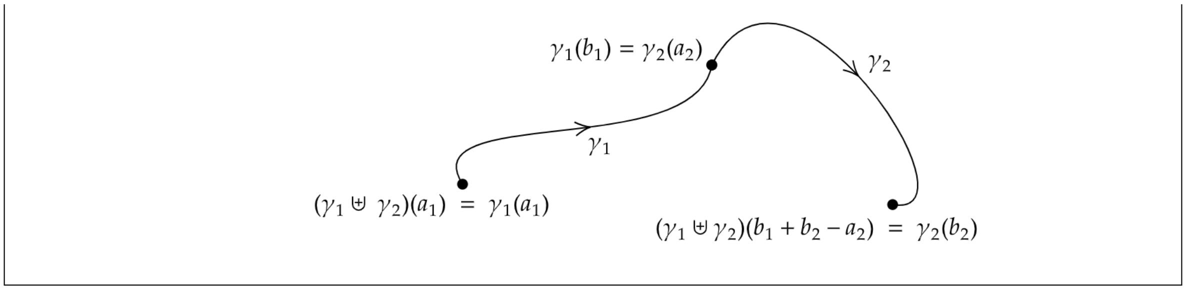
$$\begin{aligned}\gamma^- : [a,b] &\to \mathbb{C}\\ t &\mapsto \gamma^-(t) := \gamma(b+a-t)\end{aligned}$$

i.e. $\gamma^-(t) = \gamma(a+b-t)$



- If $\gamma_1 : [a_1,b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2,b_2] \to \mathbb{C}$ are two paths such that $\gamma_1(b_1) = \gamma_2(a_2)$, then **the concatenation** or **sum of the paths** γ_1, γ_2 is a path

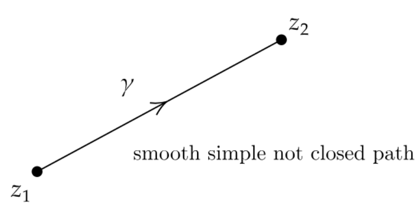
$$\begin{aligned}\gamma_1 \uplus \gamma_2 : [a_1, b_1+b_2-a_2] &\to \mathbb{C}\\ t \mapsto (\gamma_1 \uplus \gamma_2)(t) &:= \begin{cases} \gamma_1(t) & , t \in [a_1,b_1] \\ \gamma_2(t-b_1+a_2) & , t \in [b_1, b_1+b_2-a_2] \end{cases}\end{aligned}$$



Example 1.7. *1. Given two points $z_1, z_2 \in \mathbb{C}$, the path*

$$\begin{aligned} \gamma : [0,1] &\to \mathbb{C} \\ t &\mapsto (1-t)z_1 + tz_2 \end{aligned}$$

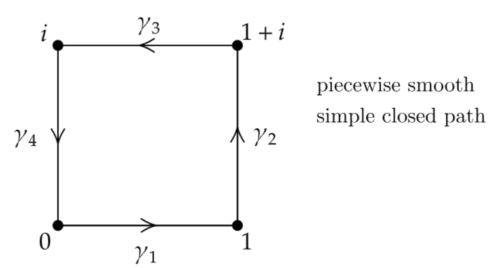
is the (standard) parametrisation of the line segment between z_1 and z_2



2. Consider the path

$$\begin{aligned} \gamma : [0,4] &\to \mathbb{C} \\ t \mapsto \gamma(t) &:= \begin{cases} t & , t \in [0,1] \\ 1 + i(t-1) & , t \in [1,2] \\ (3-t) + i & , t \in [2,3] \\ i(4-t) & , t \in [3,4] \end{cases} \end{aligned}$$

as shown in the picture

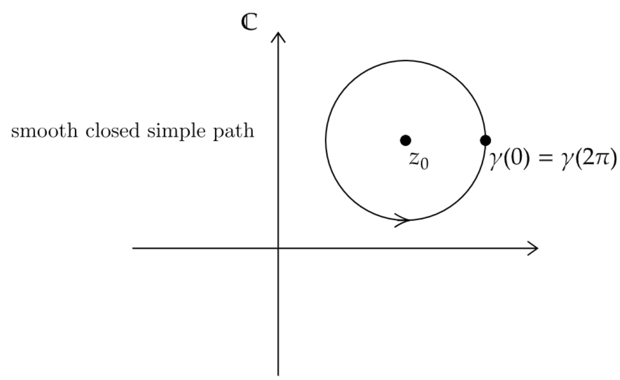


It is clear that γ is the sum of four different paths, namely

$$\begin{aligned} \gamma_1 &: [0,1] \to \mathbb{C}, t \mapsto \gamma_1(t) = t \\ \gamma_2 &: [0,1] \to \mathbb{C}, t \mapsto \gamma_2(t) = 1 + it \\ \gamma_3 &: [0,1] \to \mathbb{C}, t \mapsto \gamma_3(t) = i + (1-t) \\ \gamma_4 &: [0,1] \to \mathbb{C}, t \mapsto \gamma_4(t) = i(1-t) \end{aligned}$$

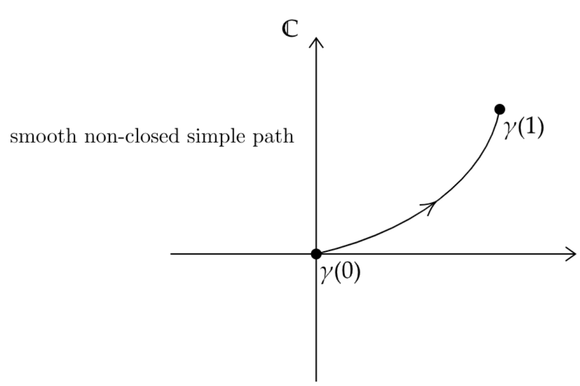
3. *A circle with center at z_0 and radius r has a parametrisation of the form*

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{C} \\ t &\mapsto z_0 + re^{it} \end{aligned}$$



4. *Consider also the path*

$$\begin{aligned} \gamma : [0, 1] &\to \mathbb{C} \\ t &\mapsto t + it^2 \end{aligned}$$



To define the complex line integrals we recall that a continuous function $g : [a, b] \to \mathbb{R}$ is Riemann integrable, i.e. $\int_a^b g(t)dt$ exists.

Definition 1.19. For a complex valued function $g \in \mathbb{C}^{[a,b]}$ we can define the integral

$$\int_a^b g(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$$

where $\forall t \in [a,b] : g(t) = u(t) + iv(t)$

Definition 1.20. Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path and $f \in \mathbb{C}^{\Omega}$ is a complex valued function, which is defined and continuous on γ. We define **the integral of** f **along** γ by

$$\int_\gamma f(z)dz := \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt$$

Since $g(t) = f\big(\gamma(t)\big)\gamma'(t) \in \mathbb{C}^{[a,b]}$ is continuous on $[a,b]$, the integral on the right is meaningful, as long as we show that it is independent of the parametrisation of γ

Let $\tilde{\gamma} : [c,d] \to \mathbb{C}$ be another parametrisation of $im(\gamma)$, such that $\tilde{\gamma}(s) = (\gamma \circ \sigma)(s)$ for some $\sigma : [c,d] \to [a,b]$ with $\sigma \in C^1\big([c,d],[a,b]\big)$ and $\sigma'(s) > 0$. Then we have

$$\int_{\tilde{\gamma}} f(z)dz = \int_c^d f\big(\tilde{\gamma}(s)\big)\tilde{\gamma}'(s)ds = \int_c^d f\Big(\gamma\big(\sigma(s)\big)\Big)\gamma'\big(\sigma(s)\big)\sigma'(s)ds$$

By letting $t = \sigma(s)$ and consequently $dt = \sigma'(s)ds$, we obtain

$$\int_c^d f\Big(\gamma\big(\sigma(s)\big)\Big)\gamma'\big(\sigma(s)\big)\sigma'(s)ds = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt = \int_\gamma f(z)dz$$

The following properties of path integrals follow from the properties of the Riemann integral.

Proposition 1.10. [SS10, Proposition I.3.1] Let $f, g \in C^0(\Omega;\mathbb{C})$, $\gamma, \gamma_1, \gamma_2$ piecewise smooth curves in Ω and $a, b \in \mathbb{C}$. Then

(i) The path integral is linear, i.e.

$$\int_\gamma \big(af(z) + bg(z)\big)dz = a\int_\gamma f(z)dz + b\int_\gamma g(z)dz$$

(ii) If γ^- is the curve γ with reverse orientation, then

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz$$

(iii) The path of integration can be split

$$\int_{\gamma_1 \uplus \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

(iv) The following estimate holds

$$\left| \int_{\gamma} f(z)dz \right| \leq L_\gamma \sup_{z \in im(\gamma)} \left| f(z) \right|$$

where for a partition $a_0 \leq ... \leq a_n$ of the interval $[a, b]$ we have

$$L_\gamma = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \left| \gamma'(t) \right| dt$$

Proof. **(i)** Follows from the linearity of the Riemann integral.

(ii) If $\gamma : [a, b] \to \mathbb{C}$, then $\gamma^- : [a, b] \to \mathbb{C}, t \mapsto \gamma(b+a-t)$ with $(\gamma^-)'(t) = -\gamma'(b+a-t)$. Hence, we have

$$\int_{\gamma^-} f(z)dz = -\int_a^b f\big(\gamma(b+a-t)\big)\gamma'(b+a-t)dt = \int_b^a f\big(\gamma(u)\big)\gamma'(u)du =$$
$$= -\int_a^b f\big(\gamma(u)\big)\gamma'(u)du = -\int_\gamma f(z)dz$$

where we used that $u = b + a - t$ and consequently that $du = -dt$

(iii) Exercise.

(iv) Consider the following steps:

$$\left| \int_\gamma f(z)dz \right| = \left| \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f\big(\gamma(t)\big)\gamma'(t)dt \right| \leq \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \left| f\big(\gamma(t)\big) \right| \left| \gamma'(t) \right| dt \leq$$
$$\leq \left(\sup_{t \in [a,b]} \left| f\big(\gamma(t)\big) \right| \right) \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \left| \gamma'(t) \right| dt$$

□

Remark 1.12. *If $S \subseteq \mathbb{C}$ is a set that can be described as the image of a path γ in $\mathbb{C}$, we will then often denote the integral of a function f along this path with $\int_S f dz$*

Definition 1.21 (Primitive)**.** Let $f \in \mathbb{C}^{\Omega}$. **A primitive of** f **on** Ω is a function $F \in \mathscr{H}(\Omega)$ such that

$$\forall z \in \Omega : F'(z) = f(z)$$

The existence of a primitive gives

Theorem 1.4. [SS10, Theorem I.3.2] Let $f \in C^0(\Omega; \mathbb{C})$ be a continuous function on an open set $\Omega \subseteq \mathbb{C}$. If f has a primitive F in Ω and γ is a curve which begins at z_1 and ends at z_2, i.e. $\gamma : [a, b] \to \mathbb{C}$ with $im(\gamma) \subseteq \Omega$, $\gamma(a) = z_1$ and $\gamma(b) = z_2$, then

$$\int_\gamma f(z)dz = F(z_2) - F(z_1)$$

An immediate Corollary is

Corollary 1.1. [SS10, Corollary I.3.3] If γ is a closed curve (i.e. $\gamma(a) = \gamma(b)$) in an open set Ω, $f \in C^0(\Omega; \mathbb{C})$ and has a primitive in Ω, then

$$\int_\gamma f(z)dz = 0$$

Proof of Theorem 1.4. Let $F = U(x, y) + iV(x, y)$ and $\gamma : [a, b] \to \mathbb{C}$. We first assume that γ is smooth. We define a function

$$\begin{aligned} G : [a, b] &\to \mathbb{C} \\ t &\mapsto F\big(\gamma(t)\big) = F\big(x(t), y(t)\big) \end{aligned}$$

and write $\gamma(t) = x(t) + iy(t)$

We need to check the compatibility of the real derivative of G and the complex derivative of F.

We have that $G \in C^0\big([a, b]; \mathbb{C}\big)$ is a continuous function, hence

$$\begin{aligned} G'(t) &= \Big(U\big(x(t), y(t)\big) + iV\big(x(t), y(t)\big)\Big)' = \\ &= \Big(U_x\big(x(t), y(t)\big)x'(t) + U_y\big(x(t), y(t)\big)y'(t)\Big) + \\ &\quad + i\Big(V_x\big(x(t), y(t)\big)x'(t) + V_y\big(x(t), y(t)\big)y'(t)\Big) \end{aligned}$$

by using the Chain Rule from Vector Analysis [EW22]. Now, applying the Cauchy-Riemann Equations for F we get

$$\begin{aligned}
&= \Big(U_x\big(x(t),y(t)\big)x'(t) - V_x\big(x(t),y(t)\big)y'(t)\Big) + \\
&\quad + i\Big(V_x\big(x(t),y(t)\big)x'(t) + U_x\big(x(t),y(t)\big)y'(t)\Big) = \\
&= \Big(U_x\big(x(t),y(t)\big) + iV_x\big(x(t),y(t)\big)\Big)x'(t) + \\
&\quad + \Big(-V_x\big(x(t),y(t)\big) + iU_x\big(x(t),y(t)\big)\Big)y'(t) = \\
&= \Big(U_x\big(x(t),y(t)\big) + iV_x\big(x(t),y(t)\big)\Big)\big(x(t) + iy(t)\big) = \\
&= F'\big(\gamma(t)\big)\gamma'(t) = \\
&= f\big(\gamma(t)\big)\gamma'(t) \text{ , since } F \text{ is a primitive of } f \text{ on } \Omega
\end{aligned}$$

Hence, we finally get

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt = \int_a^b G'(t)dt \overset{(*)}{=} G(b) - G(a) = \\
&= F\big(\gamma(b)\big) - F\big(\gamma(a)\big) = F(z_2) - F(z_1)
\end{aligned}$$

where in the step denoted by $(*)$ we used the Fundamental Theorem of Analysis on G [EW22].
If γ is piecewise smooth, then there is a dissection of $[a,b]$ of the form

$$[a,b] = \bigcup_{\ell\in\{1,\dots,n\}} [a_{\ell-1}, a_\ell]$$

with $a =: a_0$ and $b =: a_n$ in accordance with the curve; this means such that for $\ell \in \{1,\dots,n\}$ we can define $\gamma_\ell := \gamma|_{[a_{\ell-1},a_\ell]}$ and dissect the curve as follows

$$\gamma = \biguplus_{\ell=1}^{n} \gamma_\ell$$

Then

$$\int_\gamma f(z)dz = \sum_{\ell=1}^{n} \int_{\gamma_\ell} f(z)dz = \sum_{\ell=1}^{n} F(a_\ell) - F(a_{\ell-1}) = F(a_n) - F(a_0) = F(b) - F(a)$$

using the newly obtained result for the smooth case in the third step and acknowledging the telescopic character of the sum in the second last one. □

Another Corollary of Theorem 1.4 is the following:

Corollary 1.2. [SS10, Corollary I.3.4] Let $f \in \mathscr{H}(\Omega)$ on an open and connected subset $\Omega \subseteq \mathbb{C}$. If $f' = 0$, then f is constant.

Proof. We want to show that for any two points $z, w \in \Omega$, it holds $f(z) = f(w)$

Since Ω is open and connected, there is a (polygonal) path $\gamma : [0,1] \to \Omega$ connecting the two points $z, w \in \Omega$, i.e. such that $\gamma(0) = z$ and $\gamma(1) = w$

Since f is holomorphic on Ω, f is clearly a primitive of f'. We can hence use the Theorem 1.4 on f' to obtain

$$\int_\gamma f'(z)dz = f\big(\gamma(1)\big) - f\big(\gamma(0)\big) = f(w) - f(z)$$

But given that $f' = 0$, the integral on the left is equal to zero and therefore $f(z) = f(w)$ The arbitrariness of the choice of $z, w \in \Omega$ concludes the proof. $\square$

Example 1.8. *An important example is the function*

$$\begin{aligned} f : \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto \frac{1}{z} \end{aligned}$$

f has no primitive on $\mathbb{C}^*$. To see this let γ parametrise the circle centred at 0 and of radius 1, namely $C_1(0)$, i.e. $\gamma : [0, 2\pi] \to \mathbb{C}, t \mapsto e^{it}$, then

$$\int_\gamma f(z)dz = \int_0^{2\pi} f\big(e^{it}\big)ie^{it}dt = \int_0^{2\pi} \frac{1}{e^{it}}ie^{it}dt = i\int_0^{2\pi} dt = 2\pi i \neq 0$$

Example 1.9. *What is $\int_\gamma z^2dz$, if $\gamma : [0,1] \to \mathbb{C}, t \mapsto t + \pi i t^2$?*

Since $F(z) = \frac{z^3}{3}$ is a primitive of z^2, using Theorem 1.4 we have that

$$\int_\gamma z^2dz = F\big(\gamma(1)\big) - F\big(\gamma(0)\big) = \frac{(1+\pi i)^3}{3}$$

or

$$\int_\gamma z^2dz = \int_0^1 (t + \pi i t^2)^2(1 + 2\pi i t)dt = ...$$

which is much longer.

Chapter 2

Cauchy's Theorem and its applications

Cauchy's Theorem is at the heart of Complex Analysis: it "roughly" says that if f is holomorphic in an open set Ω and $im(\gamma) \subseteq \Omega$ is a closed (not necessarily simple) curve, whose "interior" is contained in Ω, then

$$\int_\gamma f(z)dz = 0$$

Cauchy's Theorem, as we will see, has many applications, e.g. Liouville's Theorem, which in return gives a proof of Fundamental Theorem of Algebra.

The interior of a path is not easy to define for a general curve. We will work around this difficulty by first proving Cauchy's Theorem for curves, whose interior is easy to define, namely for triangles and rectangles (Goursat's Theorem).

We then use Goursat's Theorem to show that a holomorphic function on an open disc has a primitive in that disc. This then will give us a Corollary: Cauchy's Theorem on a disc.

We first need the following Proposition about nested compact sets, but before that we define a useful notational tool to approach this type of scenarios, namely the diameter of a set.

Definition 2.1. Let $S \subseteq \mathbb{C}$, then we define **the diameter of** $S \subseteq \mathbb{C}$ as

$$\operatorname{diam}(S) := \sup_{a,b \in S} |a - b|$$

Proposition 2.1. [SS10, Proposition I.1.4] If

$$\Delta^{(1)} \supseteq \Delta^{(2)} \supseteq ... \supseteq \Delta^{(n)} \supseteq ...$$

is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that

$$\operatorname{diam}\left(\Delta^{(n)}\right) \xrightarrow{n\to\infty} 0$$

Then

$$\exists! z_0 \in \mathbb{C} \forall n \in \mathbb{N}^* : z_0 \in \Delta^{(n)}$$

Proof. Choose z_n any point in $\Delta^{(n)}$. For any $n, m \in \mathbb{N}^*$ such that $n \geq m \geq 1$ we have

$$|z_n - z_m| \leq \operatorname{diam}\left(\Delta^{(m)}\right) < +\infty$$

Since $\operatorname{diam}\left(\Delta^{(n)}\right) \xrightarrow{n\to\infty} 0$, this says that $(z_n)_{n\in\mathbb{N}^*} \in \prod_{n\in\mathbb{N}^*} \Delta^{(n)}$ is a Cauchy sequence, hence it converges to a limit $z_0 \in \mathbb{C}$. Then for $m \geq 1$, note that $\forall n \geq m : z_n \in \Delta^{(m)}$ since $z_n \in \Delta^{(n)} \subseteq \Delta^{(m)}$. Moreover, we have that $\Delta^{(m)}$ is compact, in particular closed and hence $\lim_{n\to\infty} z_n = z_0$ is also in $\Delta^{(m)}$ for all $m \geq 1$ and z_0 is unique; since if $z_0, z_0' \in \Delta^{(n)}$ for all $n \in \mathbb{N}^*$ and $z_0 \neq z_0'$, then $|z_0 - z_0'| > 0$, which contradicts the shrinking $\operatorname{diam}\left(\Delta^{(n)}\right) \xrightarrow{n\to\infty} 0$ □

2.1 Goursat's Theorem

Theorem 2.1 (Goursat's Theorem)**.** [SS10, Theorem II.1.1] Let $\Omega \subseteq \mathbb{C}$ be open and T a path with shape of a triangle such that $im(T) \subseteq \Omega$, whose interior is also contained in Ω. Let $f \in \mathscr{H}(\Omega)$, then

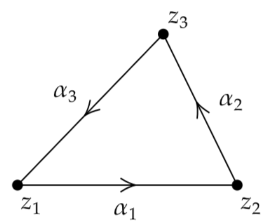
$$\int_T f(z)dz = 0$$

Proof. Note that a triangle is a closed curve, which is the union of three line segments. If T has three corners at z_1, z_2 and z_3, then we will write

$$T = \biguplus_{l=1}^{3} \alpha_l =: \langle z_1, z_2, z_3 \rangle$$

(Note that the sign $\biguplus$ does NOT represent a "flipped gravestone" ∼ Greta from Locarno[1])

[1]**Locarno** is a southern Swiss town and municipality in the district Locarno (of which it is the



with

$$\alpha_1 : [0,1] \to \mathbb{C}, t \mapsto \alpha_1(t) = z_1 + (t-0)(z_2 - z_1)$$
$$\alpha_2 : [0,1] \to \mathbb{C}, t \mapsto \alpha_2(t) = z_2 + (t-1)(z_3 - z_2)$$
$$\alpha_3 : [0,1] \to \mathbb{C}, t \mapsto \alpha_3(t) = z_3 + (t-2)(z_1 - z_3)$$

We now define the simplex

$$\Delta := \{z \in \mathbb{C} : z = t_1 z_1 + t_2 z_2 + t_3 z_3 \text{ and } 0 \leq t_1, t_2, t_3 \text{ and } t_1 + t_2 + t_3 = 1\}$$

such that it is the smallest convex set containing z_1, z_2, z_3. We hence have that $im(T) \subseteq \Delta$, in fact $im(T) = \partial\Delta$

We will inductively construct a sequence of triangular paths, so we define for all $n \in \mathbb{N}$

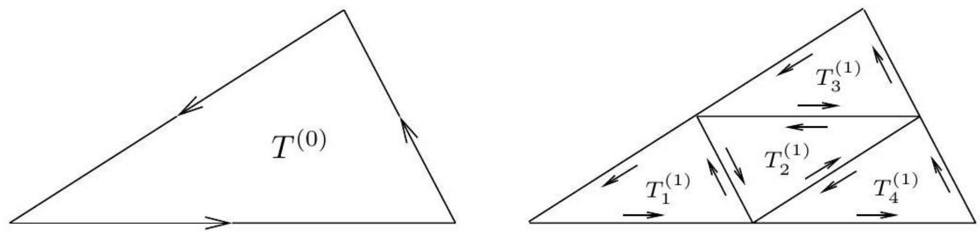
$$T^{(n)} = \langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \rangle$$

as follows:

1. Let $T^{(0)} = \langle z_1^{(0)}, z_2^{(0)}, z_3^{(0)} \rangle = \biguplus_{l=1}^{3} \alpha_l^{(0)}$
2. Assume that $T^{(n)}$ is defined as $T^{(n)} = \langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \rangle$, then $T^{(n+1)}$ is one of the the four triangular paths:

$$T_1^{(n+1)} := \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, z_2^{(n)}, \frac{z_2^{(n)} + z_3^{(n)}}{2} \right\rangle$$
$$T_2^{(n+1)} := \left\langle \frac{z_2^{(n)} + z_3^{(n)}}{2}, z_3^{(n)}, \frac{z_3^{(n)} + z_1^{(n)}}{2} \right\rangle$$
$$T_3^{(n+1)} := \left\langle \frac{z_3^{(n)} + z_1^{(n)}}{2}, z_1^{(n)}, \frac{z_1^{(n)} + z_2^{(n)}}{2} \right\rangle$$
$$T_4^{(n+1)} := \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, \frac{z_2^{(n)} + z_3^{(n)}}{2}, \frac{z_3^{(n)} + z_1^{(n)}}{2} \right\rangle$$

capital), located on the northern shore of Lake Maggiore at its northeastern tip in the canton of Ticino at the southern foot of the Swiss Alps. It has a population of about 16,000 (proper), and about 56,000 for the agglomeration of the same name including Ascona besides other municipalities.



i.e. we are at each step partitioning Δ using lines parallel to the sides and passing through their midpoints.
The triangular paths $T_k^{(n)}$ with $k \in \{1, ..., 4\}$ are all entirely contained in Δ and we have that

$$\int_{T^{(n)}} f(z)dz = \sum_{k=1}^{4} \int_{T_k^{(n+1)}} f(z)dz$$

therefore we get

$$\left| \int_{T^{(n)}} f(z)dz \right| \leq 4 \max_{k \in \{1,...,4\}} \left| \int_{T_k^{(n+1)}} f(z)dz \right|$$

We choose $T^{(n+1)}$ as one of $T_k^{(n+1)}$ with $k \in \{1, ..., 4\}$ so that

$$\left| \int_{T^{(n)}} f(z)dz \right| \leq 4 \left| \int_{T^{(n+1)}} f(z)dz \right|$$

and it then follows that

$$\left| \int_{T} f(z)dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)dz \right|$$

We also have that the closed filled triangles $\Delta^{(n)}$, defined with respect to their $T^{(n)}$ in a similar fashion as Δ with T, are nested compact sets

$$\mathbb{C} \supseteq \Delta = \Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq ...$$

Moreover, if d_n and P_n are the diameter and the perimeter of $\Delta^{(n)}$ respectively, then for all $n \in \mathbb{N}$ we have

$$d_n = \frac{d_0}{2^n} =: \frac{d}{2^n} \quad \text{and} \quad P_n = \frac{P_0}{2^n} =: \frac{P}{2^n}$$

Hence

$$\operatorname{diam}\left(\Delta^{(n)}\right) \xrightarrow{n \to \infty} 0$$

We can now apply the result anticipated by Proposition 2.1. In order to do so, though, we remember that f is holomorphic at $z_0 \in \Delta^{(n)} \subseteq \Omega$, hence it holds that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z)$$

with $\lim_{z\to z_0} \frac{|E(z)|}{|z-z_0|} = 0$. It is clear that $E(z)$ is continuous at z_0 and that $\lim_{z\to z_0} E(z) = 0$

Let $\varepsilon > 0$, we will show that $I := \left|\int_T f(z)dz\right| \leq \varepsilon P d$, which will show that $\int_T f(z)dz = 0$

To this end, we use the above equation for

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \big(f(z_0) + f'(z_0)(z - z_0)\big)dz + \int_{T^{(n)}} E(z)dz$$

Using Corollary 1.1, the first integral on the right is zero, since $g(z) = f(z_0)+f'(z_0)(z-z_0)$ has a primitive (take $G(z) = f(z_0)z + f'(z_0)\frac{(z-z_0)^2}{2}$ as such) and since the curve is closed.

Hence, we can reduce to the whole problem to the estimate

$$\left|\int_T f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} E(z)dz\right|$$

By letting

$$I_n := \left|\int_{T^{(n)}} E(z)dz\right|$$

because of the initial assumption on the asymptotic behaviour of E when approaching z_0, for the given $\varepsilon > 0$ we choose an open disc $D_\delta(z_0) \subseteq \Omega$ such that

$$\forall z \in D_\delta(z_0) : |E(z)| \leq \varepsilon |z - z_0|$$

Because $d_n \xrightarrow{n\to\infty} 0$, there exists an index $N \in \mathbb{N}$ such that $\forall n \geq N : d_n < \delta$. We also have that $z_0 \in \Delta^{(n)}$ for all $n \in \mathbb{N}$ and that $\forall n \geq N : |z - z_0| \leq d_n < \delta$

Hence, we conclude that $\forall n \geq \mathbb{N} : \Delta^{(n)} \subseteq D_\delta(z_0)$ and from it we get

$$\begin{aligned} |I| \leq 4^n |I_n| = 4^n \left|\int_{T^{(n)}} E(z)dz\right| \leq 4^n \int_{T^{(n)}} |E(z)||dz| \leq \\ \leq 4^n \varepsilon \int_{T^{(n)}} |z - z_0||dz| \leq 4^n \varepsilon P_n d_n = 4^n \varepsilon \frac{P}{2^n}\frac{d}{2^n} = \varepsilon P d \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain that $I = 0$ and conclude the proof. □

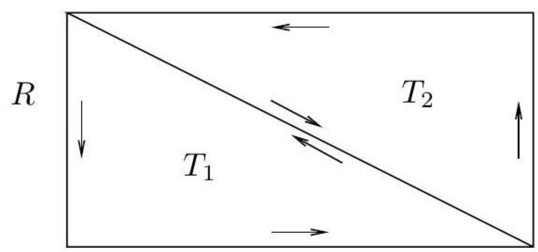
As a Corollary we get

Corollary 2.1. [SS10, Corollary II.1.2] If $f \in \mathscr{H}(\Omega)$ is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ that contains a path R with the shape of a rectangle and such that

$im(R) \subseteq \Omega$ as its interior, then

$$\int_R f(z)dz = 0$$

Proof. This follows immediately from the Goursat's Theorem 2.1 and by dividing the rectangle R into two triangles T_1, T_2 as shown in the picture.



Therefore, we obtain

$$\int_R f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz = 0$$

□

For future results, for example for the deduction of Cauchy's Integral Formula, a minor extension of this result is useful.

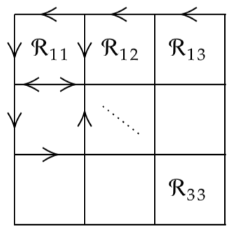
Theorem 2.2 (Goursat's Theorem stronger version)**.** If a function $f \in C^0(\Omega)$ is continuous in an open set Ω and $f|_{\Omega\setminus\{z_0\}} \in \mathscr{H}(\Omega \setminus \{z_0\})$ for some $z_0 \in \Omega$, then

$$\int_R f(z)dz = 0$$

for every closed rectangle $\mathcal{R} \subseteq \Omega$, i.e. such that $\partial\mathcal{R} = R$

Proof. Fix a closed triangle, in the topological sense, $\mathcal{R} \subseteq \Omega$. We assume that $z_0 \in \mathcal{R}$, otherwise the conclusion follows from the first version above as the integral is 0

Given a positive integer $n \in \mathbb{N}$ we subdivide $\mathcal{R}$ into n^2 congruent rectangles, such that $\partial\mathcal{R} = R$



Once again it follows that

$$\int_R f(z)dz = \sum_{l=1}^{n}\sum_{j=1}^{n}\int_{R_{lj}} f(z)dz$$

with $\partial\mathcal{R}_{lj} = R_{lj}$. If $z_0 \notin \mathcal{R}_{lj}$, then

$$\int_{R_{lj}} f(z)dz = 0$$

by the first version of Corollary 2.1. If instead $z_0 \in \mathcal{R}_{lj}$, then

$$\left|\int_{R_{lj}} f(z)dz\right| \leq MP_{R_{lj}} = M\frac{L}{n}$$

in which L is the length of the perimeter of R, $P_{R_{lj}} = \frac{L}{n}$ is the length of the perimeter of R_{lj} and $M = \max_{z\in\mathcal{R}} |f(z)| = \|f\|_{\infty,\mathcal{R}}$ is the maximum of the continuous function $|f|$ on the compact set $\mathcal{R}$

By construction, the point z_0 cannot belong to more than four subrectangles: the point in which touches most subrectangles is a common vertices of all the four. Hence we can finally consider the following estimate

$$\left|\int_R f(z)dz\right| = \left|\sum_{z_0\in\mathcal{R}_{lj}}\int_{\partial\mathcal{R}_{lj}} f(z)dz\right| \leq \sum_{z_0\in\mathcal{R}_{lj}}\left|\int_{R_{lj}} f(z)dz\right| \leq 4M\frac{L}{n} \xrightarrow{n\to\infty} 0$$

□

2.2 Local existence of primitives and Cauchy's theorem in a disc

To prove the Cauchy's Theorem in a disc, we will need the local existence of primitives. We therefore have the following Theorem:

Theorem 2.3. [SS10, Theorem II.2.1] A holomorphic function in an open disc D has a primitive in that disc.

We are going to prove the following version which assumes that f is continuous in D and that its integral along rectangles whose sides parallel to the coordinate axes vanish. Which then we are going to use to give a slightly stronger form of Cauchy's Theorem.

Theorem 2.4. Let D be an open disc in $\mathbb{C}$ and $f \in C^0(D)$ with the property that

$$\int_R f(z)dz = 0$$

for every closed rectangle $\mathcal{R} \subseteq D$ with $\partial\mathcal{R} = R$ in D and whose sides are parallel to the coordinate axes. Then f has a primitive in D

Before we prove Theorem 2.4, note that we have as a Corollary.

Theorem 2.5 (Cauchy's Theorem for a disc)**.** [SS10, Theorem II.2.2] Suppose $D \subseteq \mathbb{C}$ is an open disc in $\mathbb{C}$ and $f \in \mathscr{H}(D)$, or more generally $f \in C^0(D)$ with $f|_{D\setminus\{z_0\}} \in \mathscr{H}(D \setminus \{z_0\})$ for some $z_0 \in D$. Then

$$\int_\gamma f(z)dz = 0$$

for every closed piecewise smooth path γ in D

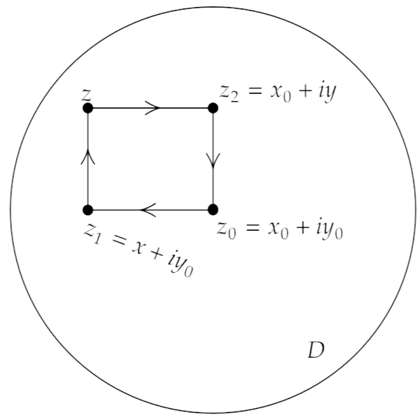
Proof. Suppose $f \in C^0(D)$ with $f|_{D\setminus\{z_0\}} \in \mathscr{H}(D \setminus \{z_0\})$ for some $z_0 \in D$, then by Goursat's Theorem 2.2

$$\int_R f(z)dz = 0$$

for every closed rectangle $\mathcal{R} \subseteq D$ with $\partial\mathcal{R} = R$ (including the ones, whose sides are parallel to axes). By Theorem 2.4 f has a primitive in D and by Theorem 1.4 and Corollary 1.1 we have that $\int_\gamma f(z)dz = 0$ for every piecewise smooth path γ in D □

Proof of Theorem 2.4. Let $f \in C^0(D)$ be continuous on the disc D and let $z_0 = x_0+iy_0$ be the center of D

For an arbitrary point $z = x+iy \in D$ such that $z \neq z_0$, let $z_1 := x+iy_0$ and $z_2 := x_0+iy$



By assumption

$$\int_{[z_0,z_1]} f(w)dw + \int_{[z_1,z]} f(w)dw + \int_{[z,z_2]} f(w)dw + \int_{[z_2,z_0]} f(w)dw = 0$$

where we denote the path along the line segment linking two points $p_1, p_2 \in D$, from p_1 to p_2, with $[p_1, p_2]$

This sum represent either $\int_R f(w)dw$ or $-\int_R f(w)dw$, depending on the location of z

We define $F \in \mathbb{C}^D$ as follows, for $z \in D$ let

$$F(z) := \int_{[z_0,z_2]} f(w)dw + \int_{[z_2,z]} f(w)dw$$

which is, by the above assumption

$$= \int_{[z_0,z_1]} f(w)dw + \int_{[z_1,z]} f(w)dw$$

Parametrizing the line segments we have

$$F(z) = i\int_{y_0}^{y} f(x_0 + it)dt + \int_{x_0}^{x} f(t + iy)dt$$

and

$$F(z) = \int_{x_0}^{x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + it)dt$$

Using both the above results and the Fundamental Theorem of Analysis [EW22]:

$$\frac{d}{dx}\int_a^x g(t)dt = g(x)$$

if $g \in C^0\big(D_r(a)\cap\mathbb{R};\mathbb{C}\big)$ with $g(t) = f(t+iy)$, then we have that $F_x(z) = f(x+iy) = f(z)$. Similarly, since

$$\frac{d}{dy}\int_a^y h(t)dt = h(y)$$

if $h \in C^0\big(D_r(a) \cap \mathbb{R}; \mathbb{C}\big)$ with $h(t) = f(x+it)$, we have $F_y(z) = if(x+iy) = if(z)$ (for both parts we used that the first integral in the equations are independent of x, y respectively). Hence, it follows that F_x, F_y exist and are continuous, so $F \in C^1(D)$
At this point, since $F_x(z) = f(z)$ and $F_y(z) = if(z)$, if we write $F(z) = u + iv$, then this gives

$$f(z) = F_x(z) = u_x + iv_x = -iF_y(z) = -i(u_y + iv_y) = v_y - iu_y$$

Hence $u_x = v_y$ and $v_x = -u_y$ and so the the Cauchy-Riemann equations hold.

Finally, we know that $F \in C^1(D)$ and that F satisfies that Cauchy Riemann Equations, therefore by Theorem 1.1 $F \in \mathscr{H}(\Omega)$ and

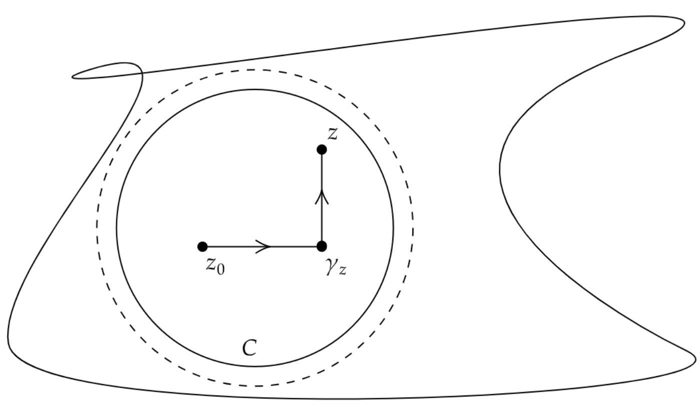
$$F'(z) = \frac{\partial F}{\partial x}(z) = f(z)$$

hence F is a primitive of f □

Corollary 2.2. [SS10, Corollary II.2.3] Let $f \in \mathscr{H}(\Omega)$ for $\Omega \subseteq \mathbb{C}$ an open set containing a circle C and its interior, then

$$\int_C f(z)dz = 0$$

Sketch of a proof. Simply consider $F(z) := \int_{\gamma_z} f(w)dw$ as in the following illustration.



□

Remark 2.1. *Corollary 2.2 is in fact valid whenever we can define the interior of a contour unambiguously and construct polygonal paths in an open neighbourhood of both the contour and its interior. In [SS10] these contours are called* ***toy contours****.*

Remark 2.2. *Note that Cauchy's Theorem 2.5 does not say anything about integrals of functions over arbitrary open sets and arbitrary closed curves. Indeed recall Example 1.8.*

Some applications of Cauchy's Theorem

We can use Cauchy's Theorem for a disc to calculate some integrals.

Example 2.1. *We can show by parametrizing the circle that for all $r > 0$ we have:*

$$\int_{C_r(z_0)} \frac{1}{z-z_0} dz = 2\pi i$$

Indeed, the circle of center z_0 and radius r has parametrization $\sigma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and hence

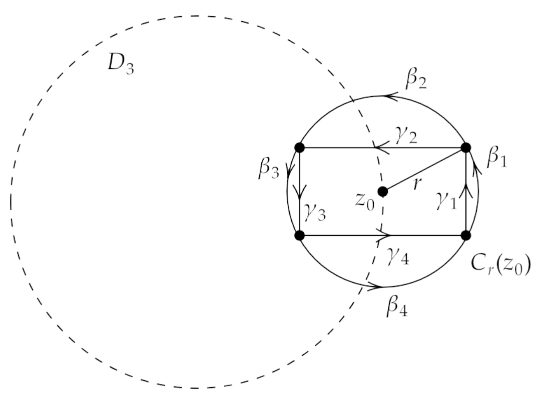
$$\int_{|w-z_0|=r} \frac{1}{z-z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i$$

Now, using Cauchy's Theorem 2.5 we can also show that

$$\int_R \frac{1}{z-z_0} dz = 2\pi i$$

for any rectangle R with center at z_0

Note that $\int_R \frac{1}{z-z_0} dz$ is not zero, since $\frac{1}{z-z_0}$ is not continuous at z_0 and hence Cauchy's Theorem 2.5 does not apply directly. We can though use it as follows:



Let $C_r(z_0)$ be the circle that circumscribes the rectangle R and

$$R = im\left(\biguplus_{k=1}^{4} \gamma_k\right) \qquad and \qquad C_r(z_0) = im\left(\biguplus_{k=1}^{4} \beta_k\right)$$

For each $k \in \{1, ..., 4\}$ we choose an open disc D_k, so that the trajectory of the closed path $\gamma_k \uplus \beta_k^-$ is in D_k and so that $f(z) = \frac{1}{z-z_0}$ is holomorphic in D_k

Now we apply Cauchy's Theorem 2.5 to $\frac{1}{z-z_0}$ *in disc* D_k *to* $\gamma_k \uplus \beta_k^-$ *in* D_k *to get*

$$\int_{\gamma_k \uplus \beta_k^-} f(z)dz = 0$$

From this follows that

$$\int_{\gamma_k} \frac{1}{z-z_0}dz = \int_{\beta_k} \frac{1}{z-z_0}dz$$

But then, considering then all paths at the same time, we get

$$\int_{\biguplus_{k=1}^4 \gamma_k} \frac{1}{z-z_0}dz = \int_{\biguplus_{k=1}^4 \beta_k} \frac{1}{z-z_0}dz$$

and hence

$$\int_R \frac{1}{z-z_0}dz = \int_{C_r(z_0)} \frac{1}{z-z_0}dz = 2\pi i$$

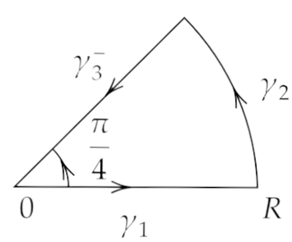
Example 2.2 (Fresnel integrals)**.** *Fresnel proved the following identity:*

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

To show this we first note that the map e^{ix^2} *has real and imaginary parts* $\cos(x^2)$ *and* $\sin(x^2)$*. If we can prove that*

$$\int_0^\infty e^{ix^2}dx = (1+i)\frac{\sqrt{2\pi}}{4}$$

then we are done. We are naturally led to define $f(z) = e^{iz^2}$*, which is holomorphic in all of* $\mathbb{C}$ *as follows.*



where

$$\begin{aligned}
\gamma_1 &: [0, R] \to \mathbb{C}, t \mapsto \gamma_1(t) = t \\
\gamma_2 &: \left[0, \frac{\pi}{4}\right] \to \mathbb{C}, t \mapsto \gamma_2(t) = Re^{it} \\
\gamma_3 &: [0, R] \to \mathbb{C}, t \mapsto \gamma_3(t) = te^{i\frac{\pi}{4}}
\end{aligned}$$

We have that for $\gamma := \gamma_1 \uplus \gamma_2 \uplus \gamma_3^-$

$$\int_\gamma e^{iz^2} dz = 0$$

This separates into all its components as follows:

$$\int_0^R e^{it} dt = -\int_{\gamma_2} e^{iz^2} dz + \int_{\gamma_3} e^{iz^2} dz = -\int_{\gamma_2} e^{iz^2} dz + e^{i\frac{\pi}{4}} \int_0^R e^{-t^2} dt$$

Claim.

$$\left| \int_{\gamma_2} e^{iz^2} dz \right| \leq \frac{\pi\left(1 - e^{-R^2}\right)}{4R} \xrightarrow{R \to +\infty} 0$$

Proof of the claim. Expanding the integral we have

$$\left| \int_{\gamma_2} e^{iz^2} dz \right| \leq \int_{\gamma_2} \left| e^{iz^2} \right| |dz| = \int_0^{\frac{\pi}{4}} \left| e^{i(Re^{it})^2} iRe^{it} \right| dt = R \int_0^{\frac{\pi}{4}} \left| e^{i(R^2 e^{2it})} e^{it} \right| dt =$$
$$= R \int_0^{\frac{\pi}{4}} \left| e^{i(R^2(\cos(2t) + i\sin(2t))} e^{it} \right| dt =$$
$$= R \int_0^{\frac{\pi}{4}} \underbrace{\left| e^{iR^2\cos(2t)} \right|}_{=1} \left| e^{-R^2\sin(2t)} \right| \underbrace{\left| e^{it} \right|}_{=1} dt =$$
$$= \int_0^{\frac{\pi}{4}} \frac{R}{e^{R^2\sin(2t)}} dt \xrightarrow{R \to \infty} \int_0^{\frac{\pi}{4}} 0 dt = 0$$

Using that the last integrand in bounded for $R \in [0, +\infty)$ and therefore applying the Dominant Convergence Theorem [Da 24] to swap the limit and the integral. □

Therefore we have

$$\lim_{R \to \infty} \int_0^R e^{it^2} dt = (1+i)\frac{\sqrt{2}}{2} \int_0^\infty e^{-t^2} dt = (1+i)\frac{\sqrt{2}}{2}\frac{\pi}{2}$$

since from [EW22] we have that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

Remark 2.3. *Note that the use of* ***compact paths*** *to solve this kind of integrals in a very useful tool. One first solves the compact case and then by letting some parameter approach its limit case of interest, if the integrand is nice enough, the complexity of the problem can be considerably reduced.*

2.3 Cauchy's Integral Formulas

We are going to prove Cauchy's Integral Formula (CIF), from which we will deduce many properties of holomorphic functions.

Theorem 2.6 (Cauchy's Integral Formula)**.** [SS10, Theorem II.4.1] Suppose $f \in \mathscr{H}(\Omega)$ is holomorphic in an open set Ω that strictly contains the closure of a disc D, i.e. $\overline{D} \subset \Omega$. If $C = \partial D$ denotes the boundary circle of this disc D with positive orientation, i.e. counterclockwise, then

$$\forall z \in D : f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

Remark 2.4. *Note that Cauchy's Integral Formula says that the values of f on D are determined by their boundary values on the circle C*

Proof. Let $z_0 \in \Omega$, since Ω is open, there exist $r > 0$ such that $\overline{D_r}(z_0) \subset \Omega$, still since Ω is open, $\forall z \in \partial D_r(z_0) : z \in \Omega$ and so $\exists \varepsilon > 0 : D_{r+\varepsilon}(z_0) \subset \Omega$. We obtain from this that $C_r(z_0) \subset D_{r+\varepsilon}(z_0)$. We let $z \in D_r(z_0)$ and define

$$g : D_{r+\varepsilon}(z_0) \to \mathbb{C}, w \mapsto g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & , w \neq z \\ f'(z) & , w = z \end{cases}$$

Then it holds that $g \in C^0\big(D_{r+\varepsilon}(z_0)\big)$ and away from z we also have that g is holomorphic, i.e. $g|_{D_{r+\varepsilon}(z_0)\setminus\{z\}} \in \mathscr{H}\big(D_{r+\varepsilon}(z_0) \setminus \{z\}\big)$

By Cauchy's Theorem 2.5 applied to g, we have

$$\int_{C_r(z_0)} g(w)dw = 0$$

i.e.

$$\int_{C_r(z_0)} \frac{f(w) - f(z)}{w - z} dw = 0$$

Note that on $C_r(z_0)$ it holds that $w \neq z$, since $z \in D_r(z_0)$. Hence

$$\int_{C_r(z_0)} \frac{f(w)}{w - z} dw = f(z) \int_{C_r(z_0)} \frac{1}{w - z} dw$$

To finish the proof we only lack the following claim:

Claim. *We state that the integral has the following form*

$$\int_{C_r(z_0)} \frac{1}{w - z} dw = 2\pi i$$

Proof of the Claim. Consider the following parametrization of $C_r(z_0)$

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{C} \\ t &\mapsto \gamma(t) = z_0 + re^{it} \end{aligned}$$

Though, $C_r(z_0)$ also has the following parametrization

$$\begin{aligned} \tilde{\gamma} : [0, 2\pi] &\to \mathbb{C} \\ s &\mapsto \tilde{\gamma}(s) = z + \rho(s)e^{it} \end{aligned}$$

where $\rho : [0, 2\pi] \to \mathbb{R}, s \mapsto \rho(s) := \left|\gamma\big(t(s)\big) - z\right| = |\tilde{\gamma}(s) - z|$ with $\sigma : [0, 2\pi] \to [0, 2\pi], s \mapsto t(s)$ as reparametrization of the control variable. Clearly ρ is smooth, moreover

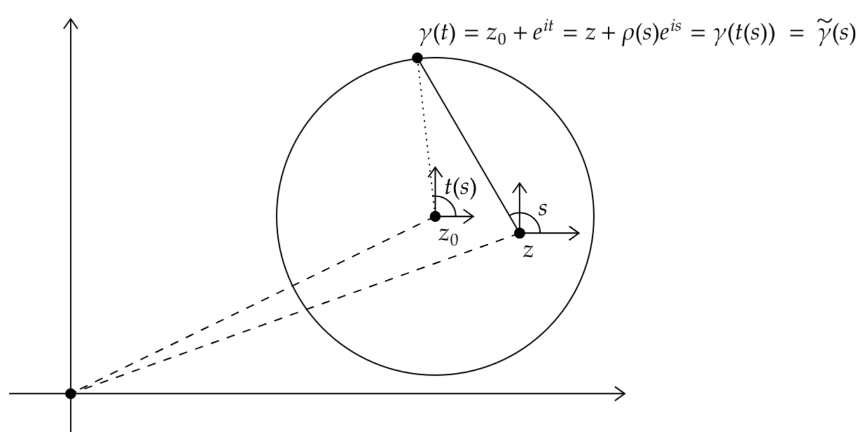
$$\tilde{\gamma}'(s) = \rho'(s)e^{is} + i\rho(s)e^{is}$$

We proceed now to develop the integral form above:

$$\begin{aligned} \int_{C_r(z_0)} \frac{1}{w - z} dw &= \int_0^{2\pi} \frac{\rho'(s)e^{is} + i\rho(s)e^{is}}{\rho(s)e^{is}} ds = \\ &= \underbrace{\int_0^{2\pi} \frac{\rho'(s)}{\rho(s)} ds}_{\text{real integral}} + i \int_0^{2\pi} ds = \\ &= \underbrace{\Big[\ln\big(\rho(s)\big)\Big]_{s=0}^{s=2\pi}}_{=0} + 2\pi i = \\ &= 2\pi i \end{aligned}$$

since $\rho(0) = \rho(2\pi)$ □

We conclude illustrating some more the new parametrization we introduced.



Here t changes with s and $\rho = \rho(s) = \left|\gamma\big(t(s)\big) - z\right|$. As mentioned before

$$\tilde{\gamma}(s) = z + \rho(s)e^{it}$$

is the new parametrisation, with

$$\sigma : [0, 2\pi] \to [0, 2\pi]$$
$$s \mapsto t(s)$$

as the change of variable. Hence we have $\tilde{\gamma} = \gamma \circ \sigma$ □

Before we give important theoretical applications of Cauchy's Theorem 2.5 and the Cauchy's Integral Formula 2.6, we'll look at one more example of contour shifting, which helps us to evaluate certain integrals.

Definition 2.2 (Fourier Transform)**.** For a function $f \in \mathbb{C}^{\mathbb{R}}$, which is Riemann integrable on every $[a, b]$ and which has converging $\int_{-\infty}^{\infty} \left|f(t)\right| dt$, its **Fourier Transform**

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

is well defined for all $\xi \in \mathbb{R}$

Example 2.3. *We'll show that $e^{-\pi x^2}$ is its own Fourier Transform. We want to show that if $f(x) = e^{-\pi x^2}$, then $\hat{f}(\xi) = e^{-\pi \xi^2}$*

We want to show that

$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} dx$$

or equivalently

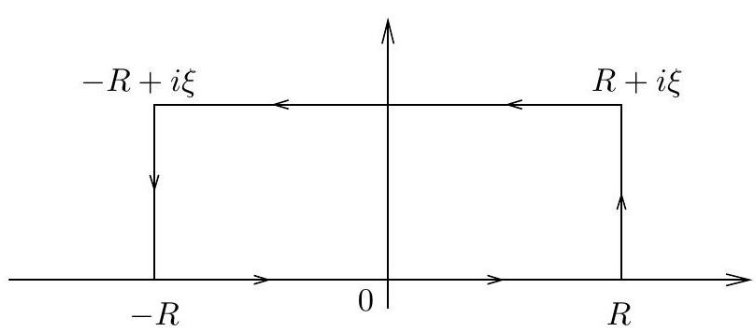
$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} e^{\pi \xi^2} dx$$

If $\xi = 0$, this gives

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$$

which we know from Real Analysis (see [EW22]).

We first suppose that $\xi > 0$ and let $f(z) = e^{-\pi z^2}$, then $f(z)$ is entire and in particular holomorphic in the piecewise smooth contour $\gamma_R = \gamma_1 \uplus \gamma_2 \uplus \gamma_3 \uplus \gamma_4$ as in the picture.



Where γ_1 is the path on the real axis, γ_2 is the right vertical path, γ_3 is the upper horizontal one and γ_4 is the left vertical one. Hence, using Cauchy's Theorem 2.5

$$\int_{\gamma_R} f(z)dz = 0$$

Note that on γ_1:

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} e^{-\pi x^2} dx$$

while on γ_3:

$$\int_{\gamma_3} f(z)dz = \int_{R}^{-R} e^{-\pi(x+i\xi)^2} dx = -\int_{-R}^{R} e^{-\pi(x^2+2\pi i x\xi)} e^{\pi\xi^2} dx = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

As $R \to \infty$, the first integral over γ_1 equals 1, while the integral over γ_3 gives

$$-e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

On the other vertical side on the right we have

$$\int_{\gamma_2} f(z)dz = \int_0^{\xi} f(R+i\xi) i dy = \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)} i dy$$

For a fixed ξ, the integral can be bounded using Proposition 1.10 with

$$\left| \int_{\gamma_2} f(z)dz \right| \le \xi \sup_{y\in[0,\xi]} \left| e^{-\pi R^2} e^{-\pi i R y} e^{\pi y^2} \right| \le C e^{-\pi R^2}$$

where C is a constant dependent on ξ. A similar bound holds for γ_4. Hence, as $R \to \infty$, both integrals vanish to 0 and we obtain that

$$0 = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz = \lim_{R\to\infty} \sum_{k=1}^{4} \int_{\gamma_k} f(z)dz = 1 + 0 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx + 0$$

We therefore obtain that

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

Next we are going to see that Cauchy's theorem 2.5 and the Cauchy's Integral Formula 2.6 will imply fundamental properties of holomorphic functions.

Namely, we are going to see that they are enough to prove:

1. If $\Omega \subseteq \mathbb{C}$ is open and $f \in \mathscr{H}(\Omega)$, then $f' \in \mathscr{H}(\Omega)$. Hence, f is infinitely often differentiable.
   Moreover, if $z_0 \in \Omega$ and $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$, then f has a power series expansion at z_0, namely
   $$\forall z \in D_r(z_0) : f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
   i.e. $f \in C^{\omega}\big(D_r(z_0)\big)$, namely f is analytic in $D_r(z_0)$
2. If f is entire, i.e. $f \in \mathscr{H}(\mathbb{C})$, then f is constant.
3. The Fundamental Theorem of Algebra holds, i.e. any polynomial $p(z) \in \mathbb{C}[z]$ of degree $n \in \mathbb{N}$, has n roots in $\mathbb{C}$ (counted with multiplicity).
4. If $f, g \in \mathscr{H}(\Omega)$ and $f(z) = g(z)$ for all z in some sequence of distinct points with a limit point in Ω, then $\forall z \in \Omega : f(z) = g(z)$. In particular, if f, g agree on an open set U of Ω, then they agree on all of Ω

We start with the following Theorem:

Theorem 2.7. [SS10, Theorem II.4.4] Suppose that $\Omega \subseteq \mathbb{C}$ is an open set and $f \in \mathscr{H}(\Omega)$. Let $z_0 \in \Omega$ and $r > 0$ be such that $\overline{D_r}(z_0) \subset \Omega$. Then f has a power series extension at z_0

$$\forall z \in D_r(z_0) : f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and the coefficients a_n are give by the formula

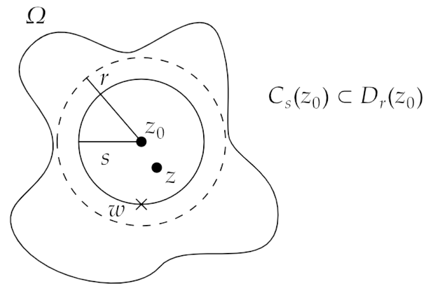
$$\forall n \in \mathbb{N} : a_n = \frac{f^{[n]}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

Moreover, the convergence of the series is absolute and uniform on $D_r(z_0)$

Proof. Let z_0 and r be defined as above, so that $\overline{D_r}(z_0) \subset \Omega$. Fix $s \in (0, r)$ and let $C_s(z_0)$ be the circle of radius s with center z_0. By setting γ a path in $D_r(z_0)$ such that $im(\gamma) = C_s(z_0) \subset \Omega$ and using Theorem 2.6, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$$

for all $z \in D_s(z_0)$



The trick is to write

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left(\frac{1}{1-\frac{z-z_0}{w-z_0}}\right)$$

Since we are integrating on γ, for $w \in im(\gamma)$ (and for $z \in D_s(z_0)$) we have for the term in the last fraction that

$$\left|\frac{z-z_0}{w-z_0}\right| = \frac{|z-z_0|}{s} < 1$$

This means that we can rewrite the whole last fraction as a geometric series (see [EW22])

$$\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

for $w \in im(\gamma)$ and $z \in D_s(z_0)$. The convergence of the series is uniform, since the bound $\frac{|z-z_0|}{s}$ for $\frac{z-z_0}{w-z_0}$ is independent of $w \in im(\gamma)$. Hence we can interchange the series and the integral (again, see [EW22]), obtaining

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma f(w)\left(\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right)dw = \\ &= \sum_{n=0}^{\infty}(z-z_0)^n\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw = \\ &= \sum_{n=0}^{\infty} a_n(z-z_0)^n \end{aligned}$$

where we defined the a_n's as

$$a_n := \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw$$

Hence, we have that in $D_s(z_0)$ the function f is the sum of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

We have seen that series in their disc of convergence are differentiable with derivatives given by termwise differentiation, as stated in Theorem 1.3. Hence, for all $z \in D_s(z_0)$ we have

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = \\ &= \sum_{k=0}^{\infty} (k+1) a_{k+1} (z - z_0)^k \end{aligned}$$

Being a power series, $f'(z)$ is also holomorphic in $D_s(z_0)$, i.e. $f'|_{D_s(z_0)} \in \mathscr{H}\big(D_s(z_0)\big)$. Inductively, we get that f is differentiable infinitely many times for $z \in D_s(z_0)$ and evaluating it and its derivatives at $z = z_0$ gives

$$\begin{aligned} a_0 &= f(z_0) \\ a_1 &= f'(z_0) \\ &\vdots \\ n! a_n &= f^{[n]}(z_0) \end{aligned}$$

Hence $a_n = \frac{f^{[n]}(z_0)}{n!}$ is independent of s and we have $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converging for all $z \in D_s(z_0)$, this for any arbitrary $s \in (0, r)$. By taking the limit for $s \to r$ we notice that γ expands to $C_r(z_0)$, which is still in Ω: the above calculations and step are still valid, especially the uniform convergence of the series, also in this case is granted, as $|z - z_0| < r$. Viewing this as (constant for each $z \in D_s(z_0)$) continuous function of s, we can extend by continuity to r. This concludes the proof. □

Remark 2.5. *In all these Theorems with $\Omega \subseteq \mathbb{C}$ open by assumption, it always follows by definition of open set in $\mathbb{C}$ that*

$$\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$$

Moreover, the above proof of Theorem 2.7 gives a method for determine the radius of convergence of the series expansion of any $f \in \mathscr{H}(\Omega)$, as done above by expanding s to r: we start by a fully contained $\overline{D}_s(z_0) \subset \Omega$ for some $s \in (0, +\infty)$ and progressively increase it until we meet $\partial\Omega$, thus external boundary of the set or some singularity points. As long as $\overline{D}_s(z_0) \subset \Omega$, the success of the operation is granted.

Remark 2.6. *Note that the proof also gives that if f is complex differentiable at z_0, then in facts it is complex differentiable infinitely many times at z_0. It follows that for every $n \in \mathbb{N}$*

$$a_n = \frac{1}{2\pi i} \int_{C_s(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw = \frac{f^{[n]}(z_0)}{n!}$$

In fact we have

Corollary 2.3 (Cauchy's Integral Formula for derivatives)**.** [SS10, Corollary II.4.2] If $f \in \mathscr{H}(\Omega)$, then f is infinitely often complex differentiable in Ω and in particular

$$\forall n \in \mathbb{N} : f^{[n]} \in \mathscr{H}(\Omega)$$

Moreover, if $z_0 \in \Omega$ and $r > 0$ are such that $\overline{D_r}(z_0) \subset \Omega$, then

$$\forall n \in \mathbb{N} \forall z \in D_r(z_0) : f^{[n]}(z) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} dw$$

Proof. The fact that f' is complex differentiable (or holomorphic) follows from the fact that, since Ω is open, $\forall z_0 \in \Omega \exists r > 0 : \overline{D}_r(z_0) \subset \Omega$. By Theorem 2.7, f has a power series expansion there, which is therefore holomorphic in $D_r(z_0)$. Since power series are infinitely often complex differentiable in their disc of convergence, we have that f is complex differentiable infinitely often at z_0. Since z_0 was arbitrary, f in infinitely often complex differentiable in Ω

To prove this, by induction on $n \in \mathbb{N}$, note that the base case with $n = 0$ is simply the Cauchy's Integral Formula in Theorem 2.6. We are now going to show the induction step, therefore suppose that

$$f^{[n-1]}(z) = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^n} dw$$

for any $z \in D_r(z_0)$. For $h \in \mathbb{C}$ small enough, so that $z + h$ and z are both away from $C_r(z_0)$, we have

$$\frac{f^{[n-1]}(z+h) - f^{[n-1]}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{h} \left(\frac{1}{\big(w-(z+h)\big)^n} - \frac{1}{(w-z)^n} \right) dw$$

We then use the equality (in $\mathbb{C}$)

$$a^n - b^n = (a-b) \left(\sum_{k=0}^{n-1} a^{n-1-k} b^k \right)$$

with $a = \frac{1}{w-(z+h)}$ and $b = \frac{1}{w-z}$ and take the limit as $h \to 0$: note that $\frac{a-b}{h} \xrightarrow{h \to 0} \frac{1}{(w-z)^2}$ and $\sum_{k=0}^{n-1} a^{n-1-k} b^k \xrightarrow{h \to 0} \frac{n}{(w-z)^{n-1}}$, to get that

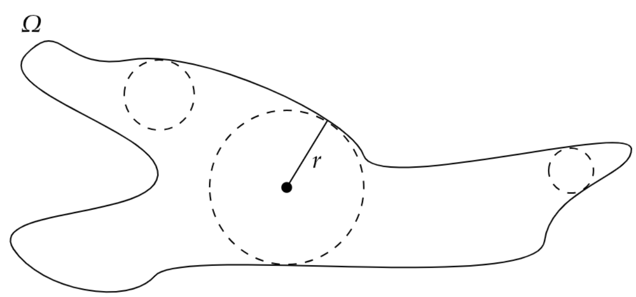
$$\begin{aligned} \lim_{h \to 0} \frac{f^{[n-1]}(z+h) - f^{[n-1]}(z)}{h} &= \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} f(w) \frac{1}{(w-z)^2} \frac{n}{(w-z)^{n-1}} dw = \\ &= \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw \end{aligned}$$

□

Remark 2.7. *Theorem 2.7 says that a holomorphic function f can be locally developed as a power series around each point of the definition domain Ω. Explicitly, for each $z_0 \in \Omega$, $\exists D_r(z_0) \subset \Omega$ and a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, which converges for all $z \in D_r(z_0)$ and represents the function f in $D_r(z_0)$*

Due to this power series expansion we have that holomorphic functions are exactly the functions which are everywhere representable as a power series (with a positive radius of convergence). Recall that any power series represents a holomorphic function in their disc of convergence. This is why we have the words "holomorphic" and "analytic" used interchangeably in various sources.

Note that the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ might not represent $f(z)$ in all of Ω, but it represents it at least in a disc whose radius is the distance from the point to the boundary of Ω



Corollary 2.4 (Cauchy Inequality)**.** [SS10, Corollary II.4.3] With the assumptions as in Theorem 2.3, we have for every $n \in \mathbb{N}$ that

$$\left|f^{[n]}(z_0)\right| \leq \frac{n!\|f\|_{\infty,C_r(z_0)}}{r^n}$$

where $\|f\|_{\infty,C_r(z_0)} := \sup_{z\in C_r(z_0)} \left|f(z)\right|$

Proof. By Corollary 2.3 we have that

$$\begin{aligned}\left|f^{[n]}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw\right| = \\ &= \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{(re^{i\theta})^{n+1}} ire^{i\theta}d\theta\right| \leq \\ &\leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{\left|f(z_0+re^{i\theta})\right|}{r^n}d\theta \leq \\ &\leq \frac{n!}{2\pi}\frac{\|f\|_{\infty,C_r(z_0)}}{r^n}\end{aligned}$$

□

An immediate Corollary of these results is the remarkable Liouville's Theorem.

Theorem 2.8 (Liouville's Theorem)**.** [SS10, Theorem II.4.5] If $f \in \mathscr{H}(\mathbb{C})$ and if f is bounded, i.e. $f \in \mathcal{B}(\mathbb{C})$, then f is constant.

Proof. Since $\mathbb{C}$ is connected, it is enough to show that $f' = 0$ (Corollary 1.2).
Let $z_0 \in \mathbb{C}$, then for all $r > 0$ we have $\overline{D}_r(z_0) \subset \mathbb{C}$ and since f is holomorphic on all of $\mathbb{C}$, we have by using Cauchy's Inequality 2.4.

$$\left|f'(z_0)\right| \leq \frac{\|f\|_{\infty,C_r(z_0)}}{r}$$

By assumption f is bounded, i.e. $\exists M \geq 0 \forall z \in \mathbb{C} : |f(z)| < M$, hence

$$\forall r > 0 : \left|f'(z_0)\right| < \frac{M}{r}$$

By letting $r \to \infty$ we get $f'(z_0) = 0$ and since z_0 was arbitrary, we get that $\forall z \in \mathbb{C}$: $f'(z) = 0$ and hence f is constant. □

Remark 2.8. *The assumption that in Liouville's Theorem 2.8 f has to be holomorphic in <u>all</u> of $\mathbb{C}$ is essential, e.g. let $\Omega = \{z \in C : Re(z) > 0\}$ and $f(z) = \frac{1}{1+z}$ on it.*

Corollary 2.5 (Fundamental Theorem of Algebra)**.** Every polynomial $p(z) = a_0 + \sum_{k=1}^{n} a_k z^k$ with $\deg(p) = n \geq 1$ has precisely n roots in $\mathbb{C}$, counted with multiplicity. If these roots are $w_1, ..., w_n$ (with possible repetitions), then

$$p(z) = a_n \prod_{k=1}^{n} (z - w_k)$$

Proof. We first show that $p(z)$ has a root in $\mathbb{C}$ for $\deg(p) \geq 1$

By contradiction, suppose that there is no such root and that consequently the function $Q(z) = \frac{1}{p(z)} \in \mathscr{H}(\mathbb{C})$

If it held $Q \in \mathcal{B}(\mathbb{C})$, then it would be a constant by Liouville's Theorem 2.8, which would then contradict that $p(z)$ is not constant, i.e. $\deg(p) \geq 1$ with $p(z) = a_0 + \sum_{k=1}^{n} a_k z^k$ where $a_n \neq 0$ and $\forall k \in \{0, ..., n\} : a_k \in \mathbb{C}$

Claim. *Q is bounded, i.e. $Q \in \mathcal{B}(\mathbb{C})$*

Proof of the Claim. For $z \neq 0$ we have

$$\begin{aligned} |p(z)| &= \left| a_0 + \sum_{k=1}^{n} a_k z^k \right| \geq \\ &\geq |a_n||z|^n - \sum_{k=1}^{n-1} |a_k||z|^k \geq \\ &\geq |z|^n \left(|a_n| - \sum_{k=1}^{n-1} |a_k||z|^{k-n} \right) \end{aligned}$$

Hence $\big|p(z)\big| \xrightarrow{|z|\to\infty} \infty$ and consequently $\big|Q(z)\big| \xrightarrow{|z|\to\infty} 0$, from this result we deduce that $\exists r > 0 : \big|Q(z)\big| \leq 1$ whenever $|z| \geq r$ (Q is a continuous function, i.e. $Q \in C^0(\mathbb{C})$), but Q is continuous and hence bounded on the compact set $\overline{D}_r(0)$, say $\big|Q\big| < m$ for some $m \in \mathbb{R}$

Choose $M := \max\{m, 1\}$, then

$$\forall z \in \mathbb{C} : \big|Q(z)\big| \leq M$$

Hence Q is constant by Liouville's Theorem 2.8 □

This contradicts the assumption that p is non-constant and hence proves the existence of one solution by contradiction.

Hence p has a root, say $w_1 \in \mathbb{C}$. Then by writing $z = (z - w_1) + w_1$ we have

$$\begin{aligned} p(z) &= a_0 + \sum_{k=1}^{n} a_k \big((z - w_1) + w_1\big)^k = \\ &= b_0 + \sum_{k=1}^{n} b_k (z - w_1)^k \end{aligned}$$

using the Binomial Theorem [EW22], new coefficients $b_{n-1}, ..., b_0$ and $b_n = a_n$. Since $p(w_1) = 0$, we must then have $b_0 = 0$

Hence

$$p(z) = (z - w_1) \left(\sum_{k=1}^{n} b_k (z - w_1)^{k-1} \right) = (z - w_1)\tilde{p}(z)$$

where $\tilde{p}$ is a polynomial with $\deg(\tilde{p}) = n - 1$. By induction on the degree of the polynomial we get the result. □

Next we discuss the principle of analytic continuation (of identities), which states that: if Ω is open and connected, $f \in \mathscr{H}(\Omega)$ and f vanishes on an infinite set $\mathscr{Z}$ of distinct points with a limit point $z_0 \in \Omega \setminus \mathscr{Z}$, then $f = 0$.

Remark 2.9. 1. *Holomorphic functions can have infinitely many zeroes, but we we are going to see that these zeroes are isolated, i.e. for each zero z_0 of a holomorphic function f there exists a neighbourhood of z_0 with no other zero.*
E.g. $\cos(z)$ *and* $\sin(z)$ *have respectively zeroes for* $z = (2k+1)\frac{\pi}{2}$ *and* $z = \pi k$, *with* $k \in \mathbb{Z}$

2. *There are holomorphic functions with no zeroes.*
E.g. the constant function c or the complex exponential e^z are examples of functions with no zeroes.

We start by the definition of a limit point.

Definition 2.3. An element $z_0 \in \mathbb{C}$ is **a limit point of a set** Ω, if

$$\exists (z_n)_{n\in\mathbb{N}^*} \in \big(\Omega \setminus \{z_0\}\big)^{\mathbb{N}^*} : \lim_{n\to\infty} z_n = z_0$$

Hence, it holds that $\forall \varepsilon > 0 : \Omega \cap \dot{D}_\varepsilon(z_0) \neq \emptyset$ and that $\forall n \in \mathbb{N}^* : z_n \neq z_0$

Example 2.4. *If* $\Omega = [-1,1] \cup \{2i\}$, *then the* $z_n \neq z_0$ *condition avoids the case* $2i$ *is a limit point of* Ω, *since otherwise we could take* $\forall n \in \mathbb{N}^* : z_n = 2i$

We next define the order of zero of f at z_0

Definition 2.4. Let $\Omega \subseteq \mathbb{C}$ be open, $f \in \mathscr{H}(\Omega)$ and $z_0 \in \Omega$, then the **order of zero of f at z_0** or **order of vanishing of f at z_0**, denoted by $\text{ord}_{z_0}(f)$ or $n_{z_0}(f)$ or $\nu_{z_0}(f)$, is either ∞, if $\forall k \geq 0 : f^{[k]}(z_0) = 0$ or it is the smallest integer $k \in \mathbb{N}$ such that $f(z_0) = f'(z_0) = ... = f^{[k-1]}(z_0) = 0$ and $f^{[k]}(z_0) \neq 0$. If $f(z_0) \neq 0$, then $k = 0$. Therefore, we define it as follows:

$$\text{ord}_{z_0}(f) := \left\{ \begin{array}{cc} \min\{k \in \mathbb{N} : f^{[k]}(z_0) \neq 0\} & , \{k \in \mathbb{N} : f^{[k]}(z_0) \neq 0\} \neq \emptyset \\ \infty & , \{k \in \mathbb{N} : f^{[k]}(z_0) \neq 0\} = \emptyset \end{array} \right\} \in \mathbb{N} \cup \{\infty\}$$

We have the following result

Proposition 2.2. [SS10, Theorem III.1.1] Let $\Omega \subseteq \mathbb{C}$ be open, $f \in \mathscr{H}(\Omega)$ and $z_0 \in \Omega$.

(i) If $\text{ord}_{z_0}(f) = \infty$, then $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ and

$$\forall r > 0 : \overline{D}_r(z_0) \subset \Omega \Longrightarrow \forall z \in D_r(z_0) : f(z) = 0$$

i.e. f is locally zero around z_0.

(ii) If $\operatorname{ord}_{z_0}(f) \neq \infty$, then $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ in which

$$\exists! h \in \mathscr{H}\big(D_r(z_0)\big) \exists! n \in \mathbb{N} \forall z \in D_r(z_0) : f(z) = (z - z_0)^n h(z)$$

where $h(z_0) \neq 0$ and $n = \operatorname{ord}_{z_0}(f)$

(iii) For any $f, g \in \mathscr{H}(\Omega)$ we have

$$\operatorname{ord}_{z_0}(f+g) \geq \min\{\operatorname{ord}_{z_0}(f), \operatorname{ord}_{z_0}(g)\}$$
$$\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$$

Proof. **(i)** Let f be holomorphic in Ω with Ω open. By Theorem 2.7 and using that Ω is open, $\exists r > 0$ is such that $\overline{D}_r(z_0) \subset \Omega$ and on it we have

$$\forall z \in D_r(z_0) : f(z) = \sum_{n=0}^{\infty} \frac{f^{[n]}(z_0)}{n!}(z - z_0)^n$$

Since $\operatorname{ord}_{z_0}(f) = \infty$, it holds that $\forall n \geq 0 : f^{[n]}(z_0) = 0$. Hence $\forall z \in D_r(z_0) :$ $f(z) = 0$

(ii) If $\operatorname{ord}_{z_0}(f) \neq \infty$, then by definition

$$\exists k \in \mathbb{N} : f(z_0) = ... = f^{[k-1]}(z_0) = 0 \text{ and } f^{[k]}(z_0) \neq 0$$

Again, using Theorem 2.7 and the fact that Ω is open, $\exists r > 0$ is such that $\overline{D}_r(z_0) \subset \Omega$, then for all $z \in D_r(z_0)$ we have the power series representation

$$\begin{aligned} f(z) &= \frac{f^{[k]}(z_0)}{k!}(z - z_0)^k + \sum_{n=k+1}^{\infty} \frac{f^{[n]}(z_0)}{n!}(z - z_0)^n = \\ &= (z - z_0)^k \underbrace{\left(\underbrace{\frac{f^{[k]}(z_0)}{k!}}_{\neq 0} + \sum_{m=1}^{\infty} \frac{f^{[m+k]}(z_0)}{(m+k)!}(z - z_0)^m \right)}_{\neq 0} = \\ &= (z - z_0)^k \left(\sum_{m=0}^{\infty} \frac{f^{[m+k]}(z_0)}{(m+k)!}(z - z_0)^m \right) \end{aligned}$$

Hence, if we define

$$\forall z \in D_r(z_0) : h(z) := \sum_{m=0}^{\infty} \frac{f^{[m+k]}(z_0)}{(m+k)!}(z - z_0)^m$$

Then $h \in \mathscr{H}\big(D_r(z_0)\big)$, since it is given by a convergent power series and also $h(z_0) = \frac{f^{[k]}(z_0)}{k!} \neq 0$, as $(z - z_0)^m = 0$ if and only if $m > 0$

Note that, since $h \in \mathscr{H}\big(D_r(z_0)\big)$, it is also continuous there and since $h(z_0) \neq 0$, it holds that

$$\exists \varepsilon \in (0, r) \forall z \in D_\varepsilon(z_0) : h(z) \neq 0$$

Moreover, h and n are unique, since if we assumed not, hence

$$f(z) = (z - z_0)^n h(z) = (z - z_0)^m g(z)$$

with f, g holomorphic and $h(z_0) \neq 0$, $g(z_0) \neq 0$, then if $m > n$ we would get

$$\begin{aligned} f(z) &= (z - z_0)^n (z - z_0)^{m-n} g(z) = \\ &= (z - z_0)^n h(z) \end{aligned}$$

and for $z \neq z_0$ that

$$h(z) = (z - z_0)^{m-n} g(z)$$

Now, taking the limit on both sides as $z \to z_0$ gives $h(z_0) = 0$, which is a contradiction, therefore $m = n$. Then $h(z) = g(z)$

(iii) Note that for any $k \in \mathbb{N}$

$$f^{[k]}(z_0) + g^{[k]}(z_0) = (f + g)^{[k]}(z_0)$$

Hence, if $f^{[k]}(z_0) = 0 = g^{[k]}(z_0)$, then also $(g + f)^{[k]}(z_0) = 0$. This implies that $\text{ord}_{z_0}(f + g) \geq \min\{\text{ord}_{z_0}(f), \text{ord}_{z_0}(g)\}$
By part (ii), instead, we write

$$\begin{aligned} f(z) &= (z - z_0)^{\text{ord}_{z_0}(f)} h_1(z) \\ g(z) &= (z - z_0)^{\text{ord}_{z_0}(g)} h_2(z) \end{aligned}$$

with $\forall z \in D_r(z_0) : h_1(z) \neq 0$ and $h_2(z) \neq 0$, then

$$fg = (z - z_0)^{\text{ord}_{z_0}(f) + \text{ord}_{z_0}(g)} h_1(z) h_2(z)$$

with $(h_1 h_2)(z_0) \neq 0$ From this, using the power series expansion of fg or the uniqueness of n and h in part (ii) we get

$$\text{ord}_{z_0}(f) + \text{ord}_{z_0}(g) = \text{ord}_{z_0}(fg)$$

□

As Corollary we get that the zeroes of an holomorphic function are isolated. More precisely we have

Theorem 2.9. Let $\Omega \subseteq \mathbb{C}$ be open, $f \in \mathscr{H}(\Omega)$ and $z_0 \in \Omega$. Assume $f(z_0) = 0$, i.e. $\text{ord}_{z_0}(f) \geq 1$. If $\text{ord}_{z_0}(f) \neq \infty$, then

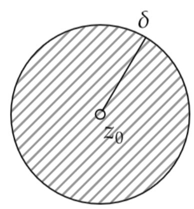
$$\exists \delta > 0 \forall z \in \dot{D}_\delta(z_0) : f(z) \neq 0$$

Proof. Using Proposition 2.2 we have that $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ in which we write $f|_{D_r(z_0)}(z) = (z - z_0)^n h(z)$ with $n = \mathrm{ord}_{z_0}(f)$ and $h(z_0) \neq 0$. Let $z \in \dot{D}_r(z_0)$, then

$$f(z) = 0 \Longleftrightarrow h(z) = 0$$

since $(z - z_0)^n \neq 0$ for $z \neq z_0$
Being $h(z_0) \neq 0$ and h is continuous on $D_r(z_0)$, we have that

$$\exists \delta \in (0, r] : |z - z_0| < \delta \Longrightarrow h(z) \neq 0$$



By the equivalence previously stated we have that

$$\forall z \in \dot{D}_\delta(z_0) : f(z) \neq 0$$

□

Now we can state the Principle of Analytic Continuation[2].

Theorem 2.10 (Principle of Analytic Continuation)**.** [SS10, Theorem II.4.8] Let $\Omega \subseteq \mathbb{C}$ be a region and let $f \in \mathscr{H}(\Omega)$. Let $\mathscr{Z} \subset \Omega$ be an infinite set with a limit point $z_0 \in \Omega$, but $z_0 \notin \mathscr{Z}$. Then

$$\Big(\forall z \in \mathscr{Z} : f(z) = 0\Big) \Longrightarrow f = 0$$

Before we give the proof, we record the following immediate Corollary.

Corollary 2.6 (Identity Theorem)**.** [SS10, Corollary II.4.9] Let $\Omega \subseteq \mathbb{C}$ be a region and $\emptyset \neq U \subseteq \Omega$ be one of its open subsets. Suppose $f, g \in \mathscr{H}(\Omega)$, then

$$\big(\forall z \in U : f(z) = g(z)\big) \Longrightarrow \big(\forall z \in \Omega : f(z) = g(z)\big)$$

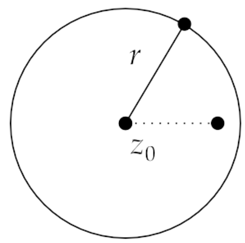
This holds more in general for a sequence $(z_n)_{n\in\mathbb{N}^*} \in \mathscr{Z}^{\mathbb{N}^*}$ of distinct points with

[2]It is useful to this purpose to recall the definition of a region in $\mathbb{C}$, see 1.11.

limit point $\lim_{n\to\infty} z_n =: z_0 \in \Omega$ such that $z_0 \notin \mathscr{Z}$

Proof. Apply Theorem 2.10 to $f - g$ to obtain this result. □

Note that if $U \subseteq \Omega$ is open and non-empty, then $\exists r > 0 \exists z_0 \in U : D_r(z_0) \subseteq U$ and the sequence $\{z_0 + \frac{r}{n+1}\}_{n=1}^{\infty}$ in $D_r(z_0) \subseteq U$ has a limit point $z_0 \in \Omega \setminus \{z_0 + \frac{r}{n+1}\}_{n=1}^{\infty}$



Remark 2.10. *1. The reason this result is called Principle of Analytic Continuation 2.10 is the following:*

If $f \in \mathscr{H}(\Omega)$ with a region Ω and $\Omega \subseteq \tilde{\Omega}$ is again a region, then there is at most one $\tilde{f} \in \mathscr{H}(\tilde{\Omega})$ such that $\forall z \in \Omega : f(z) = \tilde{f}(z)$. When such a function $\tilde{f}$ exists, we say that f **has analytic continuation** $\tilde{f}$ **to** $\tilde{\Omega}$

Note that if $g \in \mathscr{H}(\tilde{\Omega})$ is such that $\forall z \in \Omega : g(z) = f(z)$. Then $\forall z \in \Omega : \tilde{f}(z) - g(z) = 0$, hence $\forall z \in \tilde{\Omega} : \tilde{f}(z) - g(z) = 0$ by the above Theorem 2.10. Therefore $\tilde{f}$ is unique.

2. *The assumption that Ω is connected is essential, since if $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \neq \emptyset$, $\Omega_2 \neq \emptyset$ and $\Omega_1 \cap \Omega_2 = \emptyset$, then one can define $f, g : \Omega \to \mathbb{C}$ by $f|_{\Omega_1} = 1$ and $f|_{\Omega_2} = 0$ and $g = 0$. Then even though $f|_{\Omega_2} = g|_{\Omega_2}$ coincide in Ω_2, f and g do not coincide in Ω*

3. *The condition that the limit point of zeroes is in Ω is also crucial, as shown in the following Example 2.5.*

Example 2.5. *Take $\Omega = \mathbb{C}^*$ and*

$$f : \Omega \to \mathbb{C}$$
$$z \mapsto \sin\left(\frac{\pi}{z}\right) = \frac{e^{\frac{i\pi}{z}} - e^{\frac{-i\pi}{z}}}{2i}$$

It holds that $f \in \mathscr{H}(\mathbb{C}^*)$ and $f \neq 0$, since already $f(i) = \frac{e^{\pi} - e^{-\pi}}{2i} \neq 0$
Consider a sequence of zeroes given by $\{\frac{1}{n}\}_{n\in\mathbb{N}^*}$ with $\lim_{n\to\infty} \frac{1}{n} = 0$, as $f(\frac{1}{n}) = \sin(\pi n) =$

0 for all $n \in \mathbb{N}^*$. The limit point of zeroes is not in Ω, causing the result. This example shows that the zeroes can converge to a boundary point. Note that we do have that the zeroes $\{\frac{1}{n}\}_{n\in\mathbb{N}^*}$ are isolated.

Another interesting example regarding the above concept of continuation is given here at Example 2.6.
We are now going to prove the following theorem, which proves Theorem 2.10.

Theorem 2.11 (Principle of Analytic Continuation alternative version)**.** Let Ω be a region, $f \in \mathscr{H}(\Omega)$. Then the following are equivalent:

(i) $f = 0$

(ii) $\exists a \in \Omega \forall n \in \mathbb{N} : f^{[n]}(a) = 0$

(iii) $Z_f := \{z \in \Omega : f(z) = 0\}$ has a limit point in Ω

An immediate Corollary of Theorem 2.11 is

Corollary 2.7 (Identity Theorem alternative version)**.** Let Ω be a region in $\mathbb{C}$, $f, g \in \mathscr{H}(\Omega)$. Then the following are equivalent:

(i) $f = g$

(ii) $\exists a \in \Omega \forall n \in \mathbb{N} : f^{[n]}(a) = g^{[n]}(a)$

(iii) $\{z \in \Omega : f(z) = g(z)\}$ has a limit point in Ω

Proof of Theorem 2.11. Clearly **(i)$\Longrightarrow$(iii)**, since by assumption $\{z \in \Omega : f(z) = 0\} = \Omega$. In the following, we will prove (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i):

(iii)$\Longrightarrow$(ii): Let $Z_f := \{z \in \Omega : f(z) = 0\}$. By assumption Z_f has a limit point $a \in \Omega$. Since Ω is open, let $r > 0$ be a radius such that $\overline{D}_r(a) \subset \Omega$. We also have that f is continuous and using that a is a limit point in Z_f, i.e.

$$\exists (z_n)_{n\in\mathbb{N}^*} \in \left(Z_f \setminus \{a\}\right)^{\mathbb{N}^*} : \lim_{n\to\infty} z_n = a$$

by continuity of f we have that

$$0 = \lim_{n\to\infty} f(z_n) = f\left(\lim_{n\to\infty} z_n\right) = f(a)$$

Claim. *It holds that* $\forall n \in \mathbb{N} : f^{[n]}(a) = 0$

Proof of the Claim. Suppose on the contrary that

$$\exists n \in \mathbb{N}^* \forall l \in \{0, ..., n-1\} : f^{[l]}(a) = 0 \text{ but } f^{[n]}(a) \neq 0$$

Then as in the proof of Theorem 2.7, since f is analytic in $D_r(a) \subset \Omega$, i.e. $f \in C^\omega\big(D_r(a)\big)$, expanding f in a power series there

$$\forall z \in D_r(a) : f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k$$

we have that by Proposition 2.2

$$f(z) = (z-a)^n g(z)$$

with $g(a) \neq 0$ and g is analytic in $D_r(a)$, i.e. $g \in C^\omega\big(D_r(a)\big)$
Since g is continuous, $\exists \varepsilon \in (0, r) : D_\varepsilon(a) \subset D_r(a)$ such that $\forall z \in D_\varepsilon(a) : g(z) \neq 0$, then

$$f(z) = \underbrace{(z-a)^n}_{\neq 0 \text{ on } \dot{D}_\varepsilon(a)} \cdot \underbrace{g(z)}_{\neq 0 \text{ on } D_\varepsilon(a)}$$

Therefore $f(z) \neq 0$ on $\dot{D}_\varepsilon(a)$. Hence $Z_f \cap \dot{D}_\varepsilon(a) = \emptyset$, but this says that a is not a limit point of Z_f. Hence $\forall n \in \mathbb{N} : f^{[n]}(a) = 0$ □

(ii)$\Longrightarrow$(i): Let $A := \big\{z \in \Omega : \big(\forall n \in \mathbb{N} : f^{[n]}(a) = 0\big)\big\}$. By assumption $a \in A$, hence $A \neq \emptyset$. We will show that $A = \Omega$ via the characterisation of connectedness, hence that $f = 0$

Recall: for an open subset $\Omega \subseteq \mathbb{C}$, connected means that the only both open and closed subsets of Ω are $\emptyset$ and Ω (It is not possible to find two disjoint non-empty open sets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$).

Since $A \neq \emptyset$, if we can show that A is both open and closed, then $A = \Omega$. Hence:

<u>A is open:</u> To see this, let $c \in A$ and let $r > 0$ such that $\overline{D_r(c)} \subset \Omega$. Then by Theorem 2.7

$$\forall z \in D_r(c) : f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

with

$$a_n = \frac{f^{[n]}(c)}{n!} = 0$$

since $c \in A$, hence $f|_{D_r(c)} = 0$. This means that $D_r(c) \subseteq A$. Hence, for an arbitrary $c \in A$, we found a neighbourhood $D_r(c) \subseteq A$, which shows that A is open.

A is closed: We want to show that if $\{z_k\}_{k\in\mathbb{N}^*}$ is a sequence of points in A such that $\lim_{k\to\infty} z_k = c \in \Omega$, then $c \in A$, i.e. A contains all its limit points. Let $c \in \Omega$ be a limit point of a sequence $\{z_k\}_{k\in\mathbb{N}^*}$ in A. Then for any $k \in \mathbb{N}^*$ and $n \in \mathbb{N}$, it holds that

$$f^{[n]}(z_k) = 0$$

by definition of the set A. We though have that $f^{[n]}$ is continuous, hence

$$0 = \lim_{k\to\infty} f^{[n]}(z_k) = f^{[n]}\left(\lim_{k\to\infty} z_k\right) = f^{[n]}(c)$$

Since $n \in \mathbb{N}$ was arbitrary, we obtain that $f^{[n]}(c) = 0$ for every $n \in \mathbb{N}$ and therefore $c \in A$. It follows from this that A is closed in Ω, given that arbitrariness of $c \in A$

□

Remark 2.11. *1. (a) The Identity Theorem 2.7 makes it clear that the real functions*

$$\sin, \cos, \exp : \mathbb{R} \to \mathbb{R}$$

can be uniquely extended to complex numbers, via their form on the real line. E.g. $\sin(z)$ *and* $\cos(z)$ *are entire functions. For any* $z = x \in \mathbb{R}$ *we have that* $\sin^2(x) + \cos^2(x) = 1$. *Thus we define* $f \in \mathbb{C}^{\mathbb{C}}, z \mapsto f(z) = \sin^2(z) + \cos^2(z)$ *and* $g \in \mathbb{C}^{\mathbb{C}}, z \mapsto 1$. *Since* f *and* g *agree on the real line, they have to agree on all of* $\mathbb{C}$, *from which*

$$\boxed{\forall z \in \mathbb{C} : \sin^2(z) + \cos^2(z) = 1}$$

(b) The functional equations can also be transferred from reals to complex numbers.

E.g. from

$$\forall x, y \in \mathbb{R} : \exp(x + y) = \exp(x)\exp(y)$$

we first conclude

$$\forall y \in \mathbb{R} \forall z \in \mathbb{C} : \exp(z + y) = \exp(z)\exp(y)$$

and the another application of the Identity Theorem gives

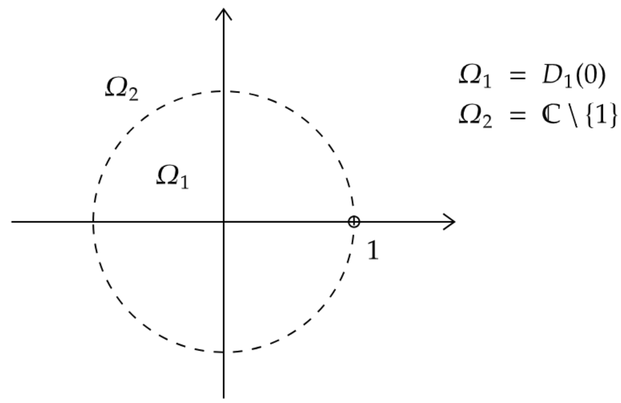
$$\boxed{\forall z, w \in \mathbb{C} : \exp(z + w) = \exp(z)\exp(w)}$$

General case: *Let* $\Omega \subseteq \mathbb{C}$ *be a region which contains a set* $U \subset \Omega$, *which itself contains a sequence of points with limit point also in* U. *Let* $F(z, w)$ *be a function defined for* $z, w \in \Omega$ *such that* $F(z, w)$ *is analytic in* z *for any* w *and vice versa. If* $F(z, w) = 0$ *whenever* $z, w \in U$, *then* $\forall z, w \in \Omega : F(z, w) = 0$

2. *The geometric series for $z \in D_1(0)$ is given by:*

$$g(z) = \sum_{n=0}^{\infty} z^n$$

has analytic continuation to $\mathbb{C} \setminus \{1\}$ given by $f(z) = \frac{1}{1-z}$



We have $g : \Omega_1 \to \mathbb{C}, z \mapsto \sum_{n=0}^{\infty} z^n$ and $f : \Omega_2 \supseteq \Omega_1 \to \mathbb{C}, z \mapsto \frac{1}{1-z}$. Therefore, if $z \in D_1(0)$, we then have

$$f(z) = g(z) = \frac{1}{1-z}$$

Example 2.6. *Let $\Omega = D_1(0)$, then it holds that*

$$\forall z \in \Omega : f(z) = \sum_{n=0}^{\infty} z^n$$

f converges on $D_1(0)$ and defines a holomorphic function there. Note that for $z = 1$, $f(z)$ does not converge, hence for any $\varepsilon > 0$, we cannot define $f(z)$ as $\sum_{n=0}^{\infty} z^n$ on $D_{1+\varepsilon}(0)$, since any such disc contains $z = 1$

Let $\tilde{\Omega} = \mathbb{C} \setminus \{1\}$, then $\Omega \subseteq \tilde{\Omega}$ and $F(z) = \frac{1}{1-z}$ is defined on all of $\tilde{\Omega}$ and it agrees with $\sum_{n=0}^{\infty} z^n$ whenever $z \in D_1(0)$, so $F(z) = \frac{1}{1-z}$ is the analytic continuation of f to $\mathbb{C} \setminus \{1\}$

Warning: This does not say that F represents $\sum_{n=0}^{\infty} z^n$ on the complement $\mathbb{C} \setminus \overline{D_1(0)}$.

Note that in the Identity Theorem 2.7 we have two holomorphic functions defined on the same set Ω. Here we have instead:

$$f : D_1(0) \to \mathbb{C} \qquad F : \mathbb{C} \setminus \{1\} \to \mathbb{C}$$

$$z \mapsto \sum_{n=0}^{\infty} z^n \qquad z \mapsto \frac{1}{1-z}$$

Remark 2.12. *Not every holomorphic function $f : \Omega \to \mathbb{C}$ can be extended to $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$, with $\Omega \subseteq \tilde{\Omega}$*

Example 2.7. *On $D_1(0)$ it holds that*

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

converges by comparison to the geometric series, since $|z|^{n!} < |z|^n$. When we look at series of holomorphic functions, for any $\varepsilon > 0$ we see that $\sum_{n=0}^{\infty} z^{n!}$ converges absolutely and uniformly on compact subsets of $D_{1-\varepsilon}(0)$, but f cannot be extended anywhere beyond $D_1(0)$

Here is another Corollary of the Identity Theorem 2.7:

Theorem 2.12. Let $f, g \in \mathscr{H}(\Omega)$ with Ω open and connected. Then

$$fg = 0 \Longrightarrow f = 0 \text{ or } g = 0$$

Proof. Suppose without loss of generality $f \neq 0$, we want to show that $g = 0$ (otherwise by renaming of the functions, it falls under this same case).

Since $f \neq 0$, so $\exists a \in \Omega : f(a) \neq 0$. By continuity of f, it holds that

$$\exists \varepsilon > 0 \forall z \in D_\varepsilon(a) \subseteq \Omega : f(z) \neq 0$$

The assumption

$$\forall z \in \Omega : f(z)g(z) = 0$$

then implies that $\forall z \in D_\varepsilon(a) : g(z) = 0$. But then

$$g|_{D_\varepsilon(a)} = 0|_{D_\varepsilon(a)} = 0$$

Using the Identity Theorem 2.7 applied to $g : \Omega \to \mathbb{C}$ and the zero function $0 : \Omega \to \mathbb{C}, w \mapsto 0$ gives

$$g|_\Omega = 0|_\Omega = 0$$

□

Remark 2.13. *The analytic functions on a non-empty open subset $\Omega \subseteq \mathbb{C}$, namely $\mathscr{H}(\Omega)$, form a commutative ring with 1. This since the sum and product of holomorphic functions are holomorphic.*

***Recall**: $(R, +, \cdot)$ is a ring with 1*

- *$(R, +)$ is an abelian group.*
- *R is a monoid under "$\cdot$", i.e.*
  - $\forall a, b, c \in R : (ab)c = a(bc)$
  - $\exists 1_R \in R \forall a \in R : a \cdot 1_R = 1_R \cdot a$
- *"$\cdot$" distributes over "$+$", i.e.*
  - $\forall a, b, c \in R : a(b + c) = ab + ac$
  - $\forall a, b, c \in R : (b + c)a = ba + ca$

The last Theorem 2.12 says that if Ω is open and connected, then the ring of analytic functions on it has no zero divisor, hence is an integral domain[3].

Our next application is the Morera's Theorem, which is a converse to Goursat's Theorem 2.1.

Recall: Goursat's Theorem 2.1 says the following: let $f : \Omega \to \mathbb{C}$ for Ω open be a holomorphic function and let T in Ω be a triangle, whose interior is also contained in Ω, then

$$\int_T f(z)dz = 0$$

Theorem 2.13 (Morera's Theorem)**.** [SS10, Theorem II.5.1] Let $\Omega \subseteq \mathbb{C}$ open and $f \in C^0(\Omega; \mathbb{C})$. Assume that for any open disc D with $\overline{D} \subset \Omega$ and any triangular path T such that $im(T) \subseteq D$ we have that

$$\int_T f(z)dz = 0$$

Then $f \in \mathscr{H}(\Omega)$, i.e. f is holomorphic on Ω

Proof. Let $z_0 \in \Omega$ and $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$. For $z \in D_r(z_0)$ we define

$$\boxed{F(z) := \int_{\gamma_z := \ell_{[z_0, z]}} f(w)dw}$$

[3] In algebra, an integral domain is a non-zero commutative ring in which the product of any two non-zero elements is non-zero. Integral domains are generalizations of the ring of integers and provide a natural setting for studying divisibility. In an integral domain, every non-zero element a has the cancellation property, that is, if $a \neq 0$, an equality $ab = ac$ implies $b = c$

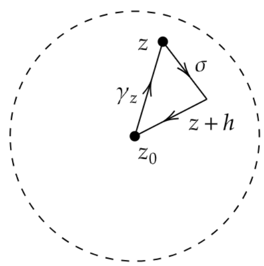
where $\gamma_z : [0,1] \to \mathbb{C}, z \mapsto (1-t)z_0 + zt$ is the line segment joining z_0 to z. Then for a small enough $h \in \mathbb{C}$ such that $z + h \in D_r(z_0)$, it holds that

$$F(z+h) - F(z) = \int_{\sigma := \ell_{[z,z+h]}} f(w)dw$$

where σ is the linear path linking z and $z+h$. Since by assumption

$$\int_T f(w)dw = 0$$

for any T in $D_r(z_0)$, in particular for $im(T) = \langle z_0, z, z+h \rangle$, we state the following claim.



Claim. Using continuity of f at z one can show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

Proof of the Claim. Consider the difference

$$\begin{aligned} F(z+h) - F(z) &= \int_{\ell_{[z,z+h]}} \big(f(w) - f(z) + f(z)\big)dw = \\ &= f(z) \underbrace{\int_{\ell_{[z,z+h]}} dw}_{=h} + \int_{\ell_{[z,z+h]}} \big(f(w) - f(z)\big)dw \end{aligned}$$

and the estimate obtained from 1.10 (iv), namely

$$\left| \int_{\ell_{[z,z+h]}} \big(f(w) - f(z)\big)dw \right| \leq \sup_{w \in \ell_{[z,z+h]}} |f(w) - f(z)| \, h$$

It follows that

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \sup_{w \in \ell_{[z,z+h]}} |f(w) - f(z)|$$

Since f is continuous at z, it holds that

$$\sup_{w\in\ell_{[z,z+h]}} |f(w)-f(z)| \xrightarrow{h\to 0} 0$$

and thus finally

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \lim_{h\to 0}\left(f(z)\frac{h}{h}\right) + \lim_{h\to 0}\sup_{w\in\ell_{[z,z+h]}}|f(w)-f(z)| = f(z)$$

□

so F is holomorphic on $D_r(z_0)$, but then F' is also holomorphic on $D_r(z_0)$, by Theorem 2.3. Since $F' = f$, it follows that f is holomorphic on $D_r(z_0)$ as above, but then f is holomorphic on all of Ω, as $z_0 \in \Omega$ was arbitrary. □

2.4 Sequences of holomorphic functions

It is known from Real Analysis that pointwise convergence of a sequence of functions leads to pathologies, such as the pointwise limit of a sequence of continuous functions not being necessarily continuous.

To avoid this we used a stronger form of convergence: the uniform convergence. For example, *the limit of a uniform convergent sequence of continuous functions is continuous.*

We also have that *uniformly convergent sequences of integrable functions converges to an integrable function.*

Hence, uniform convergence of sequences of functions has better stability properties. Uniformly convergent sequences of differentiable functions do not necessarily have differentiable limits, though.

In this regard we are going to see that sequences of complex functions have much better stability properties. As in the real case, *the uniform limit of a sequence of continuous functions is continuous* and similarly *line integrals of a uniformly convergent sequence of functions converge to the line integral of the limit function.*

Definition 2.5. A sequence $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{C}^\Omega\right)^{\mathbb{N}^*}$ of functions defined on an open set $\Omega \subseteq \mathbb{C}$ is called **uniformly convergent in Ω to the limit** $\lim_{n\to\infty} f_n = f \in \mathbb{C}^\Omega$, if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall z \in \Omega : \left|f(z) - f_n(z)\right| < \varepsilon$$
$$\iff \lim_{n\to\infty} \sup\left\{\left|f(z)-f_n(z)\right| : z\in\Omega\right\} = 0$$

Alternatively, such a sequence is denoted as $f_n \underset{n\to\infty}{\rightrightarrows} f$ or simply $f_n \rightrightarrows f$

Note that in this definition, N does not depend on z, rather only on ε

It turns out that we only need local uniform convergence or equivalently uniform convergence on compact subsets.

Definition 2.6. Let $\Omega \subseteq \mathbb{C}$ be open and $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{C}^\Omega\right)^{\mathbb{N}^*}$ a sequence of functions. $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{C}^\Omega\right)^{\mathbb{N}^*}$ is called **locally uniformly convergent** or **compactly convergent** or **uniformly convergent on compact sets**, if the following equivalent conditions are satisfied:

(i) $\forall a \in \Omega \exists \varepsilon > 0 : D_\varepsilon(a) \subseteq \Omega$ and $\left(f_n|_{D_\varepsilon(a)}\right)_{n\in\mathbb{N}^*}$ converges uniformly in $D_\varepsilon(a)$

(ii) $\forall K \subseteq \Omega$ compact $: (f_n|_K)_{n\in\mathbb{N}^*}$ converges uniformly in K

Remark 2.14. *Note that the previous conditions are equivalent.*
(i)⟹(ii): *since K is covered by finitely many discs in (i).*
(i)⟹(ii): *since Ω is open, for all $a \in \Omega$ there is a closed disc $\overline{D}$, i.e. compact, such that $a \in \overline{D} \subset \Omega$*

Remark 2.15. *Note that since continuity is a local property even in the case of real valued functions, local uniform convergence of continuous functions will imply continuity of the limit function.*

Hence, similarly to the real case one can show that:

Proposition 2.3. Let $\Omega \subseteq \mathbb{C}$ be open and $(f_n)_{n\in\mathbb{N}^*} \in \left(C^0(\Omega;\mathbb{C})\right)^{\mathbb{N}^*}$ a sequence of continuous functions. If $(f_n)_{n\in\mathbb{N}^*}$ converge uniformly to a function f in every compact subset of Ω, then $f \in C^0(\Omega;\mathbb{C})$

With this result in mind we continue to the main Theorem that we have in this section.

Theorem 2.14. [SS10, Theorem II.5.2] Let $\Omega \subseteq \mathbb{C}$ be open and $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathscr{H}(\Omega)\right)^{\mathbb{N}^*}$ a sequence of holomorphic functions. If $(f_n)_{n\in\mathbb{N}^*}$ converge uniformly to a function f in every compact subset of Ω, then $f \in \mathscr{H}(\Omega)$

Proof. Since each of the f_n's is holomorphic, they are all also continuous, hence by the above Proposition 2.3 their limit f is also continuous.

To show that f is also holomorphic we will use Morera's Theorem 2.13 and the fact that the set described by any triangle T is compact.

By Morera's Theorem 2.13, since f is already continuous, to show that f is holomorphic, it is enough to show that

$$\int_T f(w)dw = 0$$

for any open disc D such that $T \subseteq D \subseteq \Omega$ and where T is a triangle contained in D

Let $D = D_r(z_0) \subseteq \Omega$ an open disc in Ω for a z_0 and a $r > 0$. Let T be any triangle with inside contained in D. By Goursat's Theorem 2.1, we have

$$\forall n \geq 1 : \int_T f_n(w)dw = 0$$

since $f_n(z) \to f(z)$ uniformly on compact sets. Being the set described by T compact, we have that $f_n(z) \to f(z)$ uniformly on it, so

$$\left| \int_T f_n(z)dz - \int_T f(z)dz \right| \leq \int_T |f_n(z) - f(z)| \, |dz| \leq$$
$$\leq \sup_{z \in T} |f_n(z) - f(z)| \, L_T \xrightarrow{n \to \infty} 0 \cdot L_T = 0$$

for $L_T \in \mathbb{R}$ the length of the perimeter of T and since $(f_n)_{n \in \mathbb{N}^*}$ converges uniformly on the set delimited by T. We hence have that

$$\lim_{n \to \infty} \underbrace{\int_T f_n(z)dz}_{=0} = \int_T f(z)dz$$

and therefore, we obtain

$$\int_T f(z)dz = 0$$

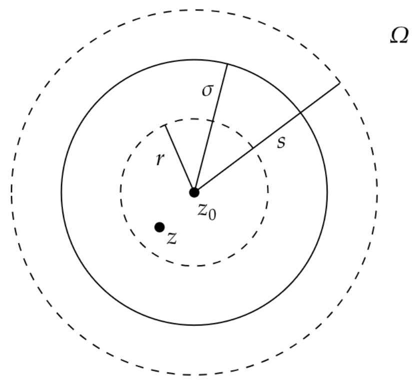
and so finally $f \in \mathscr{H}(\Omega)$, using Morera's Theorem 2.13 to conclude the proof. □

We can extend the previous result to the following generalisation.

Theorem 2.15. [SS10, Theorem II.5.3] Let $\Omega \subseteq \mathbb{C}$ be open and $(f_n)_{n \in \mathbb{N}^*} \in \left(\mathscr{H}(\Omega)\right)^{\mathbb{N}^*}$ a sequence of holomorphic functions such that $f_n \rightrightarrows f$, i.e. f_n converges uniformly to f, on every compact subset $K \subseteq \Omega$.

Then $(f'_n)_{n \in \mathbb{N}^*} \in \left(\mathscr{H}(\Omega)\right)^{\mathbb{N}^*}$ converges uniformly to $\lim_{n \to \infty} f'_n = f'$ on every compact subset of Ω

Proof. Let $z_0 \in \Omega$ and $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$, then $(f_n)_{n\in\mathbb{N}^*}$ converges uniformly to f also on $\overline{D}_r(z_0)$ as subset of Ω



Let $s > r$ such that $D_s(z_0) \subset \Omega$ and $\sigma := \frac{r+s}{2} \in (r, s)$. We have then by the Cauchy Integral Formula for derivatives 2.3 that

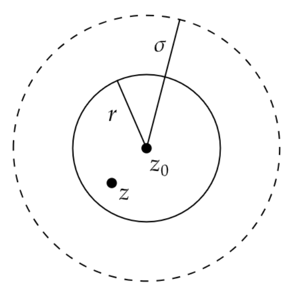
$$f'(z) = \frac{1}{2\pi i} \int_{C_\sigma(z_0)} \frac{f(w)}{(w-z)^2} dw$$

for every $z \in \overline{D}_r(z_0) \subset D_\sigma(z_0)$. Hence for $z \in \overline{D}_r(z_0)$ we have

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_{C_\sigma(z_0)} \frac{f_n(w) - f(w)}{(w-z)^2} dw \right| \leq \frac{1}{2\pi}(2\pi\sigma) \sup_{w\in C_\sigma(z_0)} \left| \frac{f_n(w) - f(w)}{(w-z)^2} \right|$$

To bound the denominator for both $w \in C_\sigma(z_0)$ and $z \in \overline{D}_r(z_0)$, it holds that

$$|w - z| = |w - z_0 + z_0 - z| = \big|w - z_0 - (z - z_0)\big| \geq \big||w - z_0| - |z - z_0|\big| = |\sigma - r|$$



Hence, we conclude that

$$|f_n'(z) - f'(z)| \leq \frac{\sigma}{(\sigma - r)^2} \sup_{w\in C_\sigma(z_0)} |f_n(w) - f(w)| \xrightarrow{n\to\infty} 0$$

and since $f_n \rightrightarrows f$ on the compact set $C_\sigma(z_0)$, we have that $f_n' \rightrightarrows f'$ on $\overline{D}_r(z_0)$

Since by definition every compact set is contained in a union of finitely many such discs, the result follows. □

These Theorems are often used to prove holomorphicity of functions defined by infinite series.

Corollary 2.8. Let $\Omega \subseteq \mathbb{C}$ be open and $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathscr{H}(\Omega)\right)^{\mathbb{N}^*}$, if

$$\forall z \in \Omega : F(z) = \sum_{n=1}^{\infty} f_n(z)$$

we then let $S_N(z) = \sum_{n=1}^{N} f_n(z)$ and so S_N is holomorphic. If the sequence $(S_N)_{N\in\mathbb{N}^*}$ converges uniformly on compact subsets of Ω, then $\lim_{N\to\infty} S_N(z) = F(z)$ is also holomorphic.

Proof. This Corollary is a direct consequence of the previous Theorem 2.15 applied on the sequence $(S_N)_{N\in\mathbb{N}^*}$ defined as follows: for each $N \in \mathbb{N}^*$

$$S_N : \Omega \to \mathbb{C}, z \mapsto S_N(z) = \sum_{n=1}^{N} f_n(z)$$

□

For series of functions, we have also the following useful Theorem of Weierstrass, called Weierstrass M-test.

Theorem 2.16 (Weierstrass M-test)**.** Let $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{C}^\Omega\right)^{\mathbb{N}^*}$ be a sequence of functions and $\emptyset \neq U \subseteq \Omega$. Suppose that

$$\exists (M_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{R}_{\geq 0}\right)^{\mathbb{N}^*} : \left(\forall n \in \mathbb{N}^* \forall z \in U : \left|f_n(z)\right| \leq M_n\right) \text{ and } \sum_{n\in\mathbb{N}^*} M_n < \infty$$

Then $\sum_{n\in\mathbb{N}^*} f_n$ converges absolutely and uniformly on U

Proof. Let $z \in \Omega$ and consider $N \in \mathbb{N}^*$, then

$$\sum_{n=1}^{N} |f_n(z)| \leq \sum_{n=1}^{N} M_n$$

by then taking the limit on both sides one gets

$$\sum_{n=1}^{\infty} |f_n(z)| = \lim_{N\to\infty} \sum_{n=1}^{N} |f_n(z)| \leq \lim_{N\to\infty} \sum_{n=1}^{N} M_n = \sum_{n=1}^{\infty} M_n < \infty$$

Hence, $\big(|f_n|\big)_{n\in\mathbb{N}^*}$ convergence absolutely and hence also in the normal sense, but being $(M_n)_{n\in\mathbb{N}^*}$ independent of z, then also uniformly. The arbitrariness of z and N give the result. □

Example 2.8. *For $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ the function $z \mapsto n^z :=$ $\exp(z \log(n))$ is an analytic function on $\mathbb{C}$*

$$|n^z| = \Big|\exp\big((x+iy)\log(n)\big)\Big| = \exp\big(x\log(n)\big) = n^x$$

Then we have

Proposition 2.4. [SS10, Proposition VI.2.1] The series that represents **the Riemann Zeta function**

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges absolutely and uniformly on every half plane

$$U_\delta := \{z \in \mathbb{C} : Re(z) \geq 1 + \delta\} \text{ with } \delta > 0$$

and is holomorphic in $\{z \in \mathbb{C} : Re(z) > 1\}$, i.e. $\zeta \in \mathscr{H}\big(\{z \in \mathbb{C} : Re(z) > 1\}\big)$

Proof. For each $\delta > 0$ it holds that, given

$$Re(z) = \sigma \geq 1 + \delta > 1$$

the series $\zeta(z)$ is uniformly bounded by

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$$

since

$$|\zeta(z)| = \left|\sum_{n=1}^{\infty} \frac{1}{n^z}\right| \leq \sum_{n=1}^{\infty} \frac{1}{|n^z|} \leq \left|\sum_{n=1}^{\infty} \frac{1}{n^{Re(z)}}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$$

Applying the previous Weierstrass M-test 2.16 we obtain the result. □

Hence $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on every half plane as $\forall \delta > 0 : Re(z) \geq 1 + \delta > 1$ and therefore defines a holomorphic function if $Re(z) > 1$ (every compact subset of $\{z \in \mathbb{C} : Re(z) > 1\}$ is contained in such a half plane in which $Re(z) \geq 1 + \delta$).

Definition 2.7. For $z \in \mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\} \subset \mathbb{C}$ (note that $\mathbb{H}$ is open), we define the **Theta function**

$$\theta : \mathbb{H} \to \mathbb{C}$$
$$z \mapsto \theta(z) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{2\pi i n^2 z}$$

A basic result for it is the following.

Proposition 2.5. For all $z \in \mathbb{H}$, the function $\theta(z)$ is well defined, i.e. the series converges and defines a holomorphic function there.

Proof. We are going to show that θ converges uniformly on any subset of the form $\mathbb{H}_\delta := \{z \in \mathbb{C} : Im(z) \geq \delta\} \subset \mathbb{C}$ with $\delta > 0$. Since any compact subset of $\mathbb{H}$ is contained in such a set, by Theorem 2.14 this will imply the result.

Let $\delta > 0$, for any $z \in \mathbb{H}_\delta$ with $z := x + iy$ and $y \geq \delta > 0$ it holds that

$$\forall n \in \mathbb{N} : |e^{2\pi i n^2 z}| = \underbrace{|e^{2\pi i n^2 x}|}_{=1} \cdot |e^{-2\pi n^2 y}| = e^{-2\pi n^2 y} \leq e^{-2\pi n y}$$

Since $y \geq \delta$, it holds that $e^{-2\pi n y} \leq e^{-2\pi n \delta} < 1$. Hence

$$\left| \sum_{n=0}^{\infty} e^{2\pi i n^2 z} \right| \leq \underbrace{\sum_{n=0}^{\infty} e^{-2\pi n \delta}}_{\text{geometric series}} < \infty$$

Hence, $\sum_{n=0}^{\infty} e^{2\pi i n^2 z}$ converges uniformly on $\mathbb{H}_\delta$ for any $\delta > 0$ and therefore also on every of their compact subset. Thus it defines a holomorphic function on $\mathbb{H}$, as argued above. □

Example 2.9 (Fourier analysis and the Theta function)**.** *The Fourier Transform serves as important tool to Number Theory, in this settings we can think of the Theta function as such a transform of a certain function. The uniqueness of the coefficients of this transform allows the Theta function to identify the squares, for instance. So,*

$$1 + 2\sum_{n=1}^{\infty} e^{2\pi i n^2 z} = \sum_{n=0}^{\infty} \alpha(m) e^{2\pi i n^2 z} \implies \alpha(m) = \begin{cases} 1 & , \ m = 0 \\ 2 & , \ m \text{ is a square} \\ 0 & , \ \text{else} \end{cases}$$

A more remarkable result is obtained when looking at, for example

$$\theta^4(z) = \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}}\sum_{n_4\in\mathbb{Z}} e^{2\pi i(n_1^2+n_2^2+n_3^2+n_4^2)z} = \sum_{m=1}^{\infty}\beta(m)e^{2\pi imz}$$

Here is interesting that

$$\beta(m) = \#\{n_1,n_2,n_3,n_4) \in \mathbb{Z}^4 : (n_1^2+n_2^2+n_3^2+n_4^2 = m\}$$

Remark 2.16. *We are going to come back to $\zeta(s)$ and $\theta(z)$ in Chapter 4: there, we are going to use $\theta(z)$ to show that $\zeta(s)$ (which is defined as the series $\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $Re(s) > 1$) has an analytic continuation to $\mathbb{C}\setminus\{1\}$. There we are also going to show the relation between these two functions allows the following identity:*

$$\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \frac{1}{2}\int_0^{\infty}\left(\theta(it)-1\right)t^{\frac{z}{2}}\frac{dt}{t}$$

for $Re(z) > 1$. One can show that in fact the RHS makes sense for any $z \in \mathbb{C}\setminus\{0,1\}$, this gives the analytic continuation of ζ. The latter identity is known as Mellin transform.

2.5 Holomorphic functions defined in terms of integrals

Finally, we also have a similar Theorem for functions defined in terms of integrals, i.e. similar to the Theorems for functions defined in terms of infinite series.

Many special functions in mathematics are defined in terms of integrals of the type

$$f(z) := \int_a^b F(z,t)dt$$

or as limits of such integrals.

Example 2.10 (Γ representation as limit)**.** *Consider in this sense*

$$\Gamma(z) := \lim_{\varepsilon\to\infty}\int_{\frac{1}{\varepsilon}}^{\varepsilon} e^{-t}t^{z-1}dt$$

We have the following Theorem

Theorem 2.17. [SS10, Theorem II.5.4] Let $\Omega \subseteq \mathbb{C}$ be open and $I = [a,b] \subset \mathbb{R}$ a compact interval. Let $F : \Omega \times I \to \mathbb{C}$ be a function with the following properties:

(i) $F \in C^0(\Omega \times I)$

(ii) $\forall t_0 \in I : f_{t_0}(z) := F(z,t_0) \in \mathscr{H}(\Omega)$

Then the function f defined by the following integral is holomorphic on Ω, hence

$$f(z) := \int_a^b F(z,t)dt \in \mathscr{H}(\Omega)$$

Proof. The idea is to use the Riemann sums to approximate the integral: let

$$f_n(z) := \frac{b-a}{n}\sum_{j=0}^{n-1} F\left(z, a + \frac{b-a}{n}j\right)$$

then $f_n(z)$ is a finite sum of holomorphic functions, hence holomorphic.

We want to show that $(f_n)_{n\in\mathbb{N}^*}$ converges to f uniformly on compact subsets. Then using Theorem 2.15 we can conclude that f is holomorphic.

Let $K \subseteq \Omega$ be compact. We use that a continuous function $F : \Omega \times I \to \mathbb{C}$, when restricted to the compact set $K \times I$, is uniformly continuous.

Hence

$$\forall\varepsilon > 0 \exists\delta > 0 \forall(z_1, t_1), (z_2, t_2) \in K \times I :$$
$$\big(|z_1 - z_2| < \delta \text{ and } |t_1 - t_2| < \delta\big) \implies |F(z_1, t_1) - F(z_2, t_2)| < \frac{\varepsilon}{b-a}$$

Let now $n \in \mathbb{N}^*$ be such that $\frac{b-a}{n} < \delta$, then let $z \in K$ and consider

$$f_n(z) - f(z) = \sum_{j=0}^{n-1} \int_{a+j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} \Bigg(\underbrace{F\left(z, a + j\frac{b-a}{n}\right)}_{\text{independent of } t} - \underbrace{F(z,t)}_{\text{dependent of } t}\Bigg)dt$$

using

$$f(z) = \int_a^b F(z,t)dt = \sum_{j=0}^{n-1} \int_{a+j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} F(z,t)dt$$

and

$$f_n(z) = \frac{b-a}{n}\sum_{j=0}^{n-1} F\left(z, a + j\frac{b-a}{n}\right) =$$
$$= \sum_{j=0}^{n-1} \int_{a+j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} F\left(z, a + j\frac{b-a}{n}\right)dt =$$
$$= \sum_{j=0}^{n-1} \frac{F\left(z, a + j\frac{b-a}{n}\right)}{\frac{b-a}{n}}$$

since the integrand is independent of t in the second last equality.

Now, for $t \in \left[a + j\frac{b-a}{n}, a + (j+1)\frac{b-a}{n}\right]$, it holds that $\left|t - \left(a + j\frac{b-a}{n}\right)\right| < \frac{b-a}{n} < \delta$ and since the "z-arguments" are equal, we also have $0 = |z - z| < \delta$. Hence

$$\left|F\left(z, a + j\frac{b-a}{n}\right) - F(z,t)\right| < \frac{\varepsilon}{b-a}$$

and finally

$$|f_n(z) - f(z)| \leq \frac{\varepsilon}{b-a}\sum_{j=0}^{n-1}\frac{b-a}{n} = \frac{\varepsilon}{b-a}n\frac{b-a}{n} = \varepsilon$$

which gives the uniform convergence of f_n to f on K, since z was arbitrary in that set. Being K arbitrary we obtain by Theorem 2.15 that f is holomorphic. □

Remark 2.17. *With some more work one can also show that f' is given by*

$$\forall z \in \Omega : f'(z) = \frac{d}{dz}\int_a^b F'(z,t)dt = \int_a^b \frac{d}{dz}F(z,t)dt = \int_a^b \underbrace{F'(z,t)}_{=f_t'(z)}dt$$

In other words, we can interchange integration and differentiation.

Remark 2.18. *Many special functions that appear as solutions of differential equation, for example Bessel functions, have integral representations,e.g. $J_n(z)$ is defined as solution of Bessel's complex differential equations:*

$$z^2\frac{d^2f(z)}{dz^2} + z\frac{df(z)}{dz} + (z^2 - n^2)f(z) = 0$$

For $n \in \mathbb{Z}$, it holds that

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin(t)}e^{-int}dt$$
$$\text{and } J_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin(t)}dt$$

with $F(z,t) = e^{iz\sin(t)}$ is continuous on $\mathbb{C} \times \mathbb{R}$. For each $t \in [-\pi, \pi]$ it holds $f_t(z) = e^{iz\sin(t)} : \mathbb{C} \to \mathbb{C}$ is holomorphic. Hence the function $\int_{-\pi}^{\pi} e^{iz\sin(t)}dt$ is holomorphic on $\mathbb{C}$ and by the previous Theorem 2.17, it results that

$$J_0'(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin(t)}\big(i\sin(t)\big)dt$$

Chapter 3

Meromorphic functions and Residue Formula

The goal is to extend Cauchy's Theorem 2.5 and the Cauchy Integral Formula 2.6 from holomorphic functions to functions which might have singularities.

Recall: Cauchy's Theorem 2.5 states that for any $f \in \mathscr{H}(\Omega)$, any closed curve γ such that the image of the curve and its interior are both contained in Ω, we have

$$\int_\gamma f(z)dz = 0$$

The Cauchy Integral Formula 2.6 states instead that for any $z \in D \subset \Omega$, with D a disc, such that $\partial D =: C$ and for any $f|_D \in \mathscr{H}(D)$, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$$

To this end, we are first going to look at isolated singularities of a function f. We will see that there are three prototypes for these:

- removable singularities
- poles
- essential singularities

given respectively by the following functions at $z = 0$: $\frac{\sin(z)}{z}, \frac{1}{z}, e^{\frac{1}{z}}$. For instance, for the first function it holds that

$$\frac{\sin(z)}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

shows that $z = 0$ is a "removable" singularity. One can view the RHS[1] as an analytic continuation to all $\mathbb{C}$ of the LHS[2] $= \frac{\sin(z)}{z}$. For the second function we have

$$\lim_{z\to 0}\left|\frac{1}{z}\right| = \infty$$

whereas for the third function $|e^{\frac{1}{z}}|$ oscillates. In fact,

- if $z \to 0$ on positive real numbers, then $|e^{\frac{1}{x}}| \xrightarrow{x\searrow 0} \infty$
- if $z \to 0$ on negative real numbers, then $|e^{\frac{1}{x}}| \xrightarrow{x\nearrow 0} 0$

These three examples of singularities are what we call a removable singularity, a pole and an essential singularity respectively.

We will prove a generalisation of Cauchy's Theorem 2.5 to functions that are holomorphic except for finitely many isolated points. This will lead us to the

Residue Formula: if $f \in \mathscr{H}(\Omega)$ for an open Ω containing a circle C and its interior, except finitely many points $z_1, ..., z_n$ with $n \in \mathbb{N}$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{N} \underbrace{Res_{z_k}(f)}_{a_{-1}}$$

where we will also see that f in a neighbourhood of z_0 has the form

$$f(z) = G(z) + \sum_{\ell=1}^{n} \frac{a_{-\ell}}{(z-z_0)^{\ell}}$$

where $G \in \mathscr{H}\big(D_r(z_0)\big)$

This Theorem, like Cauchy's Theorem 2.5 can be used to evaluate many real integrals and complex line integrals. It will also lead to many theoretical results just like Cauchy's Theorem 2.5 did.

Argument principle: which allows us to count the number of zeroes (and poles) of holomorphic (meromorphic) functions inside closed curves.

Rouché's Theorem: a holomorphic function can be perturbed slightly without changing the number of its zeroes.
Let $f, g \in \mathscr{H}(\Omega)$ with Ω open and containing C and its interior. If $\forall z \in C : |f(z)| \geq$

[1]Short form for: "Right Hand Side"
[2]Short form for: "Left Hand Side"

$|g(z)|$, then f and $f+g$ have the same number of zeroes inside C

Open mapping Theorem: let $f \in \mathscr{H}(\Omega)$ be non-constant in a region Ω, then f is an open map, i.e. image of an open set is open.

Maximum modulus principle: if f is non-constant on Ω open and connected, with compact closure $\overline{\Omega}$, and f is continuous on $\overline{\Omega}$, then

$$\sup_{z\in\Omega} |f(z)| \leq \sup_{z\in\overline{\Omega}\setminus\Omega} |f(z)|$$

Another way to say this is the following

$$\underbrace{\max_{z\in\overline{\Omega}} |f(z)|}_{\text{exists because } \overline{\Omega} \text{ is compact}} \leq \max_{z\in\overline{\Omega}\setminus\Omega} |f(z)|$$

3.1 Zeroes and poles

We start with a definition of singularities

Definition 3.1. Let $\Omega \subseteq \mathbb{C}$ be open and let $f \in \mathscr{H}(\Omega)$. The point $z_0 \in \mathbb{C} \setminus \Omega$ is called a (possible) **isolated singularity of** f if

$$\exists r > 0 : f|_{\dot{D}_r(z_0)} \in \mathscr{H}\left(\dot{D}_r(z_0)\right)$$

or if

$$\exists U_{z_0} \in \mathcal{O}_{\mathbb{C}}^{\Omega} : f|_{U_{z_0}\setminus\{z_0\}} \in \mathscr{H}\big(U_{z_0} \setminus \{z_0\}\big)$$

where $\mathcal{O}_{\mathbb{C}}^{\Omega}$ is the induced topology from $\mathbb{C}$

Note that in the above Definition 3.1 the term "possible" means that the singularity in question at that point can possibly be removed.

Example 3.1. *Let* $\tan\left(\frac{1}{z}\right)$ *be defined in* $\mathbb{C}$ *without some countable set has singularities at*

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \ldots, 0$$

Note that 0 *is not an isolated singularity of* f. *The points*

$$\frac{2}{\pi}, \frac{3}{2\pi}, \ldots, \frac{2}{(2k+1)\pi}$$

for $k \in \mathbb{N}$ *are instead isolated.*

Example 3.2. *Consider the function*

$$f : \mathbb{C}^* = \mathbb{C} \setminus \{0\} \to \mathbb{C}$$
$$z \mapsto z$$

$z = 0$ is an isolated singularity of f, because f is not defined there, but f can in fact be extended to all of $\mathbb{C}$ by defining $f(0) = 0$. In this case, $z = 0$ is a "removable singularity" of f

On the other hand, we have that

$$f : \mathbb{C}^* = \mathbb{C} \setminus \{0\} \to \mathbb{C}$$
$$z \mapsto \frac{1}{z}$$

has a singularity at $z = 0$, which cannot be removed.

Definition 3.2. Let $\Omega \subseteq \mathbb{C}$ be open. An isolated singularity $z_0 \in \mathbb{C}$ of a function $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ is called **removable** if f is **holomorphically extendable to all of** Ω, i.e.

$$\exists F \in \mathscr{H}(\Omega) \forall z \in \Omega \setminus \{z_0\} : F(z) = f(z)$$

We have the following Theorem of Riemann, sometimes called **Riemann continuation Theorem**.

Theorem 3.1 (Riemann continuation Theorem)**.** Let $\Omega \subseteq \mathbb{C}$ be open and non-empty together with $z_0 \in \Omega$. Then the following assertions for a function $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ are equivalent:

(i) f is holomorphically extendable to Ω

(ii) f is continuously extendable to Ω

(iii) f is bounded in a neighbourhood of z_0, i.e. $\exists r > 0 : f|_{\dot{D}_r(z_0)} \in \mathcal{B}\left(\dot{D}_r(z_0)\right)$

(iv) $\lim_{z \to z_0} (z - z_0) f(z) = 0$

Proof. Exercise. □

If a point satisfies one of these equivalent conditions, we call it a removable singularity. As a consequence of Riemann's continuation Theorem we have **Riemann's Theorem on removable singularities**.

Theorem 3.2 (Riemann's Theorem on removable singularities)**.** [SS10, Theorem III.3.1] Let $\Omega \subseteq \mathbb{C}$ be open, $z_0 \in \Omega$ and $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$, if $f|_{\dot{D}_r(z_0)} \in \mathcal{B}\big(\dot{D}_r(z_0)\big)$ for some $D_r(z_0) \subset \Omega$ with $r > 0$, then z_0 is a removable singularity of f, i.e.

$$\exists F \in \mathscr{H}(\Omega)\forall z \in \Omega \setminus \{z_0\} : F(z) = f(z)$$

Note that this represents (iii)$\Longrightarrow$(i) in the previous Theorem which states (iii)$\Longleftrightarrow$(i)

Example 3.3. *1. Let $f(z) = \frac{\sin(z)}{z}$ for $z \neq 0$, f has a removable singularity at $z = 0$. We can see this either using (iv) of the Riemann's continuation Theorem 3.1*

$$\lim_{z\to z_0} zf(z) = \lim_{z\to z_0} \sin(z) = 0$$

or using the extension of L'Hôpital's Rule to $\mathbb{C}$ [see Appendix B.2].

$$\lim_{z\to z_0} \frac{\sin(z)}{z} = \lim_{z\to z_0} \frac{\cos(z)}{1} = 1$$

which imply that $f(z) = \frac{\sin(z)}{z}$ is bounded in a neighbourhood of 0, i.e. $f \in \mathcal{B}\left(\dot{D}_r(0)\right)$. In fact, it holds that

$$\lim_{\substack{z\to 0\\ z\neq 0}} \frac{\sin(z)}{z} = 1$$

Finally, let $\varepsilon = \frac{1}{2}$, then $\exists r > 0 \forall |z| \in (0, r)$ it holds that

$$\left|\frac{\sin(z)}{z} - 1\right| < \frac{1}{2}$$

and therefore $f \in \mathcal{B}\left(\dot{D}_r(0)\right)$. One can also use that on $\mathbb{C}$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

and hence it follows that on $\mathbb{C}$

$$F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

is the holomorphic extension of $\frac{\sin(z)}{z}$ to all of $\mathbb{C}$

2. *Another such example is: if* $f(z) = \frac{z}{e^z-1}$ *on* $\mathbb{C}^*$*, then* f *has removable singularity at* $z = 0$*, since*
$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z}{e^z - 1} = \lim_{z\to 0} \frac{1}{e^z} = 1$$
by the same extension of L'Hôpital's rule as before. Hence f *is bounded in a neighbourhood of* 0*, i.e.* $\exists r > 0 : f \in \mathcal{B}\left(\dot{D}_r(0)\right)$

If $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ does not have a removable singularity at z_0, then f is not bounded near z_0. We can then ask whether its unboundedness is similar to the unboundedness of $\frac{1}{(z-z_0)^n}$, i.e. we can ask whether $(z - z_0)^n f(z)$ is bounded near z_0 for n sufficiently large. If such an $n \in \mathbb{N}$ exists, then z_0 is called a pole of f, let us hence first define this once for all.

Definition 3.3 (Pole)**.** Let $z_0 \in \mathbb{C}$ and $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$, if

$$\exists n \in \mathbb{N}^* \exists r > 0 : \dot{D}_r(z_0) \subset \Omega \text{ and } (z - z_0)^n \, f|_{\dot{D}_r(z_0)} \in \mathcal{B}\left(\dot{D}_r(z_0)\right)$$

then z_0 is called a **pole of** f and the natural number

$$m := \min\left\{n \in \mathbb{N}^* : \exists r > 0 : (z - z_0)^n f \in \mathcal{B}\left(\dot{D}_r(z_0)\right)\right\} \in \mathbb{N}^*$$

is called **the order of the pole of** f **at** z_0. Poles of first order are called **simple poles**.

Remark 3.1. *To compute the order of pole of* f *at* z_0*, one can can compute the order of zero of* $\frac{1}{f}$ *at* z_0*, as shown later.*

Example 3.4. *The function* $f(z) = \frac{1}{(z-z_0)^m}$ *for* $z \in \mathbb{C} \setminus \{z_0\}$ *has a unique pole of order* m *at* $z = z_0$

We will see soon that poles arise from reciprocals of holomorphic functions with zeroes.

Before we make this more precise, let us recall that zeroes of holomorphic functions are isolated and we have the following Proposition for their behaviour near a zero.

Recall (Proposition 2.2): Suppose $f \in \mathscr{H}(\Omega)$ in Ω open connected, f has a zero at a point $z_0 \in \Omega$ and does not vanish identically on Ω. Then

$$\exists r > 0 \exists! g \in \mathscr{H}\big(D_r(z_0)\big) \setminus \{0|_{D_r(z_0)}\} \exists! n \in \mathbb{N}^* : \big(\overline{D}_r(z_0) \subset \Omega\big) \text{ and}$$
$$\big(\forall z \in D_r(z_0) : f(z) = (z - z_0)^n g(z)\big) \text{ and } \big(n = \min\{n \in \mathbb{N} : f^{[n]}(z_0) \neq 0\}\big)$$

We remark that g is a unique non-vanishing holomorphic function in $\mathscr{H}\big(D_r(z_0)\big)$

The analogous Theorem for the poles is the following:

Theorem 3.3. Let $\Omega \subseteq \mathbb{C}$, $z_0 \in \Omega$ and $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$. For $m \in \mathbb{N}^*$ the following statements are equivalent:

(i) f has a pole of order m at z_0, i.e. $(z-z_0)^m f$ is bounded near z_0 and m is the smallest such integer.

(ii) $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ and $\exists g \in \mathscr{H}\big(D_r(z_0)\big) : g(z_0) \neq 0$ and

$$\forall z \in \dot{D}_r(z_0) : f(z) = \frac{g(z)}{(z-z_0)^m}$$

(iii) $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ and $\exists h \in \mathscr{H}\big(D_r(z_0)\big) \forall z \in \dot{D}_r(z_0) : h(z) \neq 0$, moreover h has a zero of order m at z_0 and such that

$$\forall z \in \dot{D}_r(z_0) : f(z) = \frac{1}{h(z)}$$

Proof. **(i)$\Longrightarrow$(ii)**: The fact that f has a pole of order m at z_0 means that $(z-z_0)^m f(z)$ is bounded near z_0 and that m is minimal, thus $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ on which f in bounded. The Riemann's Theorem on removable singularities 3.2 states that $\exists g \in \mathscr{H}\big(D_r(z_0)\big)$ such that $g(z) = (z-z_0)^m f(z)$ whenever $z_0 \neq z \in D_r(z_0)$

If $g(z_0) = 0$, then it would imply by the previous Proposition 2.2 that

$$g(z) = (z-z_0)\tilde{g}(z)$$

where $\tilde{g} \in \mathscr{H}\big(D_r(z_0)\big)$. Consequently, this would give that for $z \in D_r(z_0)$

$$\tilde{g}(z) = (z-z_0)^{m-1} f(z)$$

is bounded near z_0 and this would contradict the minimality of m. Hence $g(z_0) \neq 0$ and together with it we get that $f(z) = (z-z_0)^{-m} g(z)$ for $z \in \dot{D}_r(z_0)$

(ii)$\Longrightarrow$(iii): Suppose that $\exists r > 0$ as in (ii) and that $\exists g \in \mathscr{H}\big(D_r(z_0)\big)$ such that $g(z_0) \neq 0$ and $f(z) = (z-z_0)^{-m} g(z)$ for $z \in \dot{D}_r(z_0)$. Since $g(z_0) \neq 0$ and g is continuous, then $\exists r > 0 : \forall z \in D_r(z_0) : g(z) \neq 0$, meaning that r can be chosen adequately since the beginning.

Let for $z \in D_r(z_0)$

$$h(z) := \frac{(z-z_0)^m}{g(z)} = \frac{1}{f(z)}$$

then $h(z) \neq 0$ for $z \in \dot{D}_r(z_0)$ and therefore $h \in \mathscr{H}\big(D_r(z_0)\big)$ and with $h(z_0) = 0$. Also, we have that in $\dot{D}_r(z_0)$

$$\frac{1}{h(z)} = (z - z_0)^{-m} g(z) = f(z)$$

Note that h has a zero of order m at z_0, since $h(z) = (z - z_0)^m \frac{1}{g(z)}$ and $\forall z \in D_r(z_0)$: $\frac{1}{g(z)} \neq 0$

(iii)$\Longrightarrow$(i): Suppose that $\exists r > 0 : \overline{D}_r(z_0) \subset \Omega$ and $\exists h \in \mathscr{H}\big(D_r(z_0)\big)$ such that $h(z) \neq 0$ in $\dot{D}_r(z_0)$. Suppose also that h has a zero of order m at z_0 and

$$\forall z \in \dot{D}_r(z_0) : f(z) = \frac{1}{h(z)}$$

Since h has a zero of order m at z_0, then by Proposition 2.2 $\exists k \in \mathscr{H}\big(D_r(z_0)\big)$ such that $h(z) = (z - z_0)^m k(z)$ and

$$\exists s > 0 \forall z \in D_s(z_0) \subseteq D_r(z_0) : k(z) \neq 0$$

Since k is holomorphic and non-vanishing, $\frac{1}{k}$ is holomorphic on $D_s(z_0)$. But then

$$\forall z \in \dot{D}_s(z_0) : f(z) = \frac{1}{h(z)} = (z - z_0)^{-m} \frac{1}{k(z)}$$

would imply that $(z - z_0)^m f(z) = \frac{1}{k(z)}$ is holomorphic on $\dot{D}_s(z_0)$ and has the holomorphic extension $\frac{1}{k}$ on $D_s(z_0)$ (given that $\frac{1}{k}$ is holomorphic on $D_s(z_0)$ since $k \neq 0$ on $D_s(z_0)$).

By the Riemann's continuation Theorem 3.1 we have that

$$(z - z_0)^m f(z)$$

is bounded in a neighbourhood of z_0. Moreover, it holds that

$$(z - z_0)^{m-1} f(z) = \left(\frac{1}{k(z)}\right) \left(\frac{1}{z - z_0}\right)$$

is not bounded since

$$\frac{1}{k(z_0)} \neq 0 \text{ and } \frac{1}{z - z_0} \xrightarrow{z \to z_0} \infty$$

Hence, $m \in \mathbb{N}^*$ is minimal and f has a pole of order m at z_0 □

Example 3.5. *Consider the following examples:*

1. *The function* $f(z) = \frac{1}{e^z - 1}$ *has a pole of order* 1 *at* $z = 0$, *since*

$$\frac{1}{f(z)} = e^z - 1 = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z\left(1 + \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}\right)$$

has a zero of order 1 *at* $z = 0$ *(note that* $\frac{1}{e^z-1}$ *has simple poles at* $z = 2\pi i k$ *for* $k \in \mathbb{Z}$*).*

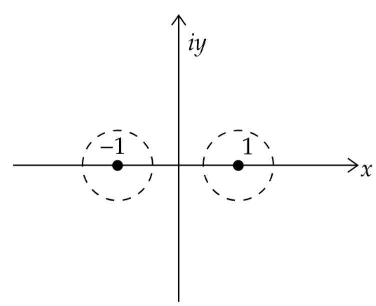
2. *The function* $f(z) = \frac{z}{z^2 - 1}$ *has poles of order* 1 *at* $z = \pm 1$, *since*

$$f(z) = (z-1)^{-1}\left(\frac{z}{z+1}\right)$$

and $h(z) = \frac{z}{z+1}$ *is holomorphic and non-vanishing on* $D_{\frac{1}{2}}(1)$. *Similarly,*

$$f(z) = (z+1)^{-1}\left(\frac{z}{z-1}\right)$$

and $\tilde{h}(z) = \frac{z}{z-1}$ *is holomorphic and non-vanishing on* $D_{\frac{1}{2}}(-1)$



The next Theorem is the analogue of the power series expansion of a holomorphic function.

Recall: if $f \in \mathscr{H}(\Omega)$ and $z_0 \in \Omega$ such that $\overline{D}_r(z_0) \subset \Omega$. Then by Theorem 2.7

$$\forall z \in D_r(z_0) : f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

For functions with poles we have

Theorem 3.4. [SS10, Theorem III.1.3] If $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ has a pole of order n at z_0, then

$$f(z) = G(z) + \sum_{k=1}^{n} \frac{a_{-k}}{(z-z_0)^k}$$

where G is a holomorphic function in a neighbourhood of z_0, i.e. $G \in \mathscr{H}\big(D_r(z_0)\big)$ for some $r > 0$

Proof. f has a pole of order n at z_0, hence $\exists r > 0$ such that $\overline{\dot{D}}_r(z_0) \subset \Omega \setminus \{z_0\}$ and we can write

$$\forall z \in \dot{D}_r(z_0) : f(z) = (z-z_0)^{-n} g(z)$$

with $g \in \mathscr{H}\big(D_r(z_0)\big)$ and with $g(z_0) \neq 0$ with Theorem 3.3

We expand g in a power series by Theorem 2.7, hence

$$\forall z \in D_r(z_0) : g(z) = \sum_{k=0}^{\infty} \frac{g^{[k]}(z_0)}{k!}(z-z_0)^k$$

It follows that for any $z \in \dot{D}_r(z_0)$ we can write

$$\begin{aligned}
f(z) &= \frac{1}{(z-z_0)^n} g(z) = \\
&= \frac{1}{(z-z_0)^n} \sum_{k=0}^{\infty} \frac{g^{[k]}(z_0)}{k!}(z-z_0)^k = \\
&= \left(\sum_{k=0}^{n-1} \frac{g^{[k]}(z_0)}{k!(z-z_0)^{n-k}} \right) + \underbrace{\sum_{k=n}^{\infty} \frac{g^{[k]}(z_0)}{k!}(z-z_0)^{k-n}}_{=G(z)} = \\
&= \left(\sum_{\ell=1}^{n} \frac{g^{[n-\ell]}(z_0)}{(n-\ell)!(z-z_0)^{\ell}} \right) + G(z)
\end{aligned}$$

where $\forall k \in \{1, ..., n\} : a_{-k} := \frac{g^{[n-k]}(z_0)}{(n-k)!}$ □

Remark 3.2. *The function $f(z) := \sum_{-n}^{\infty} a_n (z-a)^k$ is a special case of a* ***Laurent series*** *(see Serie 9 and Serie 10).*

Definition 3.4.

- The number a_{-1}, i.e. the coefficient of $(z-z_0)^{-1}$ in Theorem 3.4, is called **the residue of f at the pole** z_0, denoted by

$$\operatorname{Res}_{z_0}(f) := a_{-1}$$

- The function

$$P_{z_0}(f,z) := P^f_{z_0}(z) := \sum_{j=1}^{n} \frac{a_{-j}}{(z-z_0)^j}$$

in Theorem 3.4 is called **the principal part of f at the pole z_0**, where

$$\forall j \in \{1,...,n\} : a_{-j} := \frac{g^{[j]}(z_0)}{j!}$$

as in latter proof.

Remark 3.3. *If f has a pole of order 1 at z_0, i.e. a simple pole, then*

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)$$

Since if f has a simple pole at z_0, then

$$f(z) = \frac{a_{-1}}{z-z_0} + g(z)$$

with $g \in \mathscr{H}\big(D_r(z_0)\big)$. Hence, it follows that

$$\begin{aligned}(z-z_0)f(z) &= a_{-1} + (z-z_0)g(z)\\ \lim_{z\to z_0}(z-z_0)f(z) &= a_{-1} + \lim_{z\to z_0}(z-z_0)g(z)\\ \lim_{z\to z_0}(z-z_0)f(z) &= a_{-1} = \mathrm{Res}_{z_0}(f)\end{aligned}$$

Conversely, if $\lim_{z\to z_0}(z-z_0)f(z)$ exists and is non-zero, then $(z-z_0)f(z)$ is bounded in some neighbourhood of z_0. Hence, z_0 is a pole of f of order at most 1 by our definition of pole.

If the limit exists and is equal to 0, then it means that f has a removable singularity at z_0

More generally we have the following Theorem:

Theorem 3.5. [SS10, Theorem III.1.4] If $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ has a pole of order n at z_0, then

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\big((z-z_0)^n f(z)\big)$$

Proof. Let

$$f(z) = P^f_{z_0}(z) + G(z)$$

for $z \in D_r(z_0)$ with $P^f_{z_0}(z) = \sum_{j=1}^n \frac{a_{-j}}{(z-z_0)^j}$ and $G \in \mathscr{H}(D_r(z_0))$. Then it holds that

$$(z-z_0)^n f(z) = G(z)(z-z_0)^n + \sum_{j=1}^n a_{-j}(z-z_0)^{n-j}$$

differentiating $n-1$ times gives

$$\frac{d^{n-1}}{dz^{n-1}}\big((z-z_0)^n f(z)\big) = \frac{d^{n-1}}{dz^{n-1}}\big(G(z)(z-z_0)^n\big) + (n-1)!a_{-1}$$

and lastly

$$\lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\big((z-z_0)^n f(z)\big) = (n-1)!a_{-1} + \underbrace{\lim_{z\to z_0}(z-z_0)\tilde{G}(z)}_{=0}$$

for some $\tilde{G} \in \mathscr{H}(D_r(z_0))$ by the General Leibniz Rule [EW22]. Hence, the final result is obtained by some final algebraic manipulations, so

$$\operatorname{Res}_{z_0}(f) = a_{-1} = \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\big((z-z_0)^n f(z)\big)$$

□

Example 3.6. *Consider the examples:*

1. *For the function* $f(z) = \frac{1}{z^2+1}$ *we have a simple pole at* $z = i$

$$\operatorname{Res}_i\left(\frac{1}{z^2+1}\right) = \lim_{z\to i}(z-i)\frac{1}{z^2+1} = \lim_{z\to i}\frac{1}{z+1} = \frac{1}{2i}$$

2. *The function* $f(z) = \frac{1}{(z^2+1)^2}$ *has two poles of order* 2 *at* $z = \pm i$

$$\begin{aligned}\operatorname{Res}_{\pm i}(f) &= \lim_{z\to\pm i}\frac{1}{(2-1)!}\frac{d}{dz}\left((z\pm i)^2\frac{1}{(z^2+1)^2}\right) = \lim_{z\to\pm i}\frac{d}{dz}\left(\frac{-2}{(z\pm i)^2}\right) = \\ &= \lim_{z\to\pm i}\frac{-2}{(z\pm i)^3} = \frac{\pm 1}{4i}\end{aligned}$$

Remark 3.4. *The following one is a useful tool to calculate residues of simple poles.*

Lemma 3.1. If $f, g \in \mathscr{H}(\Omega)$ with $z_0 \in \Omega$, $f(z_0) \neq 0$ and g has a simple zero at z_0, then $\frac{f}{g}$ has a simple pole at z_0 and

$$\operatorname{Res}_{z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{f(z_0)}{g'(z_0)}$$

Proof. It is clear that if g has a simple zero at z_0, then $g(z) = (z - z_0)\tilde{g}(z)$ where $\tilde{g}(z_0) \neq 0$, non-zero and holomorphic in some $D_r(z_0) \subset \Omega$ for some $r > 0$

$$\frac{f(z)}{g(z)} = (z - z_0)^{-1} \frac{f(z)}{\tilde{g}(z)}$$

where $\frac{f(z)}{\tilde{g}(z)} \in \mathscr{H}\big(D_r(z_0)\big)$ and non-zero at z_0. So $\frac{f(z)}{g(z)}$ has a simple pole at z_0. We now apply Theorem 3.5 to $\frac{f}{g}$, so

$$\begin{aligned} \mathrm{Res}_{z_0}\left(\frac{f}{g}\right) &= \lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} \underbrace{=}_{g(z_0)=0} \lim_{z \to z_0} f(z) \frac{(z - z_0)}{g(z) - g(z_0)} = \\ &= f(z_0) \lim_{z \to z_0} \frac{(z - z_0)}{g(z) - g(z_0)} = \frac{f(z_0)}{g'(z_0)} \end{aligned}$$

since g is holomorphic at z_0 □

Example 3.7. *1. Let $g(z) = \frac{1}{z^2+1}$ with a simple pole at $z = i$, then*

$$\mathrm{Res}_i\left(\frac{1}{z^2+1}\right) = \mathrm{Res}_i\left(\frac{1}{g(z)}\right) = \frac{1}{g'(i)} = \frac{1}{2i}$$

2. We want to determine $\mathrm{Res}_i\left(\frac{z^3}{z^2+1}\right)$, hence we can either use partial fraction expansion as

$$\frac{z^3}{z^2+1} = z - \frac{z}{z^2+1} = z - \frac{1}{2}\frac{1}{z-i} - \frac{1}{2}\frac{1}{z+i}$$

and get $\mathrm{Res}_i\left(\frac{z^3}{z^2+1}\right) = -\frac{1}{2}$, or use the above Lemma 3.1

$$\mathrm{Res}_i\left(\frac{z^3}{z^2+1}\right) = \frac{f(i)}{g'(i)} = \frac{i^3}{2i} = -\frac{1}{2}$$

with $f(z) = z^3$ and $g(z) = z^2 + 1$ (notably faster in this case).

Remark 3.5. *Note that, if $f(z) = P^f_{z_0}(z) + G(z)$ for $z \in D_r(z_0)$, where $P^f_{z_0}$ is the principal part of f at z_0 and G is a holomorphic function, and $C(z_0)$ is any circle centred at z_0 and contained in $D_r(z_0)$, then*

$$\int_{C(z_0)} P^f_{z_0}(z)dz = \int_{C(z_0)} \sum_{k=1}^{n} \frac{a_{-k}}{(z - z_0)^k} dz = 2\pi i a_{-1}$$

since[3]

$$\int_{C(z_0)} \frac{1}{(z - z_0)^n} dz = \begin{cases} 0 & , n \neq 1 \\ 2\pi i & , n = 1 \end{cases}$$

[3] Using the Cauchy's Integral Formula 2.6 with the constant function $1|_{\overline{D(z_0)}} : \overline{D(z_0)} \to \mathbb{C}, z \mapsto 1$ and $C = \partial D$

By Cauchy's Theorem 2.5 we also know that if $C(z_0) \subset D_r(z_0)$, *then*

$$\int_{C(z_0)} G(z)dz = 0$$

Hence, we have that

$$\int_{C(z_0)} f(z)dz = 2\pi i a_{-1}$$

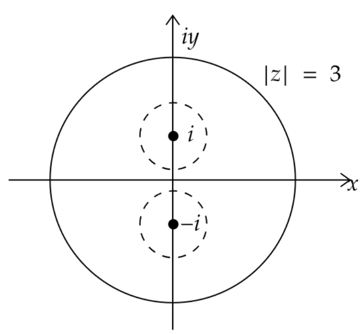
In fact, we have the following general formula

Theorem 3.6 (The Residue Formula)**.** [SS10, Theorem III.2.1] Let $\Omega \subseteq \mathbb{C}$ be open and let $S := \{z_0, ..., z_n\} \subseteq \Omega$ be a finite set. Suppose $f \in \mathscr{H}(\Omega \setminus S)$, i.e. holomorphic except for poles at $z_0, ..., z_n \in S$. Let γ be any circle contained in Ω with counterclockwise orientation and such that $im(\gamma) \cap S = \emptyset$. Let D be the open disc bounded by γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{z \in S \cap D} \operatorname{Res}_z(f)$$

Before we give the proof we will look at simple examples which use this formula to calculate integrals.

Example 3.8. *1. Let* γ *be the circle such that* $|z| = 3$ *and consider the integral*



$$\int_\gamma \frac{dz}{(z^2+1)^2} = 2\pi i \operatorname{Res}_i\left(\frac{1}{(z^2+1)^2}\right) + 2\pi i \operatorname{Res}_{-i}\left(\frac{1}{(z^2+1)^2}\right) =$$
$$= 2\pi i \left(\frac{1}{4i} + \frac{-1}{4i}\right) = 0$$

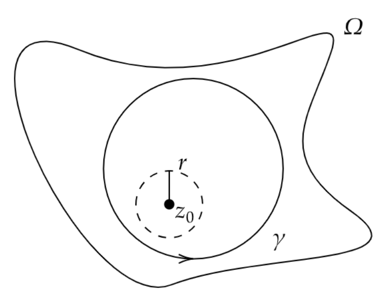
2. *The second example is*
$$\int_{C_3(0)} \frac{z^3}{(z^2+1)^2}dz = 2\pi i \operatorname{Res}_i\left(\frac{z^3}{(z^2+1)^2}\right) + 2\pi i \operatorname{Res}_{-i}\left(\frac{z^3}{(z^2+1)^2}\right) =$$
$$= 2\pi i\left(\frac{-1}{2}+\frac{-1}{2}\right) = -2\pi i$$

3. *The third example is*
$$\int_{C_{\frac{1}{2}}(1)} \frac{dz}{(z^2+1)^2} = 0$$
since there is no pole of $\frac{1}{(z^2+1)^2}$ *inside the circle given by* $|z-1| = \frac{1}{2}$

4. *The fourth example is*
$$\int_{C_2(0)} \frac{e^z}{(z^2-1)^2}dz = 2\pi i \operatorname{Res}_1\left(\frac{e^z}{(z^2-1)^2}\right) + 2\pi i \operatorname{Res}_{-1}\left(\frac{e^z}{(z^2-1)^2}\right)$$
Evaluating separately the two residues we obtain
$$\operatorname{Res}_1\left(\frac{e^z}{(z^2-1)^2}\right) = \frac{f(1)}{g'(1)} = \frac{e}{2}$$
where $f(z) = e^z$ *and* $g(z) = z^2 - 1$ *and using the Lemma 3.1, also*
$$\operatorname{Res}_1\left(\frac{e^z}{(z^2-1)^2}\right) = \frac{f(-1)}{g'(-1)} = \frac{-1}{2e}$$
by the same procedure on the other pole. Hence, we finally obtain
$$\int_{C_2(0)} \frac{e^z}{(z^2-1)^2}dz = \pi i(e - e^{-1})$$

We can now proceed to the proof of the formula.

Proof of Theorem 3.6. Let us first assume that f is holomorphic on an open set Ω containing a circle and its interior, except for a single pole at z_0 inside γ. Let D be the disc bounded by γ



By Theorem 3.4

$$f(z) = P^f_{z_0}(z) + G(z)$$

where G is holomorphic in a neighbourhood $D_r(z_0)$ of z_0 and

$$P^f_{z_0}(z) = \sum_{k=1}^{n} \frac{a_{-k}}{(z-z_0)^k}$$

is the principal part of f at z_0, note that $P^f_{z_0}(z)$ is holomorphic in all of $\mathbb{C} \setminus \{z_0\}$. Another way to say this is that the function $f(z) - P^f_{z_0}(z)$ extends holomorphically to Ω, as for the function

$$g : \Omega \to \mathbb{C}, z \mapsto g(z) = \begin{cases} f(z) - P^f_{z_0}(z) & , z \in \Omega \setminus \{z_0\} \\ G(z_0) & , z \in D_r(z_0) \end{cases}$$

is the holomorphic extension of $f(z) - P^f_{z_0}(z)$ to Ω

We then obtain that

$$\int_\gamma g(z)dz = \int_\gamma \big(f(z) - P^f_{z_0}(z)\big)dz = 0$$
$$\int_\gamma f(z)dz = \int_\gamma P^f_{z_0}(z)dz$$

and we are left to prove that $\int_\gamma P^f_{z_0}(z)dz = 2\pi i a_{-1}$. This follows from the Cauchy Integral Formula 2.6 applied to the constant function $F : \Omega \to \mathbb{C}, z \mapsto F(z) = 1$

Recall: The Cauchy Integral Formula for derivatives 2.3. Let $C = \partial D$ be any circle whose interior D is contained in Ω. Then for $F \in \mathscr{H}(\Omega)$ and any $z \in D$ it holds that

$$F^{[n]}(z) = \frac{n!}{2\pi i} \int_C \frac{F(w)}{(w-z)^{n+1}} dw$$

hence,

$$\int_\gamma \frac{dz}{(z-z_0)^n} = \frac{2\pi i}{n!} \frac{dz^{n-1}}{dz^{n-1}}(1) = \begin{cases} 0 & , n-1 \geq 1 \\ 2\pi i & , n-1 = 0 \end{cases}$$

and so we get

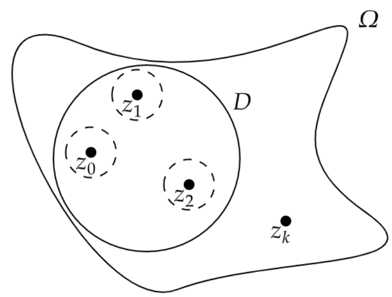
$$\int_\gamma f(z)dz = 2\pi i a_{-1} = 2\pi i_{z_0} \operatorname{Res}(f)$$

For the general case consider f holomorphic in Ω except for finitely many points $z_0, ..., z_n$ and let $P^f_{z_k}$ be the principal part at z_k for $k \in \{0, ..., n\} =: I$, which is holomorphic in $\mathbb{C} \setminus \{z_k\}$

Hence, define in $\Omega \setminus S$ the following function

$$g : \Omega \setminus S \to \mathbb{C}, z \mapsto g(z) := f(z) - \sum_{\substack{z_k \in S \\ k \in I}} P^f_{z_k}(z)$$

then for $g \in \mathscr{H}(\Omega \setminus S)$ and in fact g can be extended holomorphically to all of Ω. To see this let $z_0 \in S$, $r > 0$ such that $\overline{D}_r(z_0) \subset D$, $\dot{D}_r(z_0) \cap S = \emptyset$ and $f(z) - P^f_{z_0}(z)$ is holomorphic in $D_r(z_0)$



Then for $z \in \dot{D}_r(z_0)$

$$g(z) = \underbrace{f(z) - P^f_{z_0}(z)}_{\text{extends holomorphically to } D_r(z_0)} + \underbrace{\sum_{\substack{z_k \in S \setminus \{z_0\} \\ k \in I \setminus \{0\}}} P^f_{z_k}(z)}_{\text{holomorphic in } D_r(z_0)}$$

This gives an extension of g to $(\Omega \setminus S) \cup \{z_0\} = \Omega \setminus \{z_1, ..., z_n\}$, we can do this for each $z_k \in S$ with $k \in I$ to get a holomorphic extension of g to all Ω. By Cauchy's Theorem 2.5 we obtain that

$$\int_\gamma g(z)dz = 0$$

which in return gives

$$\int_\gamma f(z)dz = \sum_{\substack{z_k \in S \\ k \in I}} \int_\gamma P^f_{z_k}(z)dz$$

and if $\tilde{z} \in S \cap D$, then as before (for $\ell = \text{ord}_{\tilde{z}}(f)$)

$$\int_\gamma P^f_{\tilde{z}}(z)dz = \int_\gamma \sum_{j=1}^{\ell} \frac{a_{-j}}{(z - \tilde{z})^j} dz = 2\pi i a_{-1} = 2\pi i \operatorname{Res}_{\tilde{z}}(f)$$

If $\tilde{z} \in S$, but not inside D, then

$$\int_\gamma P^f_{\tilde{z}}(z)dz = 0$$

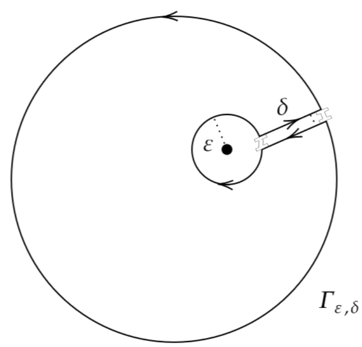
since then $P^f_{\tilde{z}}$ is holomorphic inside the disc.

Hence, we finally get

$$\int_\gamma f(z)dz = 2\pi i \sum_{\tilde{z}\in S\cap D} \mathrm{Res}_{\tilde{z}}(f)$$

□

Remark 3.6. *1. Another way to prove this is the following: first assume there is just one pole inside of γ. Consider the following contour $\Gamma_{\varepsilon,\delta}$*



inside of it f is holomorphic and we can show using Cauchy's Theorem 2.5 that

$$\int_{\Gamma_{\varepsilon,\delta}} f(z)dz = 0$$

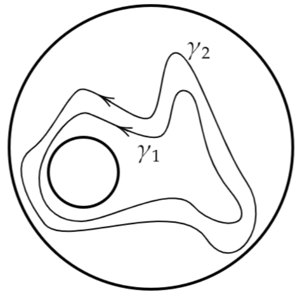
Here we went around the pole z_0 with a circle of radius $\varepsilon > 0$. The width of the corridor is $\delta > 0$
We can then make the width of the corridor narrower by letting $\delta \to 0$ and use continuity of f to show that the two sides of the corridor cancel each other. The remaining part consists of two curves, the larger circle γ and the small circle $C_\varepsilon(z_0)$ with clockwise orientation; we therefore get

$$\int_\gamma f(z)dz + \int_{C_\varepsilon(z_0)} f(z)dz = 0$$

It takes, though, some effort to make this argument rigorous.

2. *The best way to understand and generalise the Residue Formula 3.6 (and Cauchy Integral Formula 2.6) is via homotopy. It is based on the following principle.*

   Let f be holomorphic on an open set Ω. For example, the space between two circles.



The principle is that if two closed curves can be deformed to each other while remaining in Ω*, then*

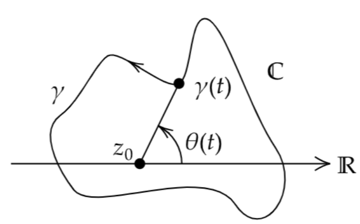
$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

We are going to get back to this soon.

3. *Assume* γ *is not a circle, then consider it a triangle, a polygon or any curve* γ*, which has a parametrisation of the form*

$$\begin{aligned} \gamma : [a,b] &\to \mathbb{C} \setminus \{z_0\} \\ t &\mapsto z_0 + r(t)e^{i\theta(t)} \end{aligned}$$

for some $r, \theta \in C^1\big([a,b], \mathbb{R}\big)$ *functions such that* $r(t) > 0$*,* $r(a) = r(b), \theta(a) = 0$ *and* $\theta(b) = 2\pi$



Moreover,

$$\boxed{r(t) = \big|\gamma(t) - z_0\big|}$$

and $\theta(t)$ *is a continuous choice of argument along the line segment* $\tilde{\gamma}(t) = \gamma(t) - z_0$*, lastly we have that*

$$\boxed{e^{i\theta(t)} = \frac{\gamma(t) - z_0}{\big|\gamma(t) - z_0\big|}}$$

Then it holds that

$$\gamma'(t) = r'(t)e^{i\theta(t)} + r(t)e^{i\theta(t)}i\theta'(t)$$

and so we obtain that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} = \int_a^b \frac{\gamma'(t)}{r(t)e^{i\theta(t)}}dt = \int_a^b \frac{r'(t)}{r(t)}dt + i\int_a^b \theta'(t)dt =$$
$$= \left[\log\left(r(t)\right)\right]_a^b + i\left[\theta(t)\right]_a^b = 0 + 2\pi i$$

(This is similar to the parametrisation of a circle using a point inside other than the centre)

Note that for

$$\int_\gamma \frac{dz}{(z-z_0)^n} = 0$$

with $n > 1$, *since* $\frac{1}{(z-z_0)^{n-1}}\frac{1}{1-n}$ *is a primitive of* $\frac{1}{(z-z_0)^n}$ *in* $\mathbb{C}\setminus\{z_0\}$ *and* γ *is a closed curve.*

Hence, for any such contour γ *we have*

$$\int_\gamma f(z)dz = 2\pi i \sum_{z\in U_\gamma\cap F} \operatorname{Res}_z(f)$$

where U_γ *is the set contoured by* γ

Before we give more theoretical applications of the Residue Theorem 3.6, let us give some applications to the evaluation of real integrals.

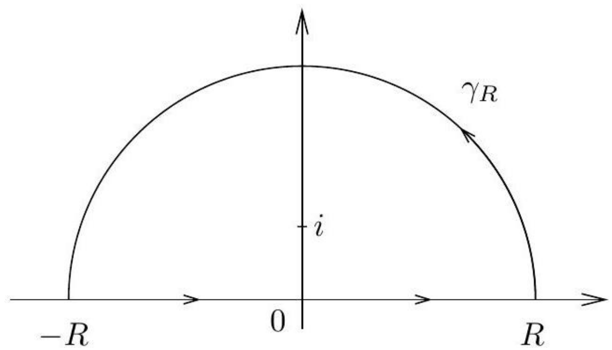
Example 3.9 (Integrals of rational functions)**.** *E.g.*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

This of course can be evaluated easily using arctangent. We though give another proof of it, using the Residue Theorem 3.6.

Idea*: To choose a function* f *and a closed contour, so that part of the contour leads to the real integral after taking limits.*

In this particular case we take $f(z) = \frac{1}{1+z^2}$ *as function and* γ_R *as the contour path.*



f has only one pole, at $z = i$ inside γ_R, hence

$$\int_{\gamma_R} f(z)dz = 2\pi i \operatorname{Res}_i(f) = 2\pi i \lim_{z\to i}(z-1)\frac{1}{1+z^2} = 2\pi i \lim_{z\to i}\frac{1}{z+i} = \pi$$

Then we also have that

$$\int_{\gamma_R} f(z)dz = \int_{-R}^{R}\frac{1}{1+x^2}dx + \int_{\Gamma_R}\frac{1}{1+z^2}dz$$

for Γ_R as the semicircular part of the closed path. As $R \to \infty$, the first integral gives

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$$

and similarly, as $R \to \infty$, we see that over the semicircle Γ_R the integral goes to zero. This is because on Γ_R: $|z^2+1| > R^2-1$ and hence $\frac{1}{z^2+1} < \frac{1}{R^2-1} \approx \frac{1}{R^2}$. Therefore, we have that

$$\left|\int_{\Gamma_R}\frac{1}{1+z^2}dz\right| < \frac{1}{R^2-1}\pi R \approx \frac{\pi}{R} \xrightarrow{R\to\infty} 0$$

Hence finally we conclude that

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \pi$$

Example 3.10. *The same technique works to calculate the integrals of the form*

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$$

where $P, Q \in \mathbb{C}[z]$, Q has no zero in the real axis and $\deg(Q) \geq \deg(P) + 2$

Note that we need this bound for the degrees of P and Q in order to get

$$\int_{\Gamma_R}\frac{P(z)}{Q(z)}dz \xrightarrow{R\to\infty} 0$$

If $\deg(Q) = n$ *and* $\deg(P) = m$ *on the semicircle, for* R *sufficiently large* Q *satisfies the inequality* $|Q(z)| > K|z|^n$ *for some* $K \in \mathbb{R}^{\geq 0}$ *and we can hence bound*

$$\left|\frac{P(z)}{Q(z)}\right| < C\frac{R^m}{R^n} = \frac{C}{R^{n-m}}$$

Hence, it holds that

$$\left|\int_{\Gamma_R} \frac{P(z)}{Q(z)} dz\right| \leq \frac{C}{R^{n-m}} R = \frac{C}{R^{n-m-1}}$$

For this reason, in order for the control bound to go to zero, we need

$$n - m - 1 > 0 \iff n > m + 1 \underset{i.e.}{\iff} n \geq m + 2$$

namely

$$\deg(Q) \geq \deg(P) + 2$$

Once we get this result, we can proceed with the calculation of the initial integral: we remember that

$$\int_{\gamma_R} \frac{P(z)}{Q(z)} dz = \int_{-R}^{R} \frac{P(x)}{Q(x)} dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz$$

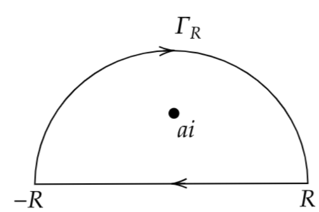
gives as $R \to \infty$

$$\int_{\gamma_R} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_{\tilde{z} \in F \cap int(\gamma_R)} \text{Res}_{\tilde{z}} \left(\frac{P}{Q}\right)$$

for F *the set of all poles of* $\frac{P}{Q}$ *(therefore we then consider only the ones inside* γ_R*).*

Example 3.11.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} dx = \frac{\pi}{2a^3}$$



Consider the function $f(z) = \frac{1}{(z^2+a^2)^2}$ *with poles at* $\pm ai$ *of order* 2 *each. Without loss of generality, we assume that* $a > 0$*, since this would only switch the poles between themselves in a manner that would way in the same result as the following.*

$$\text{Res}_{ai}\left(\frac{1}{(z^2 + a^2)^2}\right) = \lim_{z \to ai} \frac{d}{dz}\left((z - ai)^2 f(z)\right) = \lim_{z \to ai} \frac{d}{dz}\left(\frac{1}{(z + ai)^2}\right) =$$

$$= \lim_{z \to ai} \frac{-2}{(z + ai)^3} = \frac{-2}{(2ai)^3} = \frac{-2}{8a^3 i} = \frac{-i}{4a^3}$$

As above we conclude

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{1}{(z^2+a^2)^2}dz = 0$$

and we get

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}$$

Example 3.12. *The same contour can be used to evaluate integrals of rational functions times* $\sin(ax), \cos(ax)$*, for* $a \in \mathbb{R}$*, i.e. of the form*

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos(ax)dx$$

where P, Q *are polynomials in* $\mathbb{R}[x]$ *with* $\deg(Q) \geq \deg(P) + 2$

Take the function $f(z) = \frac{P(z)}{Q(z)}e^{iaz}$ *and not* $\frac{P(z)}{Q(z)}\cos(az)$*, since* $\cos(az)$ *behaves badly on the upper half plane. On the imaginary axis for example*

$$\cos(it) = \frac{e^t + e^{-t}}{2} = \frac{e^{2t}+1}{2e^t}$$

is the hyperbolic cosine, which grows exponentially. Whereas $|e^{iz}| = |e^{i(x+iy)}| = e^{-y}$*, which is bounded by 1 in the upper half plane. So, for* $Im(z) > 0 : |e^{iz}| \leq 1$

E.g. for $a > 0$ *we have*

$$\int_{-\infty}^{\infty}\frac{1}{x^2+1}\cos(ax)dx = \pi e^{-a}$$

with $f(z) = \frac{e^{iaz}}{z^2+1}$*, which has only one pole on the upper half plane at* $z = i$*, so*

$$\text{Res}_i\left(\frac{e^{iaz}}{z^2+1}\right) = \lim_{z\to i}\frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

Hence

$$\int_{\gamma_R} f(z)dz = 2\pi i\frac{e^{-a}}{2i} = \pi e^{-a}$$

Since $|e^{iaz}| < 1$ *on the upper half plane, we have that*

$$\left|\frac{e^{iaz}}{z^2+1}\right| \leq \frac{1}{R^2-1}$$

and hence that

$$\left|\int_{\Gamma_r}\frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{\pi R}{R^2-1} \xrightarrow{R\to\infty} 0$$

and consequently

$$\forall a > 0 : \int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}dx = \pi e^{-a}$$

we now note that $\frac{\cos(ax)}{x^2+1} = Re\left(\frac{e^{iax}}{x^2+1}\right)$*, hence by taking the real part of the function we get*

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx = \pi e^{-a}$$

This also shows that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1} dx = 0$$

which can also be directly seen, as $\frac{\sin(ax)}{x^2+1}$ *is an odd function.*

Example 3.13 (Integrals of trigonometric functions)**.** *The Residue Theorem 3.6 can be used to evaluate real integrals of the form*

$$\int_0^{2\pi} \frac{P\big(\cos(t), \sin(t)\big)}{Q\big(\cos(t), \sin(t)\big)} dt$$

where P, Q *are polynomials and where it holds that* $\forall x, y \in \mathbb{R} : (x^2 + y^2 = 1) \implies Q(x, y) \neq 0$ *(every pair* (x, y) *on* $C_1(0)$ *has* $Q(x, y) \neq 0$*.)*

Consider the example of

$$\forall a > 1 : \int_0^{2\pi} \frac{1}{a + \cos(\theta)} d\theta$$

The idea is to convert it to a contour integral around the unit circle. Expressing $\cos(\theta) = Re(z)$ *leads to an effective approach: the trigonometric function* $\cos(\theta), \sin(\theta)$ *can also be written in terms of* z *on the unit circle* $C_1(0)$ *as follows*

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

and

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

Hence, we can write $a + \cos(\theta) = a + \frac{z+\frac{1}{z}}{2}$ *and*

$$\int_0^{2\pi} \frac{1}{a + \cos(\theta)} d\theta = \int_{C_1(0)} \frac{1}{a + \frac{z+\frac{1}{z}}{2}} \frac{dz}{iz} = \frac{2}{i} \int_{C_1(0)} \frac{1}{z^2 + 2az + 1} dz$$

where the term $\frac{1}{iz}$ *arises naturally from the construction of the integral. The poles of the integral are at* $-a \pm \sqrt{a^2 - 1}$*, but only one of these roots is inside the unit circle, namely* $z_0 = -a + \sqrt{a^2 - 1}$

$$\mathrm{Res}_{z_0}\left(\frac{1}{z^2 + 2az + 1}\right) = \lim_{z \to z_0} \frac{1}{(z - z_1)(z - z_0)} = \lim_{z \to z_0} \frac{1}{2z + 2a} = \frac{1}{2z_0 + 2a} = \frac{1}{2\sqrt{a^2 - 1}}$$

Therefore, we finally have

$$\int_0^{2\pi} \frac{1}{a + \cos(\theta)} d\theta = \frac{2}{i} 2\pi i \frac{1}{2\sqrt{a^2 - 1}} = \frac{2\pi}{\sqrt{a^2 - 1}}$$

We now turn to more theoretical applications of the Residue Theorem 3.6. We start by giving one more description of an isolated singularity, which is a pole. Namely, we have the following Proposition.

Proposition 3.1. [SS10, Corollary III.3.2] Suppose $f \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ with Ω open has an isolated singularity at the point z_0, then

$$z_0 \text{ is a pole of } f \iff \lim_{z \to z_0} \big|f(z)\big| = \infty$$

Proof. If f has a pole of order $k \in \mathbb{N}^*$ at z_0, then by Theorem 3.3 we have that there exists some $r > 0$ for which $f(z) = (z - z_0)^{-k} h(z)$ on $\dot{D}_r(z_0)$ with a bounded function $h \in \mathscr{H}\big(D_r(z_0)\big)$ for which it holds that $h(z_0) \neq 0$. Then

$$\big|f(z)\big| = \big|(z - z_0)^{-k}\big|\big|h(z)\big| \xrightarrow{z \to z_0} \infty$$

since $\big|h(z)\big| \xrightarrow{z \to z_0} \big|h(z_0)\big| \neq 0$ and $k \geq 1$

Conversely, if $\big|f(z)\big| \xrightarrow{z \to z_0} \infty$, then we can find some $r > 0$ such that $\big|f(z)\big| \geq 1$ (or any $\varepsilon > 0$ would also be suited for the case) on $\dot{D}_r(z_0)$. In particular, $f(z) \neq 0$ on $\dot{D}_r(z_0)$ by Theorem 3.3 and there is a $h \in \mathscr{H}\big(\Omega \setminus \{z_0\}\big)$ such that

$$h(z) := \frac{1}{f(z)}$$

is holomorphic in $\dot{D}_r(z_0)$ and $\big|h(z)\big| \leq 1$ there. Furthermore, it holds that $\lim_{z \to z_0} h(z) = \lim_{z \to z_0} \frac{1}{f(z)} = 0$. By the Riemann's Theorem on removable singularities 3.2, h extends to a holomorphic function $\tilde{h}$ in $D_r(z_0)$ by defining $\tilde{h}(z_0) := \lim_{z \to z_0} \frac{1}{f(z)} = 0$ and otherwise $\tilde{h} = h|_{\dot{D}_r(z_0)}$
Therefore, if $\mathbb{N} \in \mathbb{N}$ is the order of zero of $\tilde{h}$ at z_0, then $f(z) = \frac{1}{h(z)}$ has a pole of order $\mathbb{N} \in \mathbb{N}$ at z_0 ☐

We have seen that an isolated singularity z_0 of f is removable if f is bounded near z_0, at the same time z_0 is a pole if $\big|f(z)\big| \xrightarrow{z \to z_0} \infty$

Definition 3.5 (Essential singularity)**.** An isolated singularity that is neither removable nor a pole is called an **essential singularity**.

As we saw in the very beginning, the function $e^{\frac{1}{z}}$ has a more exotic behaviour. E.g. $e^{\frac{1}{x}} \to 0$ as $x \nearrow 0$ from the negative reals, whereas $e^{\frac{1}{x}} \to \infty$ as $x \searrow 0$ from the positive reals.

In fact, any function $f \in \mathbb{C}^\Omega$ behaves exotically near an essential singularity. More precisely we have

Theorem 3.7 (Casorati-Weierstrass)**.** Suppose $f \in \mathscr{H}\big(\dot{D}_r(z_0)\big)$ and has an essential singularity at z_0
Then the image of $\dot{D}_r(z_0)$ under f, namely $f\big(\dot{D}_r(z_0)\big)$, is dense in $\mathbb{C}$

Remark 3.7. *The Casorati-Weierstrass Theorem 3.7 states that the image of a punctured disc $\dot{D}_r(z_0)$, no matter how small, effectively fills up the whole complex plane (where z_0 is an essential singularity). In fact, a remarkable Theorem of Picard says*

Theorem 3.8 (Picard's Theorem (1879))**.** If $f \in \mathscr{H}\big(\dot{D}_r(z_0)\big)$ and has an essential singularity at z_0, then $\mathbb{C} \setminus f\big(\dot{D}_r(z_0)\big)$ contains at most one point.

The function $f(z) = e^{\frac{1}{z}}$ maps each punctured disc centred at $z = 0$ to $\mathbb{C}^*$, i.e. it does not take the value 0, so the "exceptional value" permitted by Picard's Theorem 3.8 may in fact exist.

Proof of Theorem 3.7. We want to show that for the given $r > 0$

$$\forall \varepsilon > 0 \forall w \in \mathbb{C} \exists z \in \dot{D}_r(z_0) : \big|f(z) - w\big| < \varepsilon$$

and to do so we argue by contradiction and show that this will forces the singularity at z_0 to be either removable or a pole, hence contradicting the assumption that z_0 is essential.

Assume on the contrary that the image in question is not dense in $\mathbb{C}$, hence for the given $r > 0$

$$\exists \delta > 0 \exists w_0 \in \mathbb{C} \forall z \in \dot{D}_r(z_0) : \big|f(z) - w_0\big| \geq \delta$$

Let $g : \dot{D}_r(z_0) \to \mathbb{C}, z \mapsto g(z) := \frac{1}{f(z) - w_0}$, then on $\dot{D}_r(z_0)$ we have that g is bounded by $\frac{1}{\delta}$, hence has a removable singularity at z_0 by Riemann's Theorem on removable singularities 3.2. Hence, there is a holomorphic extension of g, i.e. we can define g at z_0, so that g becomes holomorphic in $D_r(z_0)$

Since $\big|f(z) - w_0\big| \geq \delta$ and $g(w) = \frac{1}{f(z) - w_0}$, clearly g has no zero in $\dot{D}_r(z_0)$, hence its reciprocal $\frac{1}{g}$ has an isolated singularity at $z_0 \in D_r(z_0)$. This singularity is either a removable one or a pole, depending on whether $\lim_{z \to z_0} \big|g(z)\big| = 0$ or not, respectively. In turn, this gives that the singularity of $f = w_0 + \frac{1}{g}$ at z_0 can be at most a pole, giving the anticipated contradiction. Note that the limit $\lim_{z \to z_0} g(z)$ exists, since g has a removable singularity at z_0 $\square$

3.2 Meromorphic functions

We now look at functions whose singularities are poles. Since at a pole $\lim_{z\to z_0} \big|f(z)\big| = \infty$, this suggest that we can add ∞ to the values of functions, including consequently the poles in their domain of definition. E.g. The function

$$\begin{aligned} f : \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto \frac{1}{z} \end{aligned}$$

can be extended to

$$\begin{aligned} \hat{f} : \mathbb{C} &\to \mathbb{C} \cup \{\infty\} \\ z &\mapsto \frac{1}{z} \end{aligned}$$

Definition 3.6.

- The set
$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
is called **the Extended Complex Plane**. Here ∞ represents a point "at infinity" and is unsigned. In $\hat{\mathbb{C}}$, we can supplement the rules in $\mathbb{C}$ by
$$\begin{array}{rrcccc} \forall z \in \mathbb{C}: & \infty \pm z & = & z \pm \infty & = & \infty \\ \forall z \in \hat{\mathbb{C}} \setminus \{0\}: & \infty \cdot z & = & z \cdot \infty & = & \infty \\ \forall z \in \mathbb{C}: & \frac{z}{\infty} & = & 0 & & \\ \forall z \in \hat{\mathbb{C}} \setminus \{0\}: & \frac{z}{0} & = & \infty & & \end{array}$$
The expressions $\infty \pm \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ and $0 \cdot \infty$ are not assigned a meaning in $\hat{\mathbb{C}}$
- A sequence $(z_n)_{n\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}^*}$ **converges to** ∞, i.e. $\lim_{n\to\infty} z_n = \infty$, if
$$\lim_{n\to\infty} |z_n| = \infty$$
where $\big(|z_n|\big)_{n\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}$. Similarly, we say that $\lim_{z\to z_0} f(z_n) = \infty$, if
$$\lim_{z\to z_0} \big|f(z_n)\big| = \infty$$

Remark 3.8. $\hat{\mathbb{C}}$ *is not a field.*

Talking now about functions, we can extend our definition of holomorphic function to a more general notion.

Definition 3.7 (Meromorphic function)**.** A function $f \in \hat{\mathbb{C}}^\Omega$ with $\Omega \subseteq \mathbb{C}$ open in $\mathbb{C}$ is called **meromorphic**, if the following conditions are satisfied:

- The set $S_f := \{z \in \Omega : f(z) = \infty\} = f^{-1}\big(\{\infty\}\big)$ has no limit point in Ω, i.e. S_f is discrete in Ω
- The points in S_f are poles of f
- The restriction of f to $\Omega \setminus S_f = \{z \in \Omega : f(z) \neq \infty\}$ is holomorphic, i.e. $f|_{\Omega \setminus S_f} \in \mathscr{H}\big(\Omega \setminus S_f\big)$

Let $\mathscr{M}(\Omega)$ denote the set of all meromorphic functions in Ω

Note that the set of poles for meromorphic functions is discrete, as the set of zeroes for holomorphic functions.

Example 3.14. *1. Let $P, Q \in \mathbb{C}[z]$ be two polynomials with no common zeroes. Note that any rational function $\frac{p}{q}$ (for p, q rational functions) can be reduced to $\frac{P}{Q}$ with no common zeroes. Let*

$$f : \mathbb{C} \to \mathbb{C}, z \mapsto f(z) := \begin{cases} \frac{P(z)}{Q(z)} & , Q(z) \neq 0 \\ \infty & , Q(z) = 0 \end{cases}$$

Then $f \in \mathscr{M}(\mathbb{C})$, since f is holomorphic outside the finite zeroes' set of $Q(z)$. If z_0 is a zero of Q, then f has a pole at z_0, as

$$\lim_{z \to z_0} \big|f(z)\big| = \lim_{z \to z_0} \frac{\big|P(z)\big|}{\big|Q(z)\big|} = \infty$$

having we assumed that $P(z_0) \neq 0$. Hence, z_0 is a pole of f

2. The function $f(z) = \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ is a meromorphic function in $\mathbb{C}$ with $S_f = \mathbb{Z}$

3. Let $f(z) = \frac{e^{\frac{1}{z}}}{z^2 - 1}$, then f is meromorphic in $\mathbb{C}^*$ with $S_f = \{\pm 1\}$, but not in $\mathbb{C}$, as $z = 0$ is an essential singularity.

If we have two functions $f, g \in \mathscr{M}(\Omega)$ with pole sets S_f and S_g, then $f + g$ is holomorphic in $\Omega \setminus (S_f \cup S_g)$ and we can define $f + g$ at points in this set using the usual definition:

$$\forall z \in \Omega \setminus (S_f \cup S_g) : (f + g)(z) = f(z) + g(z)$$

So we only have to worry about points $z \in S_f \cup S_g$ and by doing so we can extend $(f + g)|_{\Omega \setminus (S_f \cup S_g)} : \Omega \setminus (S_f \cup S_g) \to \mathbb{C}$ to a meromorphic function $f + g : \Omega \to \hat{\mathbb{C}}$. We can do this in the following manner:

If $z_0 \in S_f \cup S_g$, write for all $z \in \dot{D}_r(z_0)$ for a $r > 0$ as

$$f(z) = P^f_{z_0}(z) + \tilde{f}(z)$$
$$g(z) = P^g_{z_0}(z) + \tilde{g}(z)$$

where $P^f_{z_0}, P^g_{z_0}$ are the principal parts of f and g at z_0 (one of them can be zero if f or g does not have a pole at z_0) and where $\tilde{f}, \tilde{g} \in \mathscr{H}\big(D_r(z_0)\big)$. Then

$$(f+g)(z) = \underbrace{P^f_{z_0}(z) + P^g_{z_0}(z)}_{\text{linear combination of terms of the form } \frac{1}{(z-z_0)^l}} + \underbrace{\tilde{f}(z) + \tilde{g}(z)}_{\text{holomorphic in } D_r(z_0)}$$

so $f+g$ has a pole of order ≥ 1 at z_0 and we assign the value ∞ to that point under $f+g$, this unless $P^f_{z_0}(z) + P^g_{z_0}(z) = 0$ (which can happen). Hence, $f+g \in \mathscr{M}(\Omega)$ with $S_{f+g} \subseteq S_f \cup S_g$

We have proved part (ii) of the following Proposition:

Proposition 3.2. Let $\Omega \subseteq \mathbb{C}$ be open, then

(i) $\mathscr{H}(\Omega) \subseteq \mathscr{M}(\Omega)$

(ii) If $f, g \in \mathscr{M}(\Omega)$, then $\forall a, b \in \mathbb{C} : af + bg \in \mathscr{M}(\Omega)$. Hence, $\mathscr{M}(\Omega)$ is a $\mathbb{C}$-vector space.

(iii) If $f, g \in \mathscr{M}(\Omega)$, then $fg \in \mathscr{M}(\Omega)$

(iv) If $0 \neq f \in \mathscr{M}(\Omega)$ and the zeroes of f do not have a limit point in Ω, then $\frac{1}{f} \in \mathscr{M}(\Omega)$

Proof. **(i)** Obvious, but note that we identified a holomorphic function $f \in \mathscr{H}(\Omega)$ with the corresponding function $\tilde{f} \in \mathscr{M}(\Omega)$, where $\tilde{f} = i \circ f$ and $i : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$

(ii) The same argument for $f+g$ works with $af+bg$

(iii) Let $f = P^f_{z_0} + \tilde{f}$ and $g = P^g_{z_0} + \tilde{g}$ with $z_0 \in S_f \cup S_g$, then

$$fg = (P^f_{z_0} + \tilde{f})(P^g_{z_0} + \tilde{g}) = P^{fg}_{z_0} + G$$

where $P^{fg}_{z_0}$ is a linear combination of $\frac{1}{(z-z_0)^l}$ terms and $G \in \mathscr{H}\big(D_r(z_0)\big)$ for some $r > 0$ by Theorem 3.4 (G is the sum of the respective functions in the case of f

and g separately), so

$$fg = \left(\sum_{k=-n}^{\infty} a_k (z-z_0)^k\right)\left(\sum_{l=-m}^{\infty} b_l (z-z_0)^l\right) = \\ = \sum_{N=-(n+m)}^{\infty} \left(\sum_{\substack{k,l \\ k+l=N}} a_k b_{N-l}\right)(z-z_0)^N$$

E.g. if $f(z) = f = \frac{a_{-1}}{z-z_0} + \sum_{n\in\mathbb{N}^*} a_n (z-z_0)^n$ and $g(z) = \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{z-z_0} + \sum_{l=0}^{\infty} b_l (z-z_0)^l$, then

$$fg = \frac{b_{-2}a_{-1}}{(z-z_0)^3} + \frac{b_{-2}a_0 + a_{-1}b_{-1}}{(z-z_0)^2} + \frac{a_{-1}b_0 + b_{-1}a_0 + b_{-2}a_1}{(z-z_0)} + G$$

where G is holomorphic in some disc around z_0. Hence, we have that fg has a pole of order 3

Similarly to the case of $f+g$, we can define

$$(fg)(z) = \begin{cases} f(z)g(z) & , z \in \Omega \setminus (S_f \cup S_g) \\ \infty & , z \in S_f \cup S_g \end{cases}$$

Then fg is meromorphic, i.e. $fg \in \mathscr{M}(\Omega)$ with $S_{fg} \subseteq S_f \cup S_g$

(iv) Let $f \in \mathscr{M}(\Omega)$, if $z_0 \in \Omega \setminus S_f$ and if $f(z_0) \neq 0$, then $\frac{1}{f}$ is holomorphic at z_0, if $z_0 \in \Omega \setminus S_f$ and $f(z_0) = 0$, then $\frac{1}{f}$ has a pole of order k (equal to the order of zeroes of f at z_0) at z_0 (of order ≥ 1). If $z_0 \in S_f$, then

$$\left|\frac{1}{f(z)}\right| \xrightarrow{z\to z_0} 0$$

hence $\frac{1}{f}$ is a removable singularity at z_0. So, if the zero of f has no limit point in Ω, then the poles of $\frac{1}{f}$ have no limit point in Ω and hence $\frac{1}{f} \in \mathscr{M}(\Omega)$

□

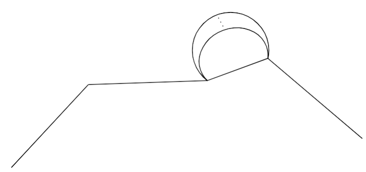
Remark 3.9. *If we assure that $f \neq 0$ in an connected component of Ω, then $\frac{1}{f} \in \mathscr{M}(\Omega)$*

Recall: If $f \in \mathbb{C}^{\Omega}$, Ω open and connected and $f \in \mathscr{H}(\Omega)$, then the zeroes of f do not have a limit point in Ω (see Theorem 2.10).

For an open and connected set Ω, the same is true for $f \in \mathscr{M}(\Omega)$

Proposition 3.3. Let Ω be open and connected, $0 \neq f \in \mathscr{M}(\Omega)$ and $Z_f := \{z \in \Omega : f(z) = 0\}$, then Z_f has no limit point in Ω

Proof. Assume on the contrary that $\exists (z_n)_{n\in\mathbb{N}^*} \in Z_f^{\mathbb{N}^*}$ of distinct points such that $\lim_{n\to\infty} z_n = b \in \Omega$



Let S_f be the set of poles of f (recall that it is countable), then $f|_{\Omega\setminus S_f} \in \mathscr{H}(\Omega \setminus S_f)$ and $\Omega \setminus S_f$ is open, connected and $f \neq 0$. Hence, by the above result we have recalled $b \notin \Omega \setminus S_f$

But $b \notin S_f$ either, since if b were a pole of f, then $\lim_{z\to b} \left|f(z)\right| = \infty$ and it would mean that for $\varepsilon > 0$ we would have $\left|f(z)\right| > 0$ for $z \in \Omega$ with $|z - b| < \varepsilon$. But this is impossible, since if $z_n \to b$, then $|z_n - b| < \varepsilon$ for $n \geq n_0$ and $f(z_n) = 0$ □

Remark 3.10. *Let $f \in \mathscr{M}(\Omega)$ and z_0 a pole of f, since S_f has no limit point in Ω, then it exists a punctured neighbourhood $\dot{D}_r(z_0)$ of z_0 for some $r > 0$ such that*

$$\dot{D}_r(z_0) \cap S_f = \emptyset$$

If the order of the pole of f at z_0 is k, then $f(z) = (z - z_0)^{-k} g(z)$ with an analytic function $g(z) \in \mathscr{H}\big(D_r(z_0)\big)$ by Theorem 3.3.

Hence, locally every meromorphic function is the quotient of two holomorphic functions. Here:

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

It is a non-trivial result that if Ω is non-empty, open and connected (i.e. a region), then we can do this globally.

For any $f \in \mathscr{M}(\Omega)$ with Ω open and connected, there exist $g, h \in \mathscr{H}(\Omega)$ such that $f = \frac{g}{h}$

Algebraically, we can state this as follows: recall that if Ω is open and connected, then $\mathscr{H}(\Omega)$ has no zero divisor, hence it is an integral domain. Consequentially, it has a quotient field (or field of fractions):

$$Q\big(\mathscr{H}(\Omega)\big) = \left\{\frac{g}{h} : g, h \in \mathscr{H}(\Omega) \text{ and } h \neq 0\right\} = \mathscr{M}(\Omega)$$

This is similar to the construction of $\mathbb{Q}$ as field of fractions of the integral domain $\mathbb{Z}$

Definition 3.8. Let $\Omega \subseteq \mathbb{C}$ open, $z_0 \in \Omega$ and $0 \neq f \in \mathscr{M}(\Omega)$. Then define the **valuation** (or **order**) **of** f **at** z_0, denoted by $\operatorname{ord}_{z_0}(f)$ or $\nu_{z_0}(f)$ to be the integer $k \in \mathbb{Z}$ such that:

- If z_0 is not a pole of f, i.e. $f(z_0) \neq \infty$, then $k \geq 0$ is the order of vanishing (zero) of f at z_0
- If $f(z_0) = \infty$, i.e. z_0 is a pole, then $k \leq -1$ is minus the order of the pole at z_0

In particular,

- If $\operatorname{ord}_{z_0}(f) > 0$, then z_0 is a zero.
- If $\operatorname{ord}_{z_0}(f) < 0$, then z_0 is a pole.
- If $\operatorname{ord}_{z_0}(f) = 0$, then $f(z_0) \neq 0$ and $f(z_0) \neq \infty$ (neither a pole nor a zero).

Combining what we know about the behaviour of functions near zeroes and poles (Theorem 2.2 and Theorem 3.3), we get:

Proposition 3.4. If $0 \neq f \in \mathscr{M}(\Omega)$ and $z_0 \in \Omega$, then

(i) $\operatorname{ord}_{z_0}(f) = k \in \mathbb{Z} \iff \exists r > 0 \exists h \in \mathscr{H}\big(D_r(z_0)\big)$:
$(h(z_0) \neq 0)$ and $\left(\forall z \in \dot{D}_r(z_0) : f(z) = (z - z_0)^k h(z)\right)$
Here we have $k < 0$ if z_0 is a pole, $k > 0$, if z_0 a zero.

(ii) $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$

(iii) If $f + g \neq 0$, then $\operatorname{ord}_{z_0}(f + g) \geq \min\big\{\operatorname{ord}_{z_0}(f), \operatorname{ord}_{z_0}(g)\big\}$

Example 3.15. *Let $f(z) = \frac{z}{(e^z-1)^2}$, z has zero of order 1 at $z = 0$, while $\left(e^2 - 1\right)^2$ has zeroes of order 2 at $z = 2\pi i n$, for $n \in \mathbb{Z}$*

$$\operatorname{ord}_{z_0}(f) = \operatorname{ord}_0(z) - \operatorname{ord}_0\left(\left(e^2 - 1\right)^2\right) = 1 - 2 = -1$$

Hence, f has a pole of order 1 at $z = 0$. For $n \neq 0$ we have

$$\operatorname{ord}_{2\pi in}(f) = \operatorname{ord}_{2\pi in}(z) - \operatorname{ord}_{2\pi in}\left(\left(e^2 - 1\right)^2\right) = 0 - 2 = -2$$

Therefore, f has a pole of order 2 at $2\pi in$, for $n \in \mathbb{Z} \setminus \{0\}$

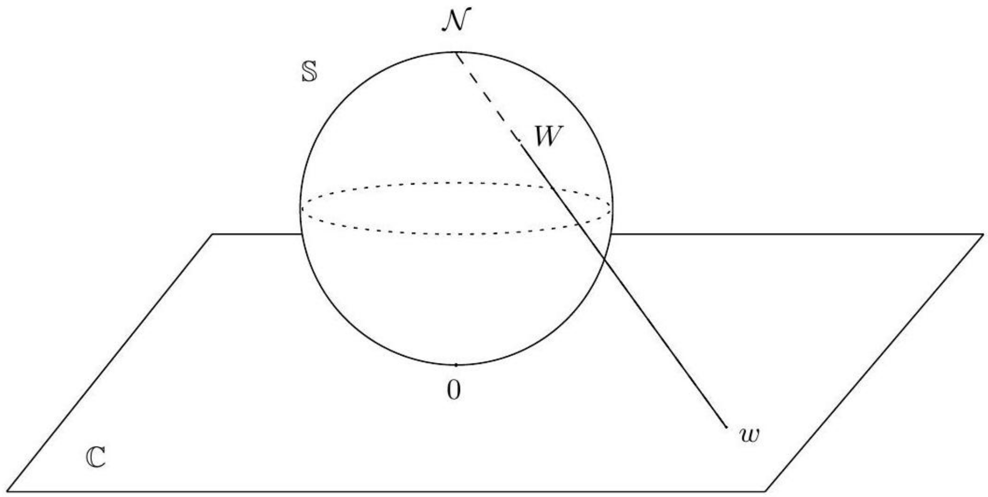
Remark 3.11 ($\hat{\mathbb{C}}$ and the stereographic Projection)**.** *Let*

$$S^2 := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\} \subset \mathbb{R}^3$$

Identifying $(x_1, x_2, 0)$ with $\mathbb{C}$ we can think of $\mathbb{C}$ sitting in $\mathbb{R}^3$ as the (x_1, x_2)-plane. Set $N = (0, 0, 1)$ and define the map

$$\pi : S^2 \setminus \{N\} \to \mathbb{C}$$

as the map that takes $p \in S^2 \setminus \{N\}$ and maps it to $\pi(p)$, which is the intersection of $\mathbb{C}$ with the ray in $\mathbb{R}^3$ that starts at N and passes through p



π is called ***the stereographic projection of*** *$S^2 \setminus \{N\}$* ***into*** *$\mathbb{C}$*

Explicitly, π is given by

$$\pi(p) = \pi(x_1, x_2, x_2) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}, 0\right) = \frac{x_1}{1 - x_3} + \frac{x_2}{1 - x_3} i$$

Note that the equation of the ray that starts at N and goes through p is:

$$N + t(p - N), \quad t \geq 0$$

and consequently

$$\pi(p) = N + t_0(p - N)$$

where t_0 is unique positive real number, so that $(0, 0, 1) + t_0(x_1, x_2, x_3 - 1)$ has third coordinate equal to 0. Solving this equation for t_0 gives the formula for $\pi(p)$ above. Defining $\pi(N) = \infty$ gives a bijection

$$\pi : S^2 \to \hat{\mathbb{C}}$$

Conversely, given $z \in \mathbb{C}$ one checks that

$$\pi^{-1}(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right) \in S^2 \setminus \{N\}$$

and $\pi^{-1}(\infty) := N$ gives the inverse map. Here we get S^2 homeomorphic to $\hat{\mathbb{C}}$, since both maps are continuous.

Before we study the values of holomorphic functions using the Residue Formula 3.6, let us mention that we can also talk about meromorphic functions on $\hat{\mathbb{C}}$ (as opposed to $\mathscr{M}(\Omega)$ with $\Omega \subseteq \mathbb{C}$).

We have already allowed ∞ as a value of meromorphic functions. We can also allow ∞ in the definition domain and study functions $f \in \hat{\mathbb{C}}^{\tilde{\Omega}}$, where $\tilde{\Omega} \subseteq \hat{\mathbb{C}}$

If a function f is analytic for large values of z, i.e. $|z| > \frac{1}{R}$ for some $R > 0$, then the function

$$g(z) := f\left(\frac{1}{z}\right)$$

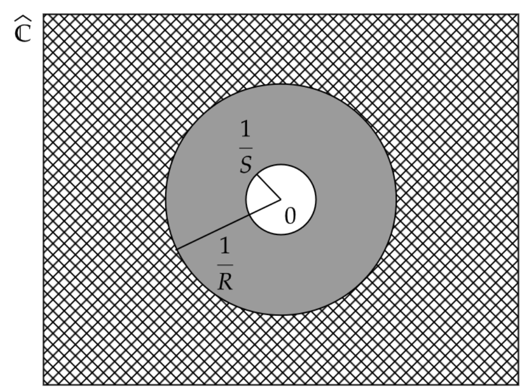
is holomorphic in a deleted neighbourhood of 0, i.e. in $\dot{D}_R(0)$

Definition 3.9 (Deleted neighbourhood at ∞)**.** We define **the delete neighbourhood at ∞** as

$$\dot{D}_R(\infty) := \left\{z \in \mathbb{C} : |z| > R^{-1}\right\}$$

This notation is designed to have for $R < S$ that

$$\dot{D}_R(\infty) \subset \dot{D}_S(\infty)$$



Definition 3.10.

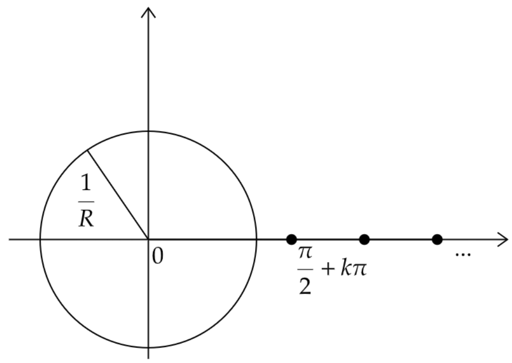
- For a function f, which is analytic for $|z| > \frac{1}{R}$ for some $R > 0$, we say that f has an **isolated singularity at** ∞, which will be called **removable**, **a pole** or **essential**, if $g(z) = f\left(\frac{1}{z}\right)$ has an isolated singularity at 0 (which is removable, a pole or essential respectively).

- A meromorphic function in the complex plane that is either holomorphic at ∞ or has a pole at ∞ **is called meromorphic in** $\hat{\mathbb{C}}$

Example 3.16. *1. An entire function is analytic in $\dot{D}_R(\infty)$ for every $R > 0$*
E.g. the function $f(z) = e^z$ has an isolated singularly at ∞, which is essential, because $e^{\frac{1}{z}}$ has an essential singularity at 0. Hence, $e^z$is not meromorphic in $\hat{\mathbb{C}}$, but it is meromorphic on $\mathbb{C}$

2. The function $p(z) \in \mathbb{C}[z]$ has a pole at ∞. If $p(z) = a_0 + \sum_{k=1}^n a_k z^k$ for some $n \in \mathbb{N}$, then $p\left(\frac{1}{z}\right) = a_0 + \sum_{k=1}^n \frac{a_k}{z^k}$ has a pole of order n at 0

3. The function $f(z) = \tan(z)$ does not have an isolated singularity at ∞: Each $\dot{D}_R(\infty)$ includes poles of f with $z = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$



Also note that $g(z) = \tan\left(\frac{1}{z}\right)$ has singularities $S := \left\{\left(\frac{\pi}{2} + k\pi\right)^{-1} : k \in \mathbb{Z}\right\}$, which accumulate at $z = 0$. Hence, the singularity of $\tan\left(\frac{1}{z}\right)$ at $z = 0$ is not isolated.

The following Theorem for meromorphic functions on $\hat{\mathbb{C}}$

Theorem 3.9. [SS10, Theorem III.3.4] If $f \in \mathscr{M}(\hat{\mathbb{C}})$, then it is a rational function. Clearly, each rational function is a meromorphic function on $\hat{\mathbb{C}}$. Hence, we have that

$$\mathscr{M}(\hat{\mathbb{C}}) = \left\{\frac{P}{Q} : P, Q \in \mathbb{C}[z]\right\}$$

Proof. Exercise. □

3.3 Applications of the Residue Theorem

The fist application is called the Argument Principle: it uses the Residue Theorem 3.6 applied to $\frac{f'}{f}$, the logarithmic derivative of f, to count the zeroes and the poles of f inside a curve.
To this end we first note the following simple Lemma:

Lemma 3.2. Let $\Omega \subseteq \mathbb{C}$ be open and connected, also $0 \neq f \in \mathscr{M}(\Omega)$, then $\frac{f'}{f} \in \mathscr{M}(\Omega)$, the **logarithmic derivative of** f, is also meromorphic in Ω. Moreover, $\frac{f'}{f}$ has poles of order 1 at all $z_0 \in \Omega$, for which $\operatorname{ord}_{z_0}(f) \neq 0$, i.e. either z_0 is a zero or a pole of f

$$\forall z_0 \in Z_f \cup S_f : \operatorname{Res}_{z_0}\left(\frac{f'}{f}\right) = \operatorname{ord}_{z_0}(f)$$

Proof. Since $f \neq 0$, Ω open and connected, being $f \in \mathscr{M}(\Omega)$, the zeroes of f do not have a limit point in Ω, and $\frac{1}{f} \in \mathscr{M}(\Omega)$

Clearly, $f' \in \mathscr{M}(\Omega \setminus S_f)$, where S_f is the set of poles of f. If $z_0 \in S_f$ is a pole of order $n \in \mathbb{N}^*$ of f, then $\exists r > 0$ by Theorem 3.3 such that

$$\forall z \in \dot{D}_r(z_0) : f(z) = (z - z_0)^{-n} h(z)$$

where $h \in \mathscr{H}\big(D_r(z_0)\big)$ and $h(z_0) \neq 0$. Then, for $z \in \dot{D}_r(z_0)$ we have

$$\begin{aligned} f'(z) &= \frac{-n}{(z-z_0)^{n+1}} h(z) + \frac{h'(z)}{(z-z_0)^n} = \\ &= \underbrace{\big(h'(z)(z-z_0) - nh(z)\big)}_{=:\tilde{h}(z)} \frac{1}{(z-z_0)^{n+1}} \end{aligned}$$

where $\tilde{h} \in \mathscr{H}\big(D_r(z_0)\big)$ and $\tilde{h}(z_0) = -nh(z_0) \neq 0$ by definition. Hence, for all $z \in \dot{D}_r(z_0)$

$$f'(z) = (z - z_0)^{-(n+1)} \tilde{h}(z)$$

Hence, f' has a pole of order $n+1$ at z_0 (Similarly, if f has a zero of order n at z_0, then f' has a zero of order $n-1$ at z_0). Hence, $f' \in \mathscr{M}(\Omega)$ and so is $\frac{f'}{f} \in \mathscr{M}(\Omega)$

So, for any $z \in \Omega$ we have

$$\operatorname{ord}_{z_0}\left(\frac{f'}{f}\right) = \operatorname{ord}_{z_0}(f') - \operatorname{ord}_{z_0}(f) = \begin{cases} -(n+1) - (-n) = -1 & , \operatorname{ord}_{z_0}(f) = -n \\ (n-1) - n = -1 & , \operatorname{ord}_{z_0}(f) = n \\ \geq 0 & , \text{otherwise} \end{cases}$$

Hence, $\frac{f'}{f}$ has a pole of order 1 at the points where $\operatorname{ord}_{z_0}(f) \neq 0$

We can also calculate the residue using

$$\forall z \in \dot{D}_r(z_0) : f(z) = (z - z_0)^n g(z)$$

where $g \in \mathscr{H}\big(D_r(z_0)\big)$, $g(z) \neq 0$ for all $z \in D_r(z_0)$ and $n = \text{ord}_{z_0}(f)$. Hence, we have: $n > 0$ if z_0 is a zero and $n < 0$ if z_0 is a pole of f

Finally, for $z \in D_r(z_0)$ we have

$$f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$$

and

$$\forall z \in D_r(z_0) : \frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} = \\ = \frac{n(z-z_0)^{n-1}g(z)}{(z-z_0)^n g(z)} + \frac{(z-z_0)^n g'(z)}{(z-z_0)^n g(z)} = \frac{n}{(z-z_0)} + \underbrace{\frac{g'(z)}{g(z)}}_{=G(z)\in\mathscr{H}\left(D_r(z_0)\right)}$$

Hence

$$\boxed{\text{Res}_{z_0}\left(\frac{f'}{f}\right) = n = \text{ord}_{z_0}(f)}$$

□

Lemma 3.2 immediately gives, using the Residue Theorem 3.6, the following result:

Theorem 3.10 (Argument Principle)**.** [SS10, Theorem III.4.1] Let $\Omega \subseteq \mathbb{C}$ open and connected, $f \in \mathscr{M}(\Omega)$ and let $im(\gamma) \subseteq \Omega$ be a circle (or any other curve such that the Residue Formula 3.6 holds). If f has no zeros or poles on γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \left(\sum_{z_0 \in Z_f \cap \text{int}(\gamma)} \text{ord}_{z_0}(f)\right) + \sum_{z_0 \in S_f \cap \text{int}(\gamma)} \text{ord}_{z_0}(f) = \sum_{\substack{z_0 \in \text{int}(\gamma) \\ \text{ord}_{z_0}(f) \neq 0}} \text{ord}_{z_0}(f)$$

where Z_f is the set of zeroes of f and S_f is the set of poles of f

Proof. This follows from the previous Lemma 3.2 and $\text{Res}_{z_0}\left(\frac{f'}{f}\right) = \text{ord}_{z_0}(f)$

$$\frac{1}{2\pi}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_k \in \text{int}(\gamma)} \text{Res}_{z_k}\left(\frac{f'}{f}\right) = \boxed{Z - P}$$

where Z is the number of zeroes of f inside γ counted with multiplicity (to express the order of zero) and P is the number of poles of f inside γ counted with multiplicity (to express the order of pole). □

Corollary 3.1. Let $f \in \mathbb{C}[z]$ be a polynomial. Choose $R > 0$ large enough, so that all zeroes of f are inside $D_R(0)$, then

$$\int_{C_R(0)} \frac{f'(z)}{f(z)} dz = \deg(f)$$

We have the following Corollary of the Argument Principle Theorem 3.10, which says that a holomorphic function, when perturbed slightly, does not change its number of zeroes.

Theorem 3.11 (Rouché's Theorem)**.** [SS10, Theorem III.4.3] Suppose $f, g \in \mathscr{H}(\Omega)$ for an open set $\Omega \subseteq \mathbb{C}$, which contains a circle C and its interior. If

$$\forall z \in C : \left|f(z)\right| > \left|g(z)\right|$$

Then f and $f + g$ have the same number of zeroes inside of C

Proof. For $t \in [0, 1]$, define $f_t(z) = f(z) + tg(z)$ so that $f_0(z) = f(z)$ and $f_1(z) = (f + g)(z)$. Note that for $z \in C$ it results that

$$\begin{aligned} \left|f_t(z)\right| = \left|f(z) + tg(z)\right| \geq \Big|\left|f(z)\right| - t\left|g(z)\right|\Big| \geq \left|f(z)\right| - t\left|g(z)\right| > \\ > \left|g(z)\right| - t\left|g(z)\right| = (1 - t)\left|g(z)\right| \geq 0 \end{aligned}$$

given the assumption. Hence, we have that $\forall z \in C : \left|f_t(z)\right| > 0$ and therefore f_t has no zero in C
Note that if we can show that

$$n_t := \frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)} dz$$

is a constant, this will show that f and $f+g$ have the same number of zeroes inside of C

By Argument Principle 3.10 applied to $f_t(z)$ (which by the above consideration has no zero in C), we have that

$$n_t = \frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)} dz$$

counts the number of zeroes of f_t inside of C, in particular it is integer valued.

Note that n_t is a continuous function of t, since $\frac{f_t'(z)}{f_t(z)}$ is jointly continuous on $[0, 1]$ for all $z \in C$, as both $f_t'(z)$ and $f_t(z)$ are continuous and $f_t(z) \neq 0$ for all $z \in C$

Recall: from Real Analysis we know that, if

$$h : [a, b] \times [c, d] \to \mathbb{R}$$

is continuous on $[a, b] \times [c, d]$, then $F(t) := \int_c^d h(t, x)dx$ is continuous on $[a, b]$.
Using this gives that

$$\frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)} dz$$

is continuous and since n_t is also integer valued, it must be a constant (otherwise the Intermediate Value Theorem [EW22] gives the existence of $t_0 \in [0, 1]$ such that n_{t_0} is not integral). Hence, n_0 is the number of zeroes inside of f and n_1 is the number of zeroes inside of $f + g$, so finally

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = n_0 = n_1 = \frac{1}{2\pi i} \int_C \frac{(f + g)'(z)}{(f + g)(z)} dz$$

□

Example 3.17. *We use Rouché's Theorem 3.11 to show that the polynomial*

$$p(z) = z^6 + 8z^4 + z^3 + 2z + 3$$

has four zeroes inside of the unit circle $C_1(0)$

The idea is to write $p = Big + Small$ *on* $C_1(0)$*, such that* $Big(z) = 8z^4 = f(z)$ *and* $Small(z) = z^6 + z^3 + 2z + 3 = g(z)$*, so:* $\left|g(z)\right| = |z^6 + z^3 + 2z + 3| < |8z^4| = \left|f(z)\right|$ *on* $C_1(0)$*, so with* $|z| = 1$

$$|z^6 + z^3 + 2z + 3| \leq |z|^6 + |z|^3 + 2|z| + 3 = 1 + 1 + 2 + 3 = 7 < 8 = 8|z|^4$$

Hence, by Rouché's Theorem 3.11, $f(z) = 8z^4$ *and* $(f + g)(z) = p(z)$ *have the same number of zeroes inside the unit circle.* f *has four zeroes (counted with multiplicity) and so does* p

Example 3.18. *Rouché's Theorem 3.11 can also be used to give a "nice[4]" or simple proof of the Fundamental Theorem of Algebra.*
Let $p(z) = z^d + \left(\sum_{k=1}^{d-1} a_k z^k\right) + a_0$ *for which, if* $|z|$ *is large enough, the term* $|z|^d$ *dominates. Choose* R *large enough, so that for* $f(z) = z^d$ *and* $g(z) = \sum_{k=1}^{d-1} a_k z^k + a_0$ *we have:*

$$\left|f(z)\right| > \left|g(z)\right|$$

on $C_R(0)$ *and hence* $p = f + g$ *and* f *have the same number of zeroes inside* $C_R(0)$*. We obtain that* f *has* d *zeroes inside* $C_R(0)$ *and so does* p

Rouché's Theorem also leads us to two other important Theorems:

[4](subjectively) "nice" means "objectively nice", if said from the Professor.

Theorem 3.12 (Open mapping Theorem)**.** [SS10, Theorem III.4.4] Let $\Omega \subseteq \mathbb{C}$ be an open and connected set. Let $f \in \mathscr{H}(\Omega)$ such that f is non-constant, then f is an open map, i.e. sends open sets to open sets in the standard topology of $\mathbb{C}$, namely form $\mathcal{O}_{\mathbb{C}}^{\Omega}$ to $\mathcal{O}_{\mathbb{C}}$

Proof. Let $z_0 \in U \subseteq \Omega$ with $U \in \mathcal{O}_{\mathbb{C}}^{\Omega}$ and $f(z_0) = w_0$. We want to show that a neighbourhood of w_0 is also contained in $f(U)$, i.e. if w is near w_0, then show that $\exists z \in U : w = f(z)$, i.e. $w \in f(U)$. If we interpret z as a zeroes of $f(z) - w$, we have also

$$f(z) - w = \underbrace{f(z) - w_0}_{\tilde{f}} + \underbrace{w_0 - w}_{\tilde{g}}$$

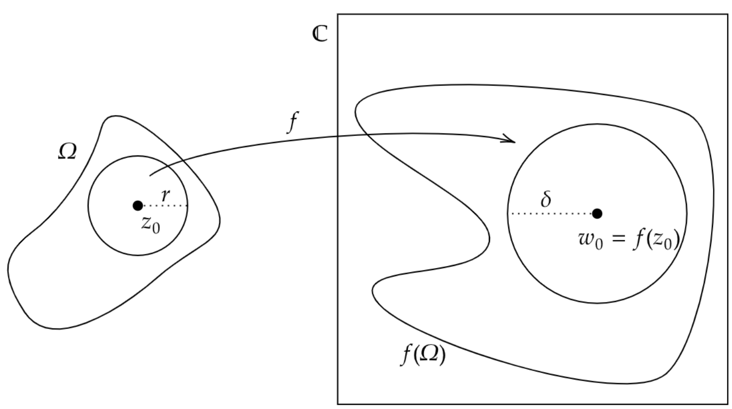
Let $r > 0$ such that $\overline{D_r}(z_0) \subset U$ and such that $\forall z \in \dot{\overline{D_r}}(z_0) : f(z) - w_0 \neq 0$; this is allowed, since the zeroes of $\tilde{f}(z) := f(z) - w_0$ are isolated. In particular, we have that $\left|f(z) - w_0\right| \neq 0$ for z on the circle $C_r(z_0)$

Since $C_r(z_0)$ is compact and $\left|f(z) - w_0\right| \neq 0$ on $C_r(z_0)$, we can find a $\delta > 0$ such that

$$\left|f(z) - w_0\right| \geq \delta$$

for all $z \in C_r(z_0)$. Let now $w \in D_\delta(w_0)$ and define

$$F : \Omega \to \mathbb{C}, w \mapsto F(z) := f(z) - w = \underbrace{f(z) - w_0}_{=:\tilde{f}} + \underbrace{w_0 - w}_{=:\tilde{g}}$$



We want to show that F has a zero inside the circle $C_r(z_0)$. This will show that $\exists z \in D_r(z_0) : f(z) = w$ and hence that $w \in f\big(D_r(z_0)\big)$

We now apply Rouché's Theorem 3.11 to $\tilde{f}, \tilde{g}$ on the circle $C_r(z_0)$, where we have that

$$\left|\tilde{f}\right| \geq \delta \quad \text{and} \quad |\tilde{g}| < \delta$$

Hence, on the circle $C_r(z_0)$, we have

$$|\tilde{f}| > |\tilde{g}|$$

and so $\tilde{f}$ and $\tilde{F} = \tilde{f} + \tilde{g}$ have the same number of zeroes inside $D_r(z_0)$. Since $\tilde{f} = f - w_0$ has a zero inside $D_r(z_0)$, namely z_0, we must conclude that $\exists z \in D_r(z_0) : F(z) = 0$, i.e. $\exists z \in D_r(z_0) : f(z) = w$, so $w \in f\big(D_r(z_0)\big)$ as wanted. $\square$

Remark 3.12. *This Theorem 3.12 states, for example, that if $f \in \mathscr{H}(D_r(z_0))$ and non-constant for some $z_0 \in \mathbb{C}$, then it is not possible to that $f(z) \in \mathbb{R}$ for all $z \in D_r(z_0)$, since any subset of $\mathbb{R}$ is not open in $\mathbb{C}$*

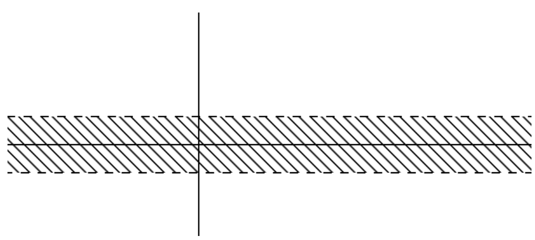
Theorem 3.13 (Maximum modulus principle)**.** [SS10, Theorem III.4.5][SS10, Corollary III.4.6] Let $\Omega \subseteq \mathbb{C}$ be open and connected and $f \in \mathscr{H}(\Omega)$ non-constant, then

$$\nexists z_0 \in \Omega \forall z \in \Omega : \big|f(z)\big| \leq \big|f(z_0)\big|$$

i.e. f cannot attain its maximum in Ω. In particular, if $\overline{\Omega}$ is bounded and $f \in C^0\big(\overline{\Omega}\big)$, then

$$\max_{z \in \overline{\Omega}} \big|f(z)\big| = \max_{z \in \partial\Omega} \big|f(z)\big|$$

Example 3.19 (Necessity of the boundedness of $\overline{\Omega}$)**.** *The assumption that $\overline{\Omega}$ is bounded, hence compact, is crucial, as shown in this example.*



Let $\Omega = \left\{z \in \mathbb{C} : Im(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$ be open in $\mathbb{C}$ and connected, it is not hard to see that the set $\overline{\Omega}$ is not bounded. Consider the function $f(z) = \exp\left(e^z\right)$ on $\overline{\Omega}$, then

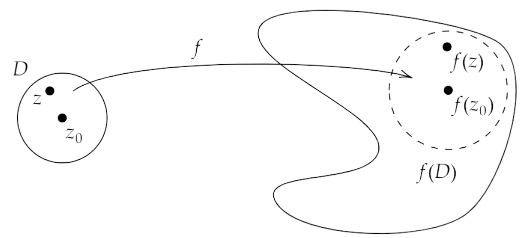
$$f|_{\partial\Omega}(z) = \exp\left(e^{x \pm i\frac{\pi}{2}}\right) = \exp(\pm i e^x)$$

Hence, for $z \in \partial\Omega$ we have $\big|f(z)\big| = 1$, but

$$f(x) = \exp(e^x) \xrightarrow{\mathbb{R} \ni x \to \infty} \infty$$

which does not exists in $\mathbb{R}$

Proof of Theorem 3.13. We first note that $\exists \max_{z\in\overline{\Omega}} \left|f(z)\right|$, since $\overline{\Omega}$ is a compact set and $f \in C^0(\overline{\Omega})$, as shown in [EW22].
Let $f \in \mathscr{H}(\Omega)$ and non-constant. Suppose also on the contrary that f attains a maximum at $z_0 \in \Omega$. By the Open mapping Theorem 3.12, if f is an open map, by letting $D = D_r(z_0) \subseteq \Omega$, then $f(D)$ is open in $\mathbb{C}$ and contains $f(z_0)$



Hence $f(D)$ contains a disc $\tilde{D}$ around $f(z_0)$ and therefore there are points $z \in D$ such that

$$\left|f(z_0)\right| < \left|f(z)\right|$$

which contradicts the assumption that $|f|$ attains its maximum at z_0 (in any disc in $\mathbb{C}$ one can find such points).

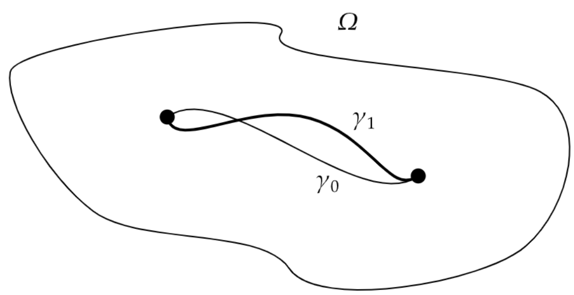
If $\overline{\Omega}$ is bounded and f is non-constant and continuous, then $\left|f(z)\right|$ attains its maximum on $\overline{\Omega}$, since it is a continuous function on a compact set, as shown in [EW22]. By the first part of the Theorem, this point where f attains its maximum cannot be inside Ω. Hence, it has to be on the boundary $\overline{\Omega} \setminus \Omega = \partial\Omega$ $\square$

3.4 Homotopy and simply connected domains

The key to understand the general form of Cauchy's Theorem 2.5 is the idea that if $f \in \mathscr{H}(\Omega)$ and if we "continuously deform" γ_0 to γ_1, while staying in Ω and keeping the endpoints fixed if the paths are not closed, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

for the two paths $\gamma_0 : [a, b] \to \Omega$ and $\gamma_1[a, b] \to \Omega$, either closed or with fixed endpoints.



Not closed curves of this type curves are called homotopic with fixed endpoints: this means that for each $s \in [0,1]$ it exists a curve γ_s in Ω parameterised by $\gamma_s(t)$, hence $\gamma_s : [a,b] \to \Omega$ such that

$$\gamma_s(a) = \gamma_0(a) = \gamma_1(a)$$
$$\gamma_s(b) = \gamma_0(b) = \gamma_1(b)$$

and such that at $s = 0$ we have $\gamma_s(t)|_{s=0} = \gamma_0(t)$ and at $s = 1$ we have $\gamma_s(t)|_{s=1} = \gamma_1(t)$. All this should be done continuously.

Definition 3.11 (Homotopy)**.** Let $\Omega \subseteq \mathbb{C}$ be open.

- Let $\gamma_0 : [a,b] \to \Omega$ and $\gamma_1 : [a,b] \to \Omega$ be two curves such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, i.e. they have the same endpoints.
  We says that γ_0 **is homotopic to** γ_1 **in** Ω **with fixed endpoints**, and denote it by $\gamma_0 \sim_\Omega \gamma_1 \operatorname{rel} \partial[a,b]$, if

  $$\exists H \in C^0\big([a,b] \times [0,1]; \Omega\big), (t,s) \mapsto H(t,s) =: \gamma_s(t)$$

  such that

  - $\forall t \in [a,b] : H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$
  - $\forall s \in [0,1] : H(t,s) = \gamma_s(t) \in C^0\big([a,b]; \Omega\big)$ is a piecewise smooth curve and $\forall s \in [0,1] : H(a,s) = \gamma_0(a) = \gamma_1(a)$ and $H(b,s) = \gamma_0(b) = \gamma_1(b)$, i.e. $\gamma_s(t)$ has the same endpoints as γ_0, γ_1

- Similarly, let $\gamma_0 : [a,b] \to \Omega$ and $\gamma_1 : [a,b] \to \Omega$ be two closed curves, we say that γ_0 **is homotopic to** γ_1 **in** Ω, denoted as $\gamma_0 \sim_\Omega \gamma_1$, if

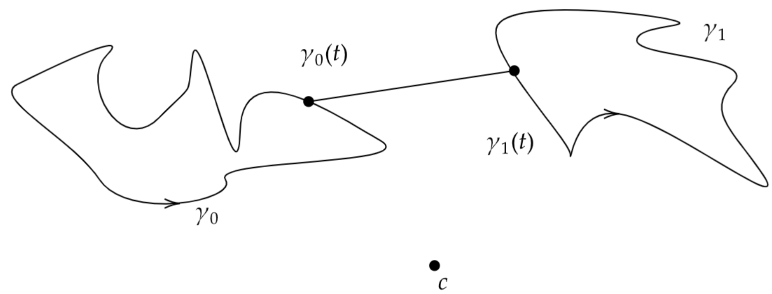
  $$\exists H \in C^0\big([a,b] \times [0,1]; \Omega\big), (t,s) \mapsto H(t,s) =: \gamma_s(t)$$

  such that

  - $\forall t \in [a,b] : H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$
  - $\forall s \in [0,1] : H(t,s) =: \gamma_s(t) \in C^0\big([a,b]; \Omega\big)$ is a piecewise smooth curve in Ω and $\forall s \in [0,1] : H(a,s) = H(b,s)$, i.e. $\gamma_s(t)$ is a closed curve in Ω for every $s \in [0,1]$

Note that if clear enough, the notation for fixed endpoints, namely "$\operatorname{rel} \partial[a,b]$" will here be omitted.

Example 3.20. *1. If $\Omega = \mathbb{C}$, then any two closed curves γ_0, γ_1 are homotopic. In particular, every closed curve is homotopic to the constant curve $\sigma_c : [a,b] \to \mathbb{C}, t \mapsto c$ for every $c \in \mathbb{C}$*



Consider the function

$$\begin{aligned} H : [a,b] \times [0,1] &\to \mathbb{C} \\ (t,s) &\mapsto (1-s)\gamma_0(t) + s\gamma_1(t) \end{aligned}$$

H is a combination of continuous functions, hence continuous. Moreover it holds that:

$$\begin{aligned} H(t,0) &= \gamma_0(t) \\ H(t,1) &= \gamma_1(t) \\ H(a,s) &= (1-s)\gamma_0(a) + s\gamma_1(a) \\ H(b,s) &= (1-s)\gamma_0(b) + s\gamma_1(b) \end{aligned}$$

Since $\gamma_0(a) = \gamma_0(b)$, $\gamma_1(a) = \gamma_1(b)$ and $\forall s \in [0,1] : H(a,s) = H(b,s)$. Hence $\gamma_s : [a,b] \to \mathbb{C}$ are all closed curves.

Note that geometrically H is defined using the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ for each fixed $t \in [a,b]$. Hence, $s \in [0,1]$ varies over the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ for each fixed $t \in [a,b]$.

For the constant curve σ_c, we can take the homotopy between σ_c and γ as

$$\begin{aligned} H : [a,b] \times [0,1] &\to \mathbb{C} \\ (t,s) &\mapsto (1-s)c + s\gamma(t) \end{aligned}$$

Note that the same definition we used for closed curves γ_0, γ_1 also gives a homotopy with fixed points, if $\gamma_0 : [a,b] \to \Omega$ and $\gamma_1 : [a,b] \to \Omega$ are two curves with fixed endpoints, i.e. with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$

2. *Note that the same formula for the homotopy works for any convex $\Omega \subseteq \mathbb{C}$. I.e. if we have two curves γ_0, γ_1 either closed or with a fixed endpoints in a convex set*

Ω, then since for a convex set the line segment between any two points is also in the set, the function defined by

$$\begin{aligned} H : [a,b] \times [0,1] &\to \mathbb{C} \\ (t,s) &\mapsto (1-s)\gamma_0(t) + s\gamma_1(t) \end{aligned}$$

gives a homotopy in Ω.
In particular, this works for Ω with the form of a disc.

3. *An example of two curves which are not homotopic in Ω: if we take $\Omega := \mathbb{C}^*$ and the curves*

$$\begin{aligned} \gamma_0 : [0,\pi] &\to \Omega \\ t &\mapsto e^{it} \end{aligned}$$

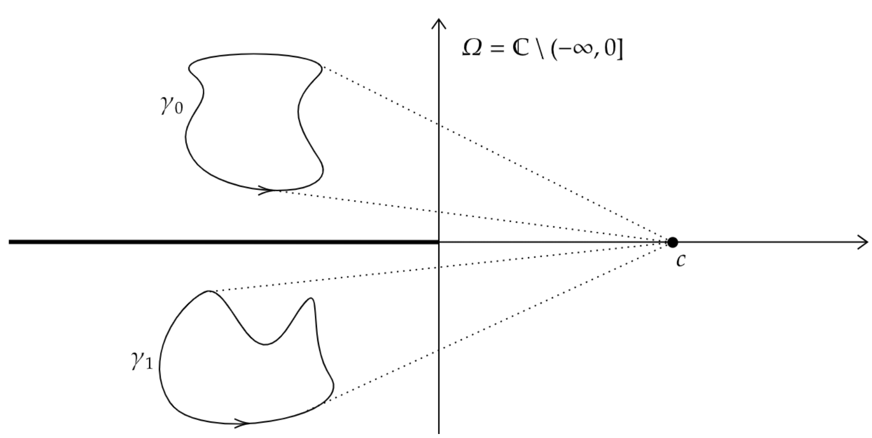
and

$$\begin{aligned} \gamma_1 : [0,\pi] &\to \Omega \\ t &\mapsto e^{-it} \end{aligned}$$

Then γ_0 and γ_1 are not homotopic in Ω

We will see a simple proof of this when we will see the homotopy version of Cauchy's Theorem 2.5. Intuitively, to deform γ_0 to γ_1 we have to go through 0, which is not in Ω though.

4. *The set $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is not convex and therefore we cannot use the previous formula, but still any two closed curves γ_0, γ_1 in Ω are homotopic, i.e. we can deform γ_0 to γ_1 in the following way. Then for $f \in \mathscr{H}(\Omega)$ we have that $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$*



The idea is to choose any point on the real line, say $c \in \mathbb{R}$, and the constant curve

$$\begin{aligned} \sigma_c : [a,b] &\to \Omega \\ t &\mapsto c \end{aligned}$$

We first deform γ_0 to c and then c to γ_1, so

$$H(t,s) := \begin{cases} c + (1-2s)\big(\gamma_0(t) - c\big) & , s \in \left[0, \frac{1}{2}\right] \\ c + (2s-1)\big(\gamma_1(t) - c\big) & , s \in \left(\frac{1}{2}, 1\right] \end{cases}$$

H is continuous, the only point to check is $s = \frac{1}{2}$, hence

$$H\left(t, \frac{1}{2}\right) = c = \lim_{s \to \frac{1}{2}} c + (2s-1)\big(\gamma_1(t) - c\big)$$

To see that the image of H is contained in Ω for all $t \in [a,b]$ and $s \in [0,1]$, check for example that if $t \in [a,b]$ and $s \in \left[0, \frac{1}{2}\right]$, then in case that $H(t,s) \notin \Omega$, it means that for some t, s the value of $H(t,s) \in \mathbb{R}^{\leq 0}$, i.e.

$$\begin{aligned} & c + (1-2s)\big(\gamma_0(t) - c\big) \leq 0 \\ \iff & (1-2s)\big(\gamma_0(t) - c\big) \leq -c \\ \iff & \gamma_0(t) \leq \frac{-c}{1-2s} + c = c\left(1 + \frac{1}{2s-1}\right) = c\left(\frac{2s}{2s-1}\right) \leq -c \end{aligned}$$

But $0 \leq s \leq \frac{1}{2} \implies 2s \geq 0$ and $2s - 1 \leq 0$. Hence, $\gamma_0(t) \in (-\infty, 0]$ but at the same time $\gamma_0(t) \in \Omega$ and this cannot happen. For $\frac{1}{2} < s \leq 1$ is similar.

Remark 3.13. *If γ_0 homotopic to γ_1 in Ω (either closed or with fixed endpoints), we write $\gamma_0 \sim_\Omega \gamma_1$ and simply write $\gamma_0 \sim \gamma_1$ if Ω is fixed and clear.*

Then $\sim$ is an equivalence relation:

- *The curve γ_0 is homotopic with itself via $H(t,s) = \gamma_0(t)$*
- *If $\gamma_0 \sim \gamma_1$ with $H(t,s)$, then $\gamma_1 \sim \gamma_0$ with $\tilde{H}(t,s) := H(t, 1-s)$*
- *If $\gamma_0 \sim \gamma_1$ with $F(t,s)$ and $\gamma_1 \sim \gamma_2$ with $G(t,s)$, then define*

$$H(t,s) := \begin{cases} F(t, 2s) & , s \in \left[0, \frac{1}{2}\right] \\ G(t, 2s-1) & , s \in \left(\frac{1}{2}, 1\right] \end{cases}$$

that gives a homotopy between $\gamma_0 \sim \gamma_2$

3.5 The Homotopy Theorem

We can now state the Homotopy Theorem.

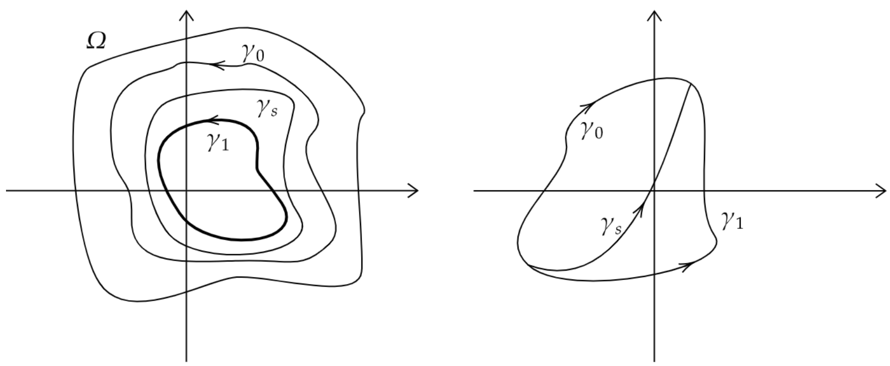
Theorem 3.14 (Homotopy Theorem)**.** [SS10, Theorem III.5.1] Let $\Omega \subseteq \mathbb{C}$ be open. Let γ_0, γ_1 be two curves in Ω that are

(i) either γ_0, γ_1 closed curves and homotopic

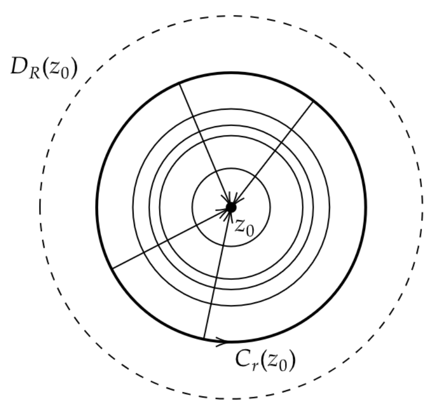
(ii) or γ_0, γ_1 have the same endpoints and are homotopic with fixed endpoints.

Then, for $f \in \mathscr{H}(\Omega)$ we have that

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$



Example 3.21. *1. Let $\Omega = D_R(z_0)$ and $R > 0$ and let $im(\gamma_0) = C_r(z_0)$ with $r \in (0, R)$ as in the following picture.*



Then γ_0 can be deformed into the point z_0 by dilation (which can be be thought as the constant curve for which $\forall t \in [a, b] : \gamma_1(t) = z_0$), so for $f \in \mathscr{H}(\Omega)$

$$\int_{C_r(z_0)} f(z)dz = \int_{\sigma_{z_0}} f(z)dz = 0$$

since $\forall t \in [a,b] : \gamma_1'(t) = 0$

Consider the homotopy

$$H : [0, 2\pi] \times [0,1] \to D_R(z_0)$$
$$(t,s) \mapsto (1-s)e^{it} + sz_0$$

In fact, in $D_R(z_0)$ *any closed curve* γ *is homotopic to a constant curve. Hence, we get*

$$\int_\gamma f(z)dz = 0$$

2. *Let* $\Omega = \mathbb{C}^*$ *and*

$$\gamma_0 : [0,\pi] \to \Omega \qquad \gamma_1 : [0,\pi] \to \Omega$$
$$t \mapsto e^{it} \qquad t \mapsto e^{-it}$$

The two paths are not homotopic with fixed endpoints, since if they were, then we would get that for $\frac{1}{z} \in \mathscr{H}(\Omega)$ *it would hold that*

$$\int_{\gamma_0} \frac{1}{z} dz = \int_{\gamma_1} \frac{1}{z} dz$$

and therefore

$$\int_{\gamma_0} \frac{1}{z} dz - \int_{\gamma_1} \frac{1}{z} dz = \int_{C_1(0)} \frac{1}{z} dz = 0$$

but

$$\int_{C_1(0)} \frac{1}{z} dz = 2\pi i \neq 0$$

We now look at the proof of the Homotopy Theorem 3.14. We will look at the case of closed curves (in [SS10] one finds instead the version where the endpoints of a curve are fixed).

Proof of the Theorem 3.14. 1. A simpler version of the proof is given with the extra assumption that the homotopy $H(t,s)$ has continuous second order partial derivatives and

$$\forall (t,s) \in [a,b] \times [0,1] : \frac{\partial^2 H}{\partial s \partial t}(t,s) = \frac{\partial^2 H}{\partial t \partial s}(t,s)$$

For this we first recall from Real Analysis:

Let $H : [a,b] \times [0,1] \to \Omega \subseteq \mathbb{C}, (t,s) \mapsto H(t,s) =: \gamma_s(t)$ be an homotopy between the two paths γ_0, γ_1 and let $h : [a,b] \times [0,1] \to \mathbb{R}$ be defined as here below.

Recall: Let $h : [a, b] \times [0, 1] \to \mathbb{R}, (t, s) \mapsto h(t, s)$
Suppose that $\frac{\partial h}{\partial s}$ exists and is continuous, if we define

$$G : [0, 1] \to \mathbb{R}$$
$$s \mapsto G(s) := \int_a^b h(t, s)dt$$

then we get that G is differentiable with

$$G'(s) = \int_a^b \frac{\partial h}{\partial s}(t, s)dt$$

We apply this to the real and imaginary part of the following:

$$I(s) := \int_a^b \underbrace{f\big(H(t,s)\big)\frac{\partial H}{\partial t}(t,s)}_{=h(t,s)}\, dt = \int_a^b f\big(\gamma_s(t)\big)\gamma_s'(t)dt = \int_{\gamma_s} f(z)dz$$

Also, note that

$$I(0) = \int_{\gamma_0} f(z)dz \quad \text{and} \quad I(1) = \int_{\gamma_1} f(z)dz$$

We want to show that $I(0) = I(1)$ by showing that $I(s)$ is constant. So consider

$$I'(s) = \int_a^b \frac{\partial}{\partial s}\left((f \circ H)(t,s)\frac{\partial H}{\partial t}(t,s)\right) dt =$$
$$= \int_a^b \left((f' \circ H)(t,s)\frac{\partial H}{\partial s}(t,s)\frac{\partial H}{\partial t}(t,s) + (f \circ H)(t,s)\frac{\partial}{\partial s}\frac{\partial H}{\partial t}(t,s)\right) dt$$

and note that what is inside the round parenthesis is also equal by assumption to

$$\frac{\partial}{\partial t}\left((f \circ H)(t,s)\frac{\partial H}{\partial s}(t,s)\right)$$

Hence, we have that

$$I'(s) = \int_a^b \frac{\partial}{\partial t}\left((f \circ H)(t,s)\frac{\partial H}{\partial s}(t,s)\right) dt =$$
$$= \left[f\big(H(t,s)\big)\frac{\partial H}{\partial s}(t,s)\right]_{t=a}^{t=b} =$$
$$= f\big(H(b,s)\big)\frac{\partial H}{\partial s}(b,s) - f\big(H(a,s)\big)\frac{\partial H}{\partial s}(a,s) = 0$$

Since H is a homotopy of closed curves for which it holds that

$$\forall s \in [0,1] : \gamma_s(a) = H(a,s) = H(b,s) = \gamma_s(b)$$

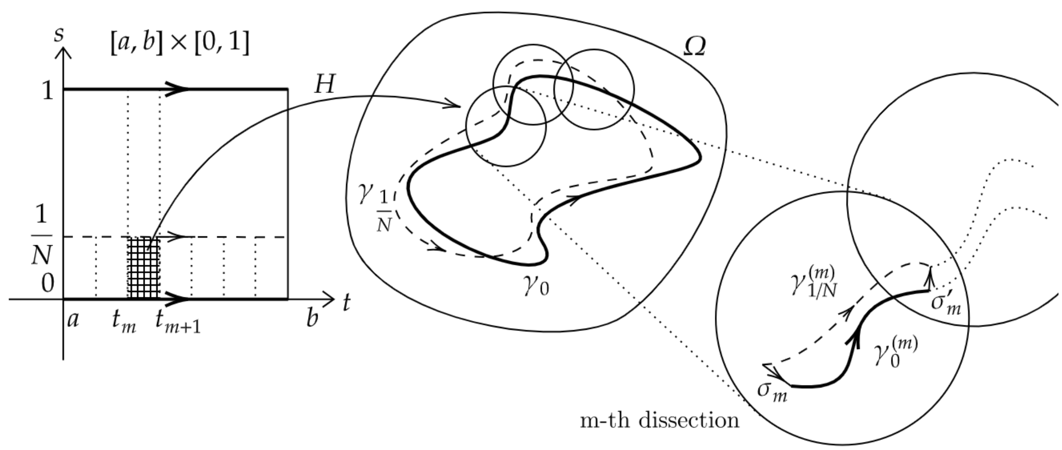
and also that

$$\forall s \in [0,1] : \frac{\partial H}{\partial s}(a,s) = \frac{\partial H}{\partial s}(b,s)$$

Finally, we have that $\forall \in [0,1] : I'(s) = 0$ and therefore I is constant. In particular, we obtain

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

2. For the general proof the idea is the following: if we make a small deformation of one of the curves $\gamma_s(t)$, say $\gamma_0(t)$ to $\gamma_{\frac{1}{N}}(t)$, so that if we look at a small piece around a point of $\gamma_0(t)$, say $t \in (t_m, t_{m+1})$ for $m \in \{0, ..., N-1\}$ and N the cardinality of the dissection of $[a,b]$ and of $[0,1]$, then we can show that these are contained in a small disc in Ω



Let $t \in (t_m, t_{m+1})$ for $m \in \{0, ..., M-1\}$ and M the cardinality of the dissection of $[a,b]$

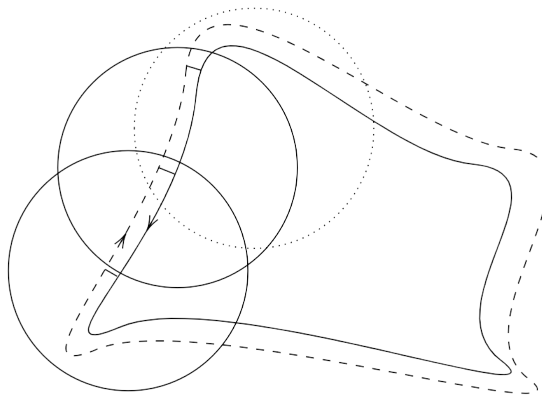
$$H(t,0) = \gamma_0(t)$$
$$H\left(t, \frac{1}{N}\right) = \gamma_{\frac{1}{N}}(t)$$

we can apply Cauchy's Theorem 2.5 in a disc to get

$$\int_{\gamma_0^{(m)}} f(z)dz = \int_{\gamma_{\frac{1}{N}}^{(m)} \uplus \sigma \uplus \sigma'} f(z)dz$$

Now, we move over to the whole curve $\gamma_0, \gamma_{\frac{1}{N}}$ using small discs contained in Ω to get

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_{\frac{1}{N}}} f(z)dz$$



To make this idea more precise, we need these two facts:

(a) If $K = im(H) = H([a,b] \times [c,d]) \subseteq \Omega$, then K is compact (since H is continuous and $[a,b] \times [0,1]$ is compact, see [EW22]).

(b) A continuous function on a compact set is uniformly continuous.

We then have that (a) implies the following Lemma:

Lemma 3.3. $\exists \varepsilon > 0 \forall z \in K \subseteq \Omega : D_\varepsilon(z) \subseteq \Omega$

Proof. Assume on the contrary that no such ε exists. Then $\forall n \in \mathbb{N}^* \exists z_n \in K : D_{\frac{1}{n}}(z_n) \nsubseteq \Omega$, i.e. $\exists w_n \in \mathbb{C} \setminus \Omega : |w_n - z_n| < \frac{1}{n}$ for any $n \geq 1$. This way we get a sequence $(z_n)_{n \in \mathbb{N}^*} \in K^{\mathbb{N}^*}$, where K is compact, hence $(z_n)_{n \in \mathbb{N}^*}$ has a subsequence $(z_{n_k})_{k \in \mathbb{N}^*}$ which converges to $\lim_{k \to \infty} z_{n_k} = z$. Since K is also closed, we have $z \in K$. Now, because of $|w_n - z_n| < \frac{1}{n}$ for any $n \geq 1$, we have for any $k \in \mathbb{N}^*$

$$|w_{n_k} - z_{n_k}| < \frac{1}{n_k}$$

so $(w_{n_k})_{k \in \mathbb{N}^*}$ also converges to z. $(w_{n_k})_{k \in \mathbb{N}^*}$ is also in $\mathbb{C} \setminus \Omega$, so its limit point is $z \in \mathbb{C} \setminus \Omega$ and this is a contradiction. □

This Lemma 3.3 together with (b) will allow us to find the small discs that are contained in Ω. This is because we can divide the rectangle $[a,b] \times [c,d]$ into small rectangles such that the images of these small rectangles are contained in small discs of radius ε

More precisely, let $(t_m)_{m\in\{0,\dots,N\}} \in [a,b]^{\{0,\dots,N\}}$ and $(s_n)_{n\in\{0,\dots,N\}} \in [0,1]^{\{0,\dots,N\}}$ represent the dissections of the two intervals and $\varepsilon > 0$. Since H is uniformly continuous on $[a,b]\times[c,d]$, then it exists an $N \in \mathbb{N}^*$ such that

$$\left|H(t,s) - H(t_m, s_n)\right| < \varepsilon$$

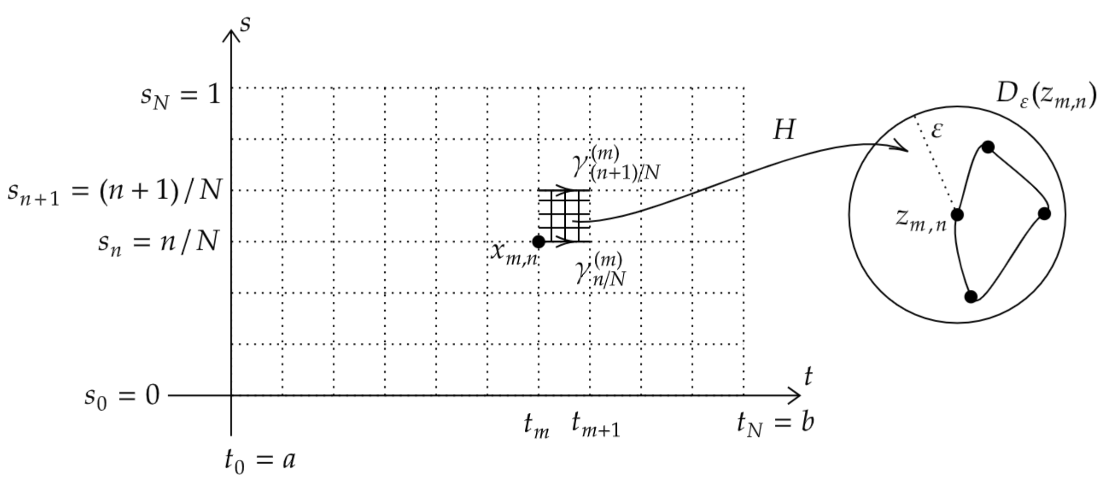
whenever

$$\left|(t,s) - (t_m, s_n)\right| < \frac{2}{N}$$

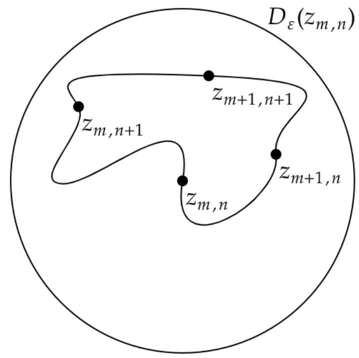
for $z_{m,n} = H(t_m, s_n)$ with $t_m := a + \frac{b-a}{N}m$ and $s_n := \frac{n}{N}$ for $m, n \in \{0, \dots, N\}$

Let $Q_{m,n} := [t_m, t_{m+1}] \times [s_n, s_{n+1}]$ for $m, n \in \{0, \dots, N-1\}$, since the diameter of $Q_{m,n}$ is $\operatorname{diam}(Q_{m,n}) = \frac{\sqrt{2}}{N} < \frac{2}{N}$, it follows that

$$H(Q_{m,n}) \subseteq D_\varepsilon(z_{m,n})$$



where $z_{m,n} := H(t_m, s_n)$ and $x_{m,n} := (t_m, s_n)$, with $z_{0,n} = H\left(a, \frac{n}{N}\right)$ and $z_{N,n} = H\left(b, \frac{n}{N}\right)$; we note that in this case $z_{0,n} = z_{N,n}$



Using induction on $n \in \mathbb{N} \cap \{0, ..., N\}$, we want to show now that

$$\int_{\gamma_{\frac{n}{N}}} f(z)dz = \int_{\gamma_0} f(z)dz$$

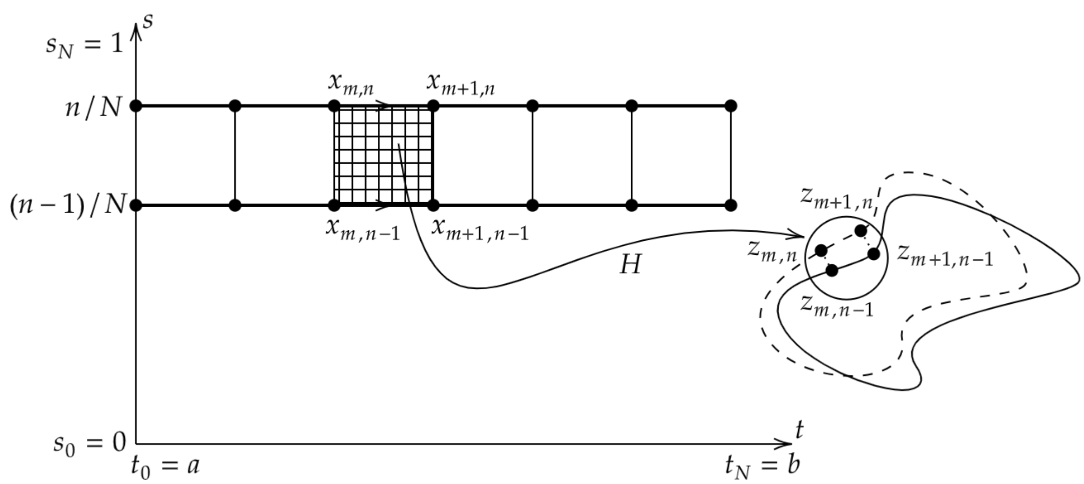
This is clearly true for $n = 0$, as $\frac{0}{N} = 0$, hence we consider $n \geq 1$ and assume that it holds

$$\int_{\gamma_{\frac{n-1}{N}}} f(z)dz = \int_{\gamma_0} f(z)dz$$

It is enough to show that

$$\int_{\gamma_{\frac{n-1}{N}}} f(z)dz = \int_{\gamma_{\frac{n}{N}}} f(z)dz$$

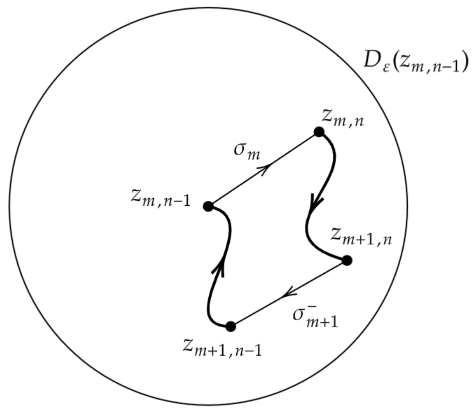
where $\gamma_{\frac{n-1}{N}} = H\left(t, \frac{n-1}{N}\right)$ and $\gamma_{\frac{n}{N}} = H\left(t, \frac{n}{N}\right)$



For each $m \in \{0, ..., N\}$, let

$$\gamma^{(m)}_{\frac{n-1}{N}} := \gamma_{\frac{n-1}{N}}\Big|_{[t_m, t_{m+1}]} \text{ and } \gamma^{(m)}_{\frac{n}{N}} := \gamma_{\frac{n}{N}}\Big|_{[t_m, t_{m+1}]}$$

and let $\sigma_m := \ell_{[z_{m,n-1},z_{m,n}]}$ be line segment between $z_{m,n-1}$ and $z_{m,n}$, thus $\sigma_{m+1} = \ell_{[z_{m+1,n-1},z_{m+1,n}]}$ be the line segment between $z_{m+1,n-1}$ and $z_{m+1,n}$; as special case we consider $\sigma_0 = [z_{0,n-1}, z_{0,n}]$ and $\sigma_N = [z_{N,n-1}, z_{N,n}]$



Now we apply Cauchy's Theorem 2.5 in the disc $D_\varepsilon(z_{m,n-1})$ and obtain

$$\int_{\gamma^{(m)}_{\frac{n}{N}}} f(z)dz - \int_{\sigma_{m+1}} f(z)dz - \int_{\gamma^{(m)}_{\frac{n-1}{N}}} f(z)dz + \int_{\sigma_m} f(z)dz = 0$$

Summing over m to get the full $\gamma_{\frac{n}{N}}$, we have that

$$\begin{aligned}\int_{\gamma_{\frac{n-1}{N}}} f(z)dz &= \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{\frac{n-1}{N}}} f(z)dz = \\ &= \left(\sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{\frac{n}{N}}} f(z)dz\right) + \sum_{m=0}^{N-1} \left(\int_{\sigma_m} f(z)dz - \int_{\sigma_{m+1}} f(z)dz\right) = \\ &= \int_{\gamma_{\frac{n}{N}}} f(z)dz + \underbrace{\int_{\sigma_0} f(z)dz - \int_{\sigma_N} f(z)dz}_{=0} = \\ &= \int_{\gamma_{\frac{n}{N}}} f(z)dz\end{aligned}$$

We have $\sigma_0 = \sigma_N$, since $\gamma_{\frac{n-1}{N}}(a) = H\left(a, \frac{n-1}{N}\right) = H\left(b, \frac{n-1}{N}\right) = \gamma_{\frac{n-1}{N}}(b)$ and similarly $\gamma_{\frac{n}{N}}(a) = \gamma_{\frac{n}{N}}(b)$, i.e. the curves γ_s are closed. Hence, we proved by induction that the integral over γ_0 and over γ_1 are equal, which concludes the proof.

□

In spaces like $\mathbb{C}, \mathbb{C}\backslash(-\infty, 0]$ or any convex set, we observed that any two closed curves, or any two curves with the same endpoints, are homotopic. This leads us to the following definition.

Definition 3.12. An open set $\Omega \subseteq \mathbb{C}$ is called **simply connected**, if it is connected and any two curves with the same endpoints are homotopic in Ω

E.g. $\mathbb{C}, \mathbb{C} \setminus (-\infty, 0]$ and $D_r(z_0)$ are simply connected, while $\mathbb{C}^*$ is not.

As Corollary of the Homotopy Theorem 3.14 we have that

Theorem 3.15. [SS10, Theorem III.5.2] Any holomorphic function $f \in \mathscr{H}(\Omega)$ on a simply connected region $\Omega \subseteq \mathbb{C}$ has a primitive. In particular, we have that

$$\int_\gamma f(z)dz = 0$$

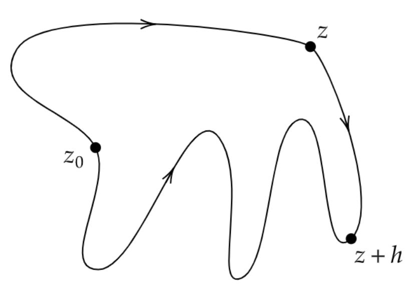
for any closed curve γ in Ω and that any two primitives differ by a constant.

Proof. Let $\Omega \subseteq \mathbb{C}$ be simply connected and fix $z_0 \in \Omega$, we define $F \in \mathbb{C}^\Omega$ such that for any $z \in \Omega$

$$\boxed{F(z) := \int_{\gamma_z} f(w)dw}$$

where γ_z is a curve connecting z_0 to z. Note that this is a well-defined function, since using the Homotopy Theorem 3.14 and Ω simply connected, any two curves $\gamma_z, \tilde{\gamma}_z$ between z_0 and z satisfy $\gamma_z \sim \tilde{\gamma}_z$; therefore they give the same value of

$$\int_{\gamma_z} f(z)dz = \int_{\tilde{\gamma}_z} f(z)dz$$



Let $z \in \Omega$, if we choose h small enough, so that the image of the line segment $im(\ell_{[z,z+h]}) \subseteq \Omega$, then

$$F(z+h) - F(z) = \int_{\ell_{[z,z+h]}} f(w)dw$$

Arguing as in the proof of Theorem 2.3 or using continuity of f as below, we get that

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$$

which shows that F is a primitive of f in Ω, as z was arbitrary. I.e.

$$\begin{aligned}F(z+h)-F(z)&=\int_{\ell_{[z,z+h]}}\big(f(w)-f(z)+f(z)\big)dw=\\&=f(z)\underbrace{\int_{\ell_{[z,z+h]}}dw}_{=h}+\int_{\ell_{[z,z+h]}}\big(f(w)-f(z)\big)dw\end{aligned}$$

so

$$\left|\int_{\ell_{[z,z+h]}}\big(f(w)-f(z)\big)dw\right|\leq h\sup_{w\in im\left(\ell_{[z,z+h]}\right)}\big|f(w)-f(z)\big|$$

hence

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|\leq\sup_{w\in im\left(\ell_{[z,z+h]}\right)}\big|f(w)-f(z)\big|$$

Being f is of C^0- class, it implies that $\sup_{w\in im\left(\ell_{[z,z+h]}\right)}\big|f(w)-f(z)\big|\xrightarrow{h\to 0}0$ □

3.6 Complex Logarithm

For $z\in\mathbb{C}^*$ we want to define the logarithm of $z=re^{i\theta}$ and we want it to be the inverse function of the exponential function, i.e. $w=\log(z)$ if $e^w=z$. A natural candidate is

$$\boxed{\log(z)=\log(r)+i\theta}$$

where $\log(r)$ is the usual logarithm $\log:\mathbb{R}^+\to\mathbb{R}$ of the positive real number r. The problem is that this is not single valued, as θ is only unique up to an integer multiple of 2π. Indeed, the argument is multivalued.

Example 3.22. *Let $z=1\in\mathbb{C}$, it holds that $e^0=1$, but also that for any $w\in 2\pi i\mathbb{Z}$ the equality $e^w=1$ holds.*

We want a holomorphic function $\ell\in\mathscr{H}(\Omega)$, which satisfies

$$\exp\circ\ell=id$$

throughout its domain of definition Ω

Definition 3.13. Let $\Omega \subset \mathbb{C}$ be open. **A** (fixed) **branch of the logarithm on** Ω, denoted by $\log_\Omega$, is a function in $\mathscr{H}(\Omega)$ such that

$$\forall z \in \Omega : \exp\big(\log_\Omega(z)\big) = z$$

If Ω is clear from the text, sometimes this function is also denoted by log

Note that any function $f \in \mathscr{H}(\Omega)$ that meets that condition is a branch of logarithm on Ω, but when one is fixed, then it it denoted by $\log_\Omega$

Remark 3.14. 1. *Since* $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$*, in order for* $\log_\Omega$ *to exist we need that* $\boxed{0 \notin \Omega}$

2. *If* $\Omega = \mathbb{C}^*$*, even though* $\exp \in \big(\mathbb{C}^*\big)^{\mathbb{C}}$ *is surjective, there is no branch of the logarithm on* Ω*. Indeed, if there were a* $f \in \mathscr{H}(\Omega)$ *such that* $\forall z \in \Omega : \exp\big(f(z)\big) = z$*, then differentiating on both sides would give*

$$f'(z) \underbrace{\exp\big(f(z)\big)}_{=z} = 1$$

for all $z \in \Omega$*, which then gives* $f'(z) = \frac{1}{z}$ *for all* $z \in \Omega$*, i.e.* $\frac{1}{z}$ *has a primitive in* $\mathbb{C}^*$*, which would imply that*

$$\int_{C_1(0)} \frac{1}{z} dz = 0$$

which we know it is not.

3. *If* Ω *is open and connected and* $\ell = \log_\Omega \in \mathbb{C}^\Omega$ *is a logarithm, then* $\tilde{\ell}$ *is also a logarithm on* Ω *if and only if*

$$\tilde{\ell} = \ell + 2\pi i n$$

for some $n \in \mathbb{Z}$*. Indeed, if* $\tilde{\ell}$ *is a logarithm function, then* $\exp\big(\tilde{\ell}(z)\big) = z$ *and* $\exp\big(\ell(z)\big) = z$
Hence, for all $z \in \Omega$ *we have* $\exp\big(\tilde{\ell}(z) - \ell(z)\big) = 1$*, so for all* $z \in \Omega$ *we also have*

$$\tilde{\ell}(z) - \ell(z) \in 2\pi i\mathbb{Z}$$

i.e. $\frac{\tilde{\ell}(z)-\ell(z)}{2\pi i}$ *is a constant integer valued function on* Ω*, which is connected. Hence, its image under* $\frac{\tilde{\ell}(z)-\ell(z)}{2\pi i}$ *is connected and is also a subset of* $\mathbb{Z}$*, therefore it is a single point* $n \in \mathbb{Z}$*. Conversely, if* $\tilde{\ell} = \ell + 2\pi i n$ *for some* $n \in \mathbb{Z}$*, then*

$$\exp\big(\tilde{\ell}(z)\big) = \exp\big(\ell(z)\big)\exp(2\pi i n) = \exp\big(\ell(z)\big) = z$$

Theorem 3.16. [SS10, Theorem III.6.1] Let $\Omega \subset \mathbb{C}^*$ be a simply connected set, then there exists a branch of the logarithm on Ω, i.e. a function $F \in \mathscr{H}(\Omega)$ such that $\forall z \in \Omega : \exp\big(F(z)\big) = z$

Proof. Since $0 \notin \Omega$, then $\frac{1}{z} \in \mathscr{H}(\Omega)$ and since Ω is simply connected, we have that then $\frac{1}{z}$ has a primitive in Ω that we call $f(z)$

Let $G : \mathbb{C}^\Omega$ such that $z \mapsto G(z) := z\exp\big(-f(z)\big)$, since $f'(z) = \frac{1}{z}$ we have that

$$G'(z) = -zf'(z)\exp\big(-f(z)\big) + \exp\big(-f(z)\big) = -\exp\big(-f(z)\big) + \exp\big(-f(z)\big) = 0$$

Since Ω is connected, it follows that $G(z) = ze^{-f(z)} = a$ (constant) necessarily, for some $a \in \mathbb{C}$. Moreover, since $\exp \neq 0$, $z \neq 0$, also $a \neq 0$, so we obtain that it exists $b \in \mathbb{C}$ such that $a = \exp(b)$ and therefore, by algebraic manipulations on G, that

$$\exp\big(f(z)\big) = \frac{z}{a}$$

Hence, let $F \in \mathbb{C}^\Omega$ such that $z \mapsto F(z) := f(z) + b$, then

$$\exp\big(F(z)\big) = \exp\big(f(z) + b\big) = \underbrace{\exp\big(f(z)\big)}_{=\frac{z}{a}}\underbrace{\exp(b)}_{=a} = z$$

and so $F(z)$ is a branch of the logarithm on Ω □

Definition 3.14. Let $\mathbb{C}^- := \mathbb{C}\backslash(-\infty, 0]$. **The principal branch of the logarithm on** $\mathbb{C}^-$ is the unique function $\log_{\mathbb{C}^-} \in \mathscr{H}(\mathbb{C}^-)$ such that $\log_{\mathbb{C}^-}(1) = 0$. This particular $\log_{\mathbb{C}^-}$ is also denoted by Log (with capital L) or $\mathrm{Log}_{\mathbb{C}^-}$.

Proposition 3.5. If $z = re^{i\theta} \in \mathbb{C}^-$ with $r > 0$ and $\theta \in (-\pi, \pi)$, then the principal branch of the logarithm is given by the formula

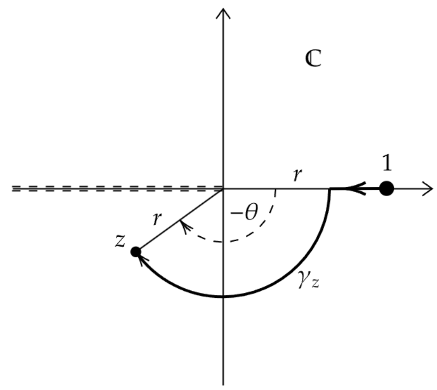
$$\mathrm{Log}_{\mathbb{C}^-}(z) = \log(r) + i\theta = \log\big(|z|\big) + i\,\mathrm{Arg}(z)$$

The common notation to use for the Principal branch is Log

Proof. Let $f \in \mathscr{H}(\Omega)$ be such that

$$f(z) := \int_{\gamma_z} \frac{1}{w} dw$$

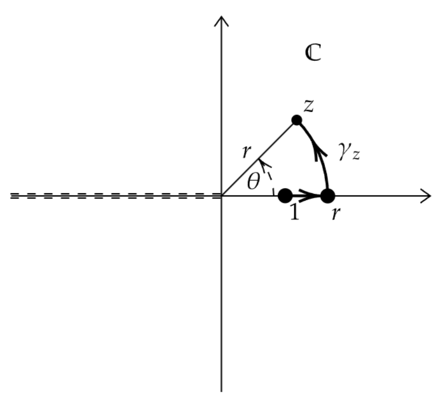
hence, a primitive of $\frac{1}{z}$ by Theorem 3.15, where we take a path γ_z in $\mathbb{C}^-$ which starts at 1 and ends at z. Note that $\int_{\gamma_1} \frac{1}{w} dw = 0$, hence $\log_{\mathbb{C}^-}(1) = 0$. This implies that this branch of logarithm is equal to the principal one (we are going to use the usual notation for this branch from now on).



If $z = re^{i\theta}$ with $r < 1$, take the path γ_z that goes on the real line from 1 to r, then on the circular arc to z, so

$$\operatorname{Log}(z) = \underbrace{-\int_r^1 \frac{dx}{x}}_{\text{on the } x\text{-axis}} + \underbrace{\int_0^{-\theta} \frac{-ire^{-it}}{re^{-it}} dt}_{\text{on the arc } z = re^{-it} \text{ for } t \in (0, -\theta)} = \underbrace{\log(r)}_{\text{the real one}} + i\theta$$

Instead, if $r > 1$, we take the path



and so similar calculations give the result (Exercise). □

Notation: From now on, if not specified, log = Log

Remark 3.15. *1. The identity*

$$\operatorname{Log}(z) + \operatorname{Log}(w) = \operatorname{Log}(zw)$$

does not hold in general for all $z, w, zw \in \mathbb{C}^-$, but if $w = re^{i\alpha}$, $z = se^{i\beta}$ and $zw = sre^{i\theta}$ with $\alpha, \beta, \theta \in (-\pi, \pi)$, then exists $\gamma \in \{-2\pi, 0, 2\pi\}$ such that

$$\theta = \alpha + \beta + \gamma$$

Then

$$\begin{aligned}\mathrm{Log}(zw) &= \log(rs) + i\theta = \\ &= \log(r) + \log(s) + i(\alpha + \beta + \gamma) = \\ &= \big(\log(r) + i\alpha\big) + \big(\log(s) + i\beta\big) + i\gamma = \\ &= \mathrm{Log}(w) + \mathrm{Log}(z) + i\gamma\end{aligned}$$

In particular,

$$\mathrm{Log}(z) + \mathrm{Log}(w) = \mathrm{Log}(zw) \iff \gamma = 0 \iff \alpha + \beta \in (-\pi, \pi)$$

The condition is satisfied whenever $Re(w) > 0$ and $Re(z) > 0$, hence in the real positive half plane $\mathbb{H}$.

2. *For the principal branch of the logarithm, Log, we have the following Taylor expansion*

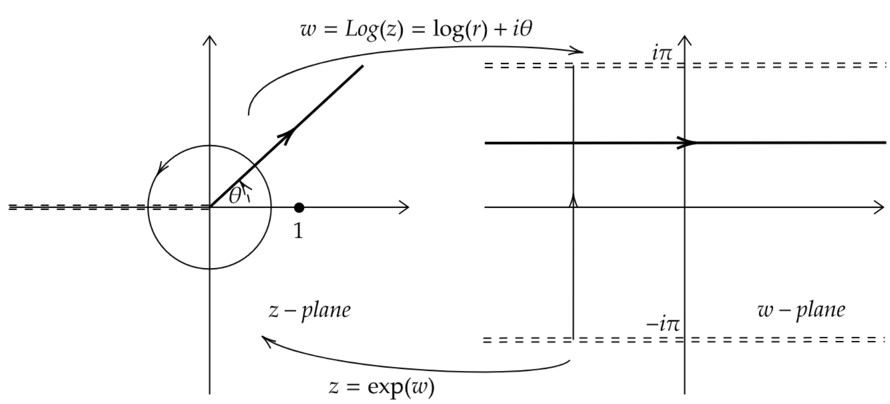
$$\boxed{\forall z \in D_1(1) : \mathrm{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n}$$

To see this, we differentiate both sides: on the left we have $\frac{1}{z}$, while on the right we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^{n-1} = \sum_{n=0}^{\infty}(1-z)^n = \frac{1}{1-(1-z)} = \frac{1}{z}$$

Hence, log and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$ only differ by a constant. Looking at $z = 1$ we obtain that the constant is 0 and with it the result.

3. *We have*



The image of a punctured circle around 0

$$\{z \in \mathbb{C}^- : |z| < r \text{ and } \operatorname{Arg}(z) \in (-\pi, \pi)\}$$

is the vertical interval

$$\{z \in \mathbb{C} : Re(w) = \log|z| \text{ and } Im(w) \in (-\pi, \pi)\}$$

where if $r < 1$*, then* $Re(w) < 0$ *or if* $r > 1$*, then* $Re(w) > 0$

The image of $\{z \in \mathbb{C} : \operatorname{Arg}(z) = \theta\}$*, a ray from* 0 *to* ∞*, is the horizontal line* $\{w \in \mathbb{C} : Im(w) = \theta\}$

4. *We can define a holomorphic branch of the logarithm on any cut plane of the form*

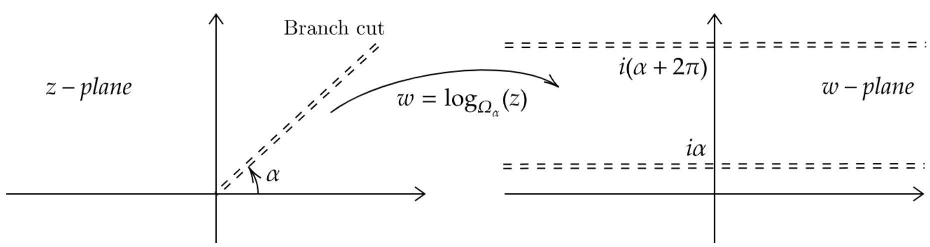
$$\Omega_\alpha := \mathbb{C} \setminus \Big(\{z \in \mathbb{C} : \operatorname{Arg}(z) = \alpha\} \cup \{0\}\Big)$$

for $\alpha \in [-\pi, \pi]$*, such that*

$$\boxed{w = \log_{\Omega_\alpha}(z) = \log|z| + i\theta \quad , \text{ with } \theta \in (\alpha, \alpha + 2\pi)}$$

and consequently

$$\boxed{\theta = \operatorname{Arg}(z) + \alpha}$$



5. *Let* $\Omega \subseteq \mathbb{C}^*$ *be simply connected and* $\log_\Omega \in \mathbb{C}^\Omega$ *a branch of the logarithm. Let* $\alpha \in \mathbb{C}$ *and* $z \in \Omega$*. We define*

$$\boxed{z^\alpha := \exp\big(\alpha \log_\Omega(z)\big) =: [z^\alpha]}$$

Note that this definition depends on the choice of the branch of the logarithm $\log_\Omega$*: if we choose another branch of the logarithm, call it* ℓ*, as* $\ell = \log_\Omega + 2\pi i k$ *for some* $k \in \mathbb{Z}$*, then*

$$[z^\alpha]_\ell = \exp\Big(\alpha\big(\log_\Omega(z) + 2\pi i k\big)\Big) = z^\alpha e^{2\pi i k \alpha}$$

If we set $\Omega = \mathbb{C}^-$*, if we choose the principal branch of the logarithm with* $\operatorname{Log}(1) = 0$ *and* $\alpha = \frac{1}{m}$ *for* $m \in \mathbb{N}$*, then*

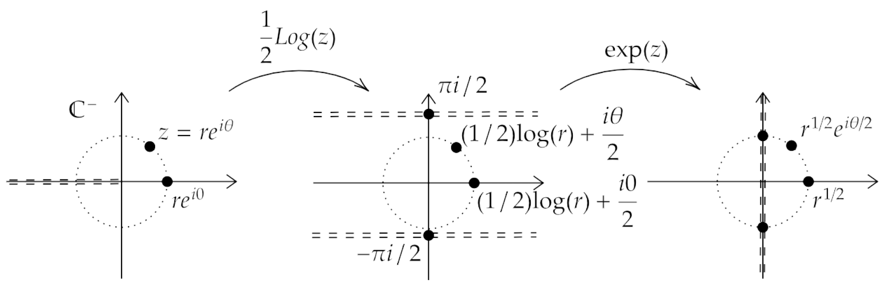
$$z^{\frac{1}{m}} = e^{\frac{1}{m}\operatorname{Log}(z)}$$

satisfies

$$\left(z^{\frac{1}{m}}\right)^m = \prod_{j=1}^{m} e^{\frac{1}{m}\operatorname{Log}(z)} = e^{m\frac{1}{m}\operatorname{Log}(z)} = e^{\operatorname{Log}(z)} = z$$

Example 3.23. *Let* Log *be the principal branch of the logarithm on* $\mathbb{C}^-$*, then*

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\operatorname{Log}(z)}$$



Note that for $z \in \mathbb{R}^+$*, the value of* $z^{\frac{1}{2}}$ *is the usual positive square root.*

If we choose another branch of the logarithm in $\mathbb{C}^-$*, for some* $k \in \mathbb{Z}$*, e.g.*

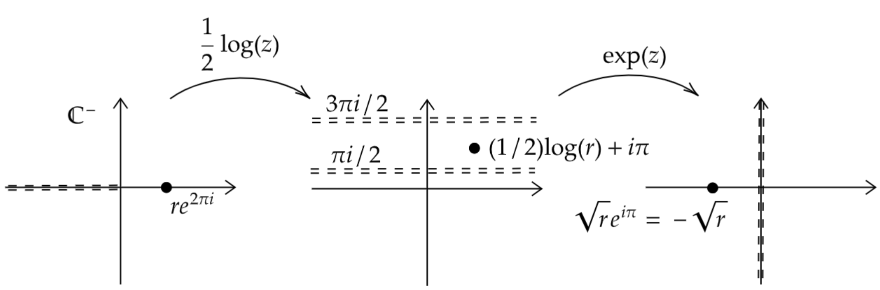
$$\boxed{\log_{\mathbb{C}^-,k}(z) := \log(r) + i(\theta + 2\pi k)}$$

then

$$z^{\frac{1}{2}} = \exp\left(\frac{1}{2}\log_{\mathbb{C}^-,k}\right) = r^{\frac{1}{2}}e^{i\frac{\theta+ik\pi}{2}} = r^{\frac{1}{2}}e^{i\frac{\theta}{2}}e^{ik\pi} = r^{\frac{1}{2}}e^{i\frac{\theta}{2}}(-1)^k = (-1)^k\left[z^{\frac{1}{2}}\right]$$

i.e. infinitely many choices for the branch of the logarithm give only two different choices for $\sqrt{z}$ *(where we wrote* $\left[z^{\frac{1}{2}}\right]$ *for the principal branch of the square root).*

For the choice $\log_{\mathbb{C}^-,1}(z) = \log(r) + i(\theta + 2\pi)$ *we have (always with* $r > 0$ *and* $\operatorname{Arg}(z) \in (\pi, 3\pi)$*)*



Definition 3.15 (Logarithm of a function)**.** Let $\Omega \subseteq \mathbb{C}$ open and $f \in \mathscr{H}(\Omega)$, a function $g \in \mathscr{H}(\Omega)$ such that

$$f(z) = e^{g(z)}$$

is called a logarithm of f and (if fixed) is denoted by $\log_\Omega(f)$ or $\log(f)$ in the sense of composition, i.e. $\log_\Omega(f) = \log_\Omega \circ f$

Remark 3.16. *The logarithm of a function is in general not a branch of logarithm on any subset* $\Omega \subseteq \mathbb{C}$

Finally, we have that if $f \in \mathscr{H}(\Omega)$ on a simply connected region Ω ad f is non-vanishing in all of Ω, then f has a logarithm in Ω,

Theorem 3.17. [SS10, Theorem III.6.2] Let $\Omega \subseteq \mathbb{C}$ be a simply connected region. If $f \in \mathscr{H}(\Omega)$ non-vanishing in all of Ω, then $\exists g \in \mathscr{H}(\Omega)$, called logarithm of f, i.e. $\log(f)$, such that

$$f(z) = e^{g(z)}$$

Proof. Exercise. Define g a primitive of $\frac{f'}{f}$ □

Corollary 3.2. If $f \in \mathscr{H}(\Omega)$, non-vanishing in all of a simply connected region $\Omega \subseteq \mathbb{C}$, then f has a square root in Ω, i.e.

$$\exists h \in \mathscr{H}(\Omega) : h^2(z) = f(z)$$

Proof. Let

$$h(z) = \exp\left(\frac{1}{2}\log(f)\right) = \exp\left(\frac{1}{2}g(z)\right)$$

from Theorem 3.17, then

$$h^2 = \exp\big(g(z)\big) = f(z)$$

□

Before we move to conformal maps in the next section, we mention that there are various ways to look at simply connected regions. This is taken up in the book in the Appendix B [SS10].

We have seen that if Ω is simply connected (i.e. such that any two curves in Ω with same endpoints are homotopic), then for all closed curve γ in Ω and for all $f \in \mathscr{H}(\Omega)$ we have

$$\int_\gamma f(z)dz = 0$$

Definition 3.16. A region $\Omega \subseteq \mathbb{C}$ is called **holomorphically simply connected**, if for all closed curve γ in Ω and all $f \in \mathscr{H}(\Omega)$

$$\int_\gamma f(z)dz = 0$$

Clearly, we have with Cauchy's Theorem 2.5 or the Homotopy Theorem 3.14 that

$$\Omega \text{ simply connected} \Longrightarrow \Omega \text{ holomorphically simply connected}$$

In fact, the converse is also true, as we have

Theorem 3.18. Let Ω be a region, then

$$\Omega \text{ holomorphically simply connected} \Longleftrightarrow \Omega \text{ simply connected}$$

The other direction

$$\Omega \text{ holomorphically simply connected} \Longrightarrow \Omega \text{ simply connected}$$

uses the Riemann Mapping Theorem (which we will see soon).
For bounded regions we also have

Theorem 3.19. If Ω is a bounded region in $\mathbb{C}$, then

$$\Omega \text{ is simply connected} \Longleftrightarrow \mathbb{C} \setminus \Omega \text{ is connected}$$

The proof of the direction: Ω bounded and simply connected $\Rightarrow \mathbb{C} \setminus \Omega$ connected, uses the notion of winding numbers, which we are going to briefly discuss as next, since it also leads to the natural generalisation of the Residue Theorem 3.6.

Remark 3.17. *In the above Theorem 3.19, the assumption that Ω is bounded in $\mathbb{C}$ is important, since the infinite strip is simply connected and unbounded, its complement has 2 components. However, if the complement is taken in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the conclusion holds if Ω is bounded or not.*

3.7 Winding numbers

See Appendix B in [SS10].

We have seen that for $f \in \mathscr{H}(\Omega)$, for some $\Omega \subseteq \mathbb{C}$ simply connected, if γ_1, γ_2 are two closed curves such that $\gamma_1 \sim_\Omega \gamma_2$, then by Theorem 3.14

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

We want to generalise it to $f \in \mathscr{M}(\Omega)$, hence we want to understand the integral

$$\int_\gamma f dz$$

for some γ in Ω and $f \in \mathscr{M}(\Omega)$

Recall: If $f \in \mathscr{M}(\Omega)$, $z_0 \in \Omega$, $im(\gamma) = \partial D_r(z_0)$ and $\overline{D}_r(z_0) \subset \Omega$, then

$$\int_\gamma f(z)dz = 2\pi i \sum_{w \in (S_f \cap \text{int}(\gamma))} \text{Res}_w(f)$$

with $\forall t \in [0, 2\pi] : \gamma(t) = z_0 + r(t)e^{i\theta(t)}$ in $\Omega \subseteq \mathbb{C}$ for Ω open and for some functions r, θ of class C^1 such that $\forall t \in [0, 2\pi] : r(t) > 0$ and $r(0) = r(2\pi)$, $\theta(0) = \theta(2\pi)$
The same proof we gave for the Residue Formula 3.6 for a circle works also here for

$$\int_\gamma f(z)dz = 2\pi i \sum_{w \in (S_f \cap \text{int}(\gamma))} \text{Res}_w(f)$$

The Homotopy Theorem 3.14 gives the following first generalisation of the Residue Theorem 3.6.

Proposition 3.6. Let $\Omega \subseteq \mathbb{C}$ be open and let $f \in \mathscr{M}(\Omega)$. Let $V := \Omega \setminus S_f$ so that $f \in \mathscr{H}(V)$. Then

(i) If γ_1, γ_2 are two closed curves in $V \subseteq \Omega$, which are homotopic in V, then

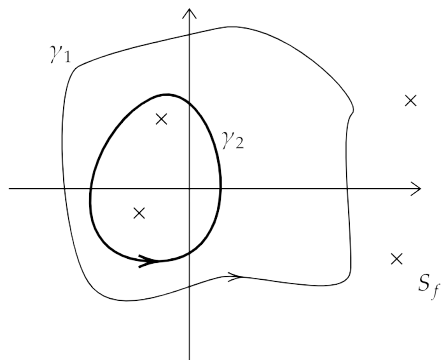
$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

(ii) In particular, if γ_2 is a circle (with counterclockwise orientation), then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz = 2\pi i \sum_{w \in S_f \cap \text{int}(\gamma_2)} \text{Res}_w(f)$$

Proof. **(i)** This is a special case of the Homotopy Theorem 3.14, since $f \in \mathscr{H}(V)$ and $\gamma_1 \sim_V \gamma_2$

(ii) Follows from the previous point and the Residue Formula 3.6.



□

To look at more general curves, we first introduce the winding number of a curve.

Definition 3.17. [SS10, Appendix B p.347] Let $z_0 \in \mathbb{C}$ and γ a piecewise smooth closed curve in $\mathbb{C}$, such that $z_0 \notin im(\gamma)$. **The winding number of** γ **around** z_0 is defined as

$$w_\gamma(z_0) := \text{ind}_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}$$

The winding number is also called **the index of** γ **around** z_0 and denoted by $\text{ind}_\gamma(z_0)$

Why is this called the winding number?

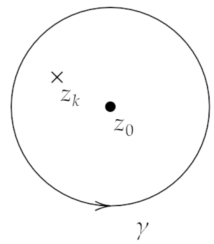
Remark 3.18. *To get a feeling for why this is called the winding number:*

1. *If $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi n]$ and for $n \in \mathbb{N}$, i.e. the circle with center z_0 traced n times counterclockwise. Then*

$$w_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_0^{2\pi n} \frac{ire^{it}}{re^{it}}dt = \frac{1}{2\pi}\int_0^{2\pi n} dt = n \in \mathbb{N}$$

2. *On the other hand, if $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$, but we are looking at a point $z_0 \neq z_1 \in \mathbb{C}$*

$$w_\gamma(z_1) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_1} = \sum_{\tilde{z} \in S_f \cap \text{int}(\gamma)} \text{Res}_{\tilde{z}}\left(\frac{1}{z - z_1}\right) = \begin{cases} 0 & , z_1 \notin \text{int}(\gamma) \\ 1 & , z_1 \in \text{int}(\gamma) \end{cases}$$



So at least, when γ is a circle, then the integral $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_1}$ indeed tells us whether γ wraps around z_1 or not.

"To get an intuition": For a general smooth $\gamma : [a, b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$ and z_0 inside the path, the following imprecise and really not completely correct argument might give an insight as to why it is called winding number.

$$w_\gamma(z_0) = \int_\gamma \frac{dz}{z - z_0} = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0} dt$$

From Real Analysis we might be tempted to write this last integral as

$$\log\big(\gamma(t) - z_0\big)\Big|_a^b$$

But of course this is not correct, because $\gamma(t) - z_0$ is complex valued and if γ wraps around a point z_0, then we cannot define an analytic branch of $\log\big(\gamma(t) - z_0\big)$ on $\mathbb{C}\setminus\{z_0\}$

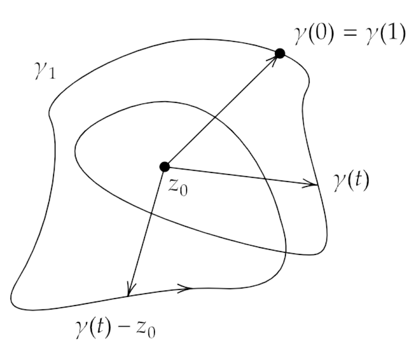
If we think of $\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$ and recall that the difficulty in defining the logarithm comes from choosing the correct value of the $\mathrm{Arg}(z)$, we can look at

$$\int_\gamma \frac{1}{z - z_0} dz = \log\big(\gamma(b) - z_0\big) - \log\big(\gamma(a) - z_0\big) =$$

$$= \log\big|\gamma(b) - z_0\big| + i\,\mathrm{Arg}\big(\gamma(b) - z_0\big) - \Big(\log\big|\gamma(a) - z_0\big| + i\,\mathrm{Arg}\big(\gamma(a) - z_0\big)\Big) =$$

$$= i\Big(\mathrm{Arg}\big(\gamma(b) - z_0\big) - \mathrm{Arg}\big(\gamma(a) - z_0\big)\Big)$$

The ambiguity in defining $\mathrm{Arg}\big(\gamma(t) - z_0\big)$ for $t = a, t = b$ must be an integral multiple of 2π and this integer counts the number of times γ wraps around z_0



We have indeed the following Proposition, which shows that $\omega_\gamma(z)$ is always an integer.

Proposition 3.7. [SS10, Proposition B.1.3] Let γ be a closed curve in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus im(\gamma)$, which is open. Then the map

$$\begin{aligned} w_\gamma : \Omega &\to \mathbb{C} \\ z &\mapsto \frac{1}{2\pi i} \int_\gamma \frac{du}{u-z} \end{aligned}$$

takes values in $\mathbb{Z}$ and is continuous. Hence it is constant on any connected subset of Ω. Moreover $w_\gamma(z) = 0$, if $|z|$ is large enough.

Proof. Suppose $\gamma : [a,b] \to \mathbb{C}$ is a parametrization of the curve and let

$$\begin{aligned} G : [a,b] &\to \mathbb{C} \\ t &\mapsto G(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s)-z} ds \end{aligned}$$

Note that $G(b) = 2\pi i w_\gamma(z)$ and $G(a) = 0$

The Fundamental Theorem of Analysis [EW22] tells us that G is continuous (except possibly at finitely many points) and differentiable on (a,b) (except at those already mentioned points) with

$$G'(t) = \frac{\gamma'(t)}{\gamma(t)-z}$$

Let $H(t) = \big(\gamma(t) - z\big)e^{-G(t)}$ in $\mathbb{C}^{[a,b]}$, then

$$H'(t) = \gamma'(t)e^{-G(t)} - \underbrace{\big(\gamma(t)-z\big)G'(t)}_{=\gamma'(t)} e^{-G(t)} = 0$$

by the latter result. Hence, H is constant by Corollary 1.2 and so $H(t) = \big(\gamma(t) - z\big)e^{-G(t)} = c$ for some $c \in \mathbb{C}$. We have that $\forall t \in [a,b] : \gamma(t) - z = ce^{G(t)}$ and

$$c = c\underbrace{e^{G(a)}}_{=1} = \gamma(a) - z = \gamma(b) - z = ce^{G(b)}$$

From this it follows that $e^{G(b)} = 1$ and so $G(b) \in 2\pi i\mathbb{Z}$ ($c \neq 0$, since $\gamma(t) \neq z$ and $e^{-G(t)} \neq 0$).

Since $G(b) = 2\pi i w_\gamma(z)$, this shows that $w_\gamma(z)$ is integer valued: being

$$w_\gamma(z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)}{\gamma(s)-z} ds$$

the integral a continuous function, it is a continuous function of $z \in \Omega \setminus im(\gamma)$. Being also integer valued, $w_\gamma(z)$ is constant in any open connected subset of $\Omega \setminus im(\gamma)$

Finally, if $M := \max_{t\in[a,b]} \left|\gamma(t)\right|$ (it exists, since $[a,b]$ is compact and γ piecewise of class C^0) and $|z| > M$, then

$$\left|w_\gamma(z)\right| = \frac{1}{2\pi}\left|\int_\gamma \frac{dw}{w-z}\right| \leq \frac{1}{2\pi}\frac{L(\gamma)}{|z|-M}$$

where $L(\gamma)$ denotes the length of γ. Since

$$|w-z| > \big||z|-|w|\big| \geq |z| - M$$

we then have

$$\left|w_\gamma(z)\right| \leq \frac{1}{2\pi}\frac{L(\gamma)}{|z|-M} \xrightarrow{|z|\to\infty} 0$$

Hence $|w_\gamma(z)| < 1$ once $|z|$ is large enough, but being an integer means that $w_\gamma(z) = 0$, if $|z|$ is large enough. □

We can now give the general Residue Formula.

Theorem 3.20 (Generalised Residue Formula)**.** Let $\Omega \subseteq \mathbb{C}$ be open and simply connected, $f \in \mathscr{M}(\Omega)$ and $V = \Omega \setminus S_f$. Let γ be a closed curve in V. Then we have

$$\int_\gamma f(z)dz = 2\pi i \sum_{z\in S_f} w_\gamma(z)\,\mathrm{Res}_z(f)$$

Proof. For any $z_0 \in S_f$, let $P^f_{z_0}$ be the principal part of f at z_0

$$P^f_{z_0}(z) = \sum_{j=1}^{N(z_0)} \frac{a_{-j}(z_0)}{(z-z_0)^j}$$

for some $a_{-j}(z_0) \in \mathbb{C}$ and with $N(z_0) = -\operatorname{ord}_{z_0}(f)$ (since f has a pole at z_0, the order of that point is negative).

Case 1: Let S_f be finite, then

$$\tilde{f} := f - \sum_{z_0\in S_f} P^f_{z_0}$$

has removable singularities at $z_0 \in S_f$, hence has a holomorphic extension to Ω Hence, we have that $\int_\gamma \tilde{f}dz = 0$, as Ω is simply connected.

Hence

$$\int_\gamma f dz = \int_\gamma \left(\sum_{z_0 \in S_f} P^f_{z_0}(z) \right) dz = \sum_{z_0 \in S_f} \int_\gamma P^f_{z_0}(z) dz$$

Recall: $\int_\gamma \frac{dz}{(z-z_0)^j} = 0$ if $j \neq 1$, since $\frac{1}{(z-z_0)^j}$ has primitive $\frac{-1}{(z-z_0)^{j-1}} \frac{1}{j-1}$ and γ is closed.

So

$$\begin{aligned}
\int_\gamma f(z) dz &= \int_\gamma \left(\sum_{z_0 \in S_f} \frac{a_{-1}(z_0)}{z - z_0} \right) dz = \\
&= \sum_{z_0 \in S_f} \int_\gamma \frac{a_{-1}(z_0)}{z - z_0} dz = \\
&= \sum_{z_0 \in S_f} 2\pi i a_{-1}(z_0) \underbrace{\int_\gamma \frac{dz}{z - z_0}}_{= w_\gamma(z_0)} = \\
&= \sum_{z_0 \in S_f} 2\pi i a_{-1}(z_0) w_\gamma(z_0) = \\
&= 2\pi i \sum_{z_0 \in S_f} \text{Res}_{z_0}(f) w_\gamma(z_0)
\end{aligned}$$

Case 2: S_f is infinite. Pick $R > 0$ such that $w_\gamma(z) = 0$ if $|z| \geq R$ and γ is homotopic (so that for the homotopy in question H it hold that $im\Big(H\big([a,b] \times [0,1]\big)\Big) \subseteq D_R(0)$) to the constant cure in $\Omega \cap D_R(0)$ (since Ω is simply connected $\gamma \sim_\Omega \sigma$ (constant curve), which only involves a bounded set). Then $S_f \cap D_R(0)$ is finite, since S_f is a discrete set.

Let

$$\tilde{f} := f|_{\Omega \cap D_R(0)} - \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} P^f_{z_0}\big|_{\Omega \cap D_R(0)} \in \mathscr{H}\big(\Omega \cap D_R(0)\big)$$

we have then that $\int_\gamma \tilde{f} = 0$ since γ is homotopic to the constant curve in $\Omega \cap D_R(0)$

Hence

$$\int_\gamma f dz = \sum_{\substack{z_0 \in S_f \\ |z_0|<R}} \int_\gamma P^f_{z_0} dz =$$
$$= 2\pi i \sum_{\substack{z_0 \in S_f \\ |z_0|<R}} \mathrm{Res}_{z_0}(f) w_\gamma(z_0) =$$
$$= 2\pi i \sum_{z_0 \in S_f} \mathrm{Res}_{z_0}(f) w_\gamma(z_0)$$

Since for $|z_0| \geq R$, $w_\gamma(z_0) = 0$

□

3.8 Cauchy Integral Formula

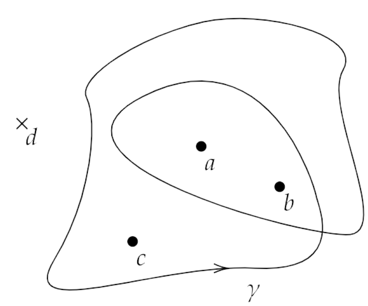
Corollary 3.3. Let Ω be open and simply connected, $f \in \mathscr{H}(\Omega)$ and γ a closed cure in Ω, then

$$\forall z \in \Omega \setminus im(\gamma) : \frac{1}{2\pi i} \int \frac{f(w)}{w - z} dw = f(z) w_\gamma(z)$$

Proof. This is the generalised Residue Theorem 3.20 applied to the function $\frac{f(w)}{w-z} = g(w)$, which is meromorphic in Ω and has a simple pole at $w = z$ and residue $f(z)$ □

Example 3.24. *Let f be meromorphic except for poles at $z = a, b, c, d$, then*

$$\int_\gamma f(z) dz = 2\,\mathrm{Res}_a(f) + 2\,\mathrm{Res}_b(f) + 1\,\mathrm{Res}_c(f)$$



3.9 Conformal maps and the Riemann mapping Theorem

See Chapter 8 in [SS10].

Motivating questions:

1. Given two open sets $U, V \subseteq \mathbb{C}$, when does there exists a holomorphic bijection between them, i.e. when is there a bijective $f \in V^U$ such that $f \in \mathscr{H}(U, V)$?

   We are going to see that the inverse map $f^{-1} \in U^V$ is automatically also holomorphic (compare open sets using holomorphic functions).

2. Given an open set $\Omega \subseteq \mathbb{C}$, what conditions guarantee that there is a holomorphic bijection from Ω to $\mathbb{D}$ (where $\mathbb{D}$ is the unit disc)?

   Why $\mathbb{D}$? $\mathbb{D}$ has a very nice geometric structure and we developed most properties of holomorphic functions for $\mathbb{D}$ first. If there is a holomorphic bijection between Ω and $\mathbb{D}$ we can hope to transfer questions about holomorphic functions on Ω to holomorphic functions on $\mathbb{D}$

Remark 3.19. *1. We'll start by examples of simple maps and show for example that there is a holomorphic bijection between $\mathbb{D}$ and the upper half plane $\mathbb{H}$.*

We can then compose simple maps to get more examples of holomorphic bijections.

2. We will then prove Schwarz's Lemma, which says any $f \in \mathbb{R}^{\mathbb{D}}$ such that $f(0) = 0$ must satisfy

   (a) $\forall z \in \mathbb{D} : \left|f(z)\right| \leq |z|$

   (b) If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then f is a rotation.

   (c) $|f'(0)| \leq 1$ and if equality holds, then f is a rotation.

3. Schwarz's Lemma will then give us all holomorphic bijections of $\mathbb{D}$ to itself.

4. Then we will get to Riemann Mapping Theorem, which says that if $\Omega \neq \mathbb{C}$ or $\Omega \neq \emptyset$ and is simply connected, then there is a holomorphic bijection between Ω and $\mathbb{D}$
More precisely, for any $z_0 \in \Omega$ there exists a unique $f \in \mathbb{D}^{\Omega}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$

Remark 3.20. *The Riemann Mapping Theorem says that there are only three kinds of simply connected domains in $\mathbb{C}$ (up to holomorphic bijections) $\emptyset, \mathbb{C}$ and $\mathbb{D}$*

Note that there can be no holomorphic bijection $f \in \mathbb{D}^{\mathbb{C}}$ between $\mathbb{C}$ and $\mathbb{D}$, since in that case f would be bounded and entire; hence by Liouville's Theorem 2.8 f is constant.

Note that for Ω to be connected is also a necessary condition, since $\mathbb{D}$ is connected. The same is true for simply connected, since if $f \in V^U$ is a holomorphic bijection with U simply connected, then so is V

Definition 3.18. Let $U, V \subseteq \mathbb{C}$ be two open sets in $\mathbb{C}$.

- An injective map $f \in \mathscr{H}(U, V)$ is called a **conformal map from** U **to** V. These are denoted by $Mon_{\mathscr{H}}(U, V)$
- If f is bijective, then it is called a **conformal equivalence** or **biholomorphic**, or a **holomorphic isomorphism** and U and V are said to be **conformally equivalent**. The set of such functions is denoted by $Isom_{\mathscr{H}}(U, V)$
- If $U = V$, a conformal equivalence is called a **(conformal) automorphism** and the the of all such (conformal) automorphisms is denoted by $Aut_{\mathscr{H}}(\mathbb{D})$.

Remark 3.21. *Note that there is a small difference in the definition of conformal compared to the book [SS10]: in the book f is taken to be bijective.*
Also, from now on, unless otherwise specified, we are going to assume that $U, V \subseteq \mathbb{C}$ and that they are open in $\mathbb{C}$

Proposition 3.8. [SS10, Proposition VIII.1.1] If $f \in V^U$ is conformal (i.e. is holomorphic and injective), then

$$\forall z \in U : f'(z) \neq 0$$

The inverse of f, which is defined on the image of f, is holomorphic, i.e. $f \in im(f)^U \subseteq V^U$ is a conformal equivalence and $f^{-1} \in U^{im(f)}$ is also a conformal equivalence.

Proof. Suppose f is injective and holomorphic, but on the contrary $\exists z_0 \in U : f'(z_0) = 0$ We want to show that f cannot be injective. Let $h : U \to V, z \mapsto h(z) := f(z) - f(z_0)$, this implies that $h(z_0) = 0$ and $h'(z_0) = 0$

- If $k = \operatorname{ord}_{z_0}\big(f(z) - f(z_0)\big)$, then by our assumption $k \geq 2$
- If $k = \infty$, then $\forall z \in U : f(z) - f(z_0) = 0$, hence f is constant and cannot be injective.

Therefore, we can assume that $k < \infty$, by Theorem 2.7 we have that $\exists r > 0$ such that for all $z \in D_r(z_0)$

$$
\begin{aligned}
f(z)-f\left(z_0\right)= & \frac{f^{(k)}\left(z_0\right)}{k!}\left(z-z_0\right)^k+G(z)\left(z-z_0\right)^{k+1}= \\
& =a\left(z-z_0\right)^k+G(z)\left(z-z_0\right)^{k+1}
\end{aligned}
$$

where $\frac{f^k\left(z_0\right)}{k!}=: a \neq 0$ and $z \in D_r\left(z_0\right)$. Since the zeroes of $f^{\prime}$ are isolated, we can also choose $r>0$ such that $f^{\prime}(z) \neq 0$ for $z \in \dot{D}_r\left(z_0\right)$

The idea is to use Rouché's Theorem 3.11 to show that for any $w \in \mathbb{C}$

$$
g_w: U \rightarrow V, z \mapsto g_w(z)=f(z)-f\left(z_0\right)-w
$$

has the same number of zeroes as $a\left(z-z_0\right)^k-w$ in some disc around z_0

Since $a\left(z-z_0\right)^k=w$ has k solutions, we will have that $g_w(z)=f(z)-f\left(z_0\right)-w$ has k zeroes for z sufficiently close to z_0. Denote these zeroes of g_w by $z_1, \ldots, z_k$. If $w \neq 0$, then those zeroes are not equal to z_0 (if for some k we had that $z_k=z_0$, then $0=g_w\left(z_0\right)=f\left(z_0\right)-f\left(z_0\right)+w=w \neq 0$).

Since $f^{\prime}(z) \neq 0$ for $z \in \dot{D}_r\left(z_0\right)$, we have that $g_w^{\prime}(z)=f^{\prime}(z) \neq 0$ for $z \in \dot{D}_r\left(z_0\right)$

Hence, each zero has order 1 and they are distinct, but that means that there exist k distinct points $z_1, z_2, \ldots, z_k$ such that $f\left(z_i\right)=f\left(z_0\right)+w$, i.e. f is not injective.

To show that in some neighbourhood of z_0 the function $g_w(z)=f(z)-f\left(z_0\right)-w$ has k zeroes, we write the following expansion for $z \in D_r\left(z_0\right)$

$$
\begin{aligned}
f(z)-f\left(z_0\right)-w & =a\left(z-z_0\right)^k+G(z)\left(z-z_0\right)^{k+1}-w \\
& =\left(a\left(z-z_0\right)^k-w\right)+G(z)\left(z-z_0\right)^{k+1}
\end{aligned}
$$

We apply Rouché's Theorem 3.11 as follows: let $c:=\sup_{\left|z-z_0\right|=\frac{r}{2}}|G(z)|$, c exists since G is continuous. Pick $s \in\left(0, \min \left\{\frac{r}{2}, 1\right\}\right)$ and assume that $|w|<|a|\left(\frac{s}{2}\right)^k$. On $C_s(z_0)$, by inverse triangular inequality one has

$$
\left|a\left(z-z_0\right)^k-w\right| \geq|a| s^k-|a|\left(\frac{s}{2}\right)^k \geq|a|\left(\frac{s}{2}\right)^k
$$

and that

$$
\left|G(z)\left(z-z_0\right)^{k+1}\right| \leq c s^{k+1}
$$

So, if $|a|\left(\frac{s}{2}\right)^k>c s^{k+1}$, i.e. $s<\frac{|a|}{c 2^k}$, then we can apply Rouché's Theorem 3.11 to get that $g(z)=f(z)-f\left(z_0\right)-w$ has the same number of zeroes in $D_s(z_0)$ as $a\left(z-z_0\right)^k-w$, for $|w| < |a| \left(\frac{s}{2}\right)^k$ $\left(\text{and } s < \min\left\{\frac{|a|}{c2^k}, \frac{r}{2}, 1\right\}\right)$ as wanted.

Note that if $w = re^{i\theta}$, then the zeroes of $a\left(z - z_0\right)^k - w$ are at $\{z_n\}_{n=0}^{k-1}$, where $z_n - z_0 = \left|\frac{w}{a}\right|^{\frac{1}{k}} e^{i\left(\frac{\theta + 2\pi n}{k}\right)}$ for $n \in \{0, ..., k-1\}$, but then $|z_n - z_0| = \left(\frac{|w|}{|a|}\right)^{\frac{1}{k}} < \frac{s}{2} < s$. Hence, all k roots of $a\left(z - z_0\right)^k - w$ are inside $D_s(z_0)$

The rest is straightforward: $f \in f(U)^U$ is clearly bijective. Without loss of generality, assume that $f(U) = V$. The inverse function $f^{-1} \in U^V$ is continuous, since $f \in V^U$ is an open map.
Let $w_0 \in V$ and $w \in V$ close enough to w_0. We write $w = f(z)$ and $w_0 = f(z_0)$. If $w \neq w_0$, then we have

$$\frac{f^{-1}(w) - f^{-1}\left(w_0\right)}{w - w_0} = \frac{z - z_0}{f(z) - f\left(z_0\right)} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}$$

Since $f'\left(z_0\right) \neq 0$ and f^{-1} is continuous, we have

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}\left(w_0\right)}{w - w_0} = \lim_{z \to z_0} \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'\left(z_0\right)}$$

Hence $f^{-1} \in \mathscr{H}(V)$ with $V = im(f)$ □

Remark 3.22.

1. *Proposition 3.8 says that if $f \in V^U$ is a conformal equivalence, then $f^{-1} \in U^V$ is automatically a conformal equivalence.*
2. *The conformal equivalence is an equivalence relation:*
    - *$u \sim_c u$, since $id : U \to U, u \mapsto u$ as the identity map is bijective and holomorphic*
    - *If $U \sim_c V$ with $f \in V^U$, then $V \sim_c U$ with $f^{-1} \in U^V$*
    - *If $U \sim_c V$ and $V \sim_c W$ with $f \in V^U$ and $g \in W^V$ respectively, then $g \circ f \in W^U$ gives a conformal equivalence between U and W*
3. *Conformal equivalence allows to transfer the holomorphic functions on one set to the holomorphic functions on the other set.*

Corollary 3.4. If $f \in V^U$ is a conformal equivalence, then the map

$$T : \mathscr{H}(V) \to \mathscr{H}(U)$$
$$\phi \mapsto \phi \circ f$$

where $\phi \in \mathbb{C}^V$ is a holomorphic function on V, is a liner isomorphism of vector spaces with inverse

$$\begin{aligned} T^{-1} : \mathscr{H}(U) &\to \mathscr{H}(V) \\ \varphi &\mapsto \varphi \circ f^{-1} \end{aligned}$$

where $\varphi \in \mathbb{C}^U$ is a holomorphic function on U, i.e. T is an isomorphism of vector spaces, so it holds that

$$T(a\phi_1 + b\phi_2) = aT(\phi_1) + bT(\phi_2)$$

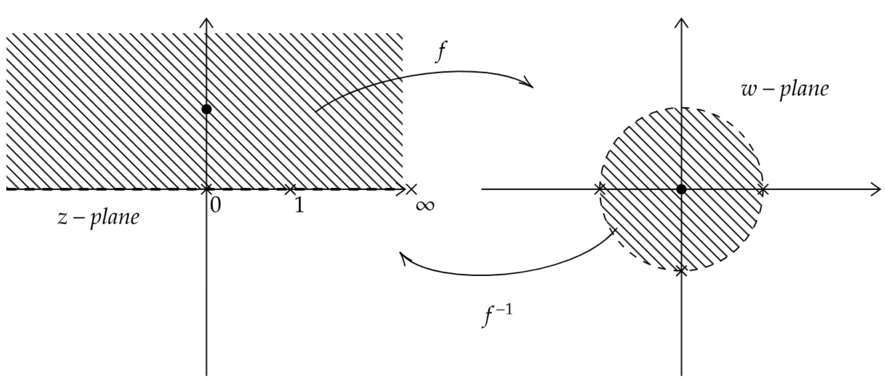
for $a, b \in \mathbb{C}$ and $\phi_1, \phi_2 \in \mathscr{H}(V)$

Example 3.25 (The disc and the upper half plane)**.** *Let* $\mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\}$ *be the upper half plane and* $\mathbb{D} := D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$ *be the unit disc. Then the map*

$$\begin{aligned} f : \mathbb{H} &\to \mathbb{D} \\ z &\mapsto \frac{z - i}{z + i} \end{aligned}$$

is a conformal equivalence, so

$$f^{-1}(w) = i\frac{1 + w}{1 - w}$$



This example shows that the property that a set is bounded is not preserved under conform equivalence.

Proof of the Example 3.25. First note that for any $z \in \mathbb{H}$ it holds:

$$|f(z)| = \left|\frac{z - i}{z + i}\right| < 1$$

since the distance from z to i is shorter than the distance from z to $-i$, which is in the lower half plane.

Moreover, f is clearly holomorphic, since $\forall z \in \mathbb{H} : z + i \neq 0$. Similarly, the map $g : \mathbb{D} \to \mathbb{H}, z \mapsto g(w) := i\frac{1+w}{1-w}$ is holomorphic.

Lastly, to see that $g(w) \in \mathbb{H}$ for any $w \in \mathbb{D}$, we look at

$$\begin{aligned}
im\big(g(w)\big) &= \frac{i\left(\frac{1+w}{1-w}\right) - \overline{\left(i\left(\frac{1+w}{1-w}\right)\right)}}{2i} = \\
&= \frac{1}{2}\left(\frac{1+w}{1-w} + \frac{1+\bar{w}}{1-\bar{w}}\right) = \frac{1}{2}\left(\frac{(1-\bar{w})(1+w)+(1-w)(1+\bar{w})}{|1-w|^2}\right) \\
&= \frac{1-|w|^2}{|1+w|^2} > 0 \quad \text{, since} \quad |w| < 1
\end{aligned}$$

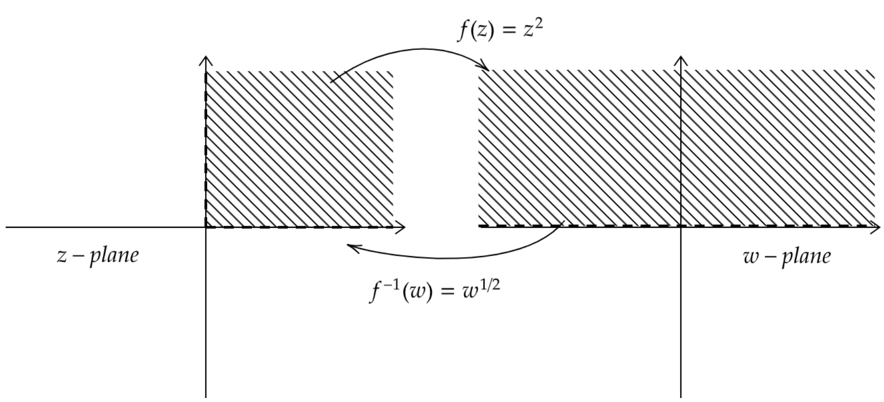
Hence g indeed goes from $\mathbb{D}$ to $\mathbb{H}$. Finally a direct calculation verifies that $f(g(w)) = w$ and $(g \circ f)(z) = z$ and so $g = f^{-1}$ □

Note that the map f from Example 3.25 takes the real line to the boundary of the disc with $f(0) = -1, f(1) = -i$ and $f\,(\text{"}\infty\text{"}) = 1$

Example 3.26 (The map $z \mapsto z^2$)**.** *Let* $U := \left\{z \in \mathbb{C} : \operatorname{Arg}(z) \in \left(0, \frac{\pi}{2}\right)\right\}$*, so*

$$\begin{aligned}
f : U &\to \mathbb{H} \\
z &\mapsto z^2
\end{aligned}$$

maps the first quadrant to $\mathbb{H}$



In that regard, the map

$$\begin{aligned}
g : \mathbb{H} &\to U \\
z &\mapsto z^{\frac{1}{2}} = \exp\left(\frac{1}{2}\operatorname{Log}(z)\right)
\end{aligned}$$

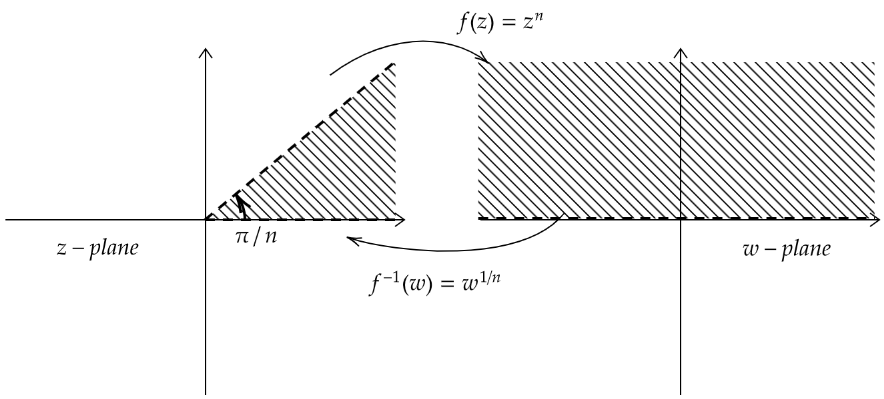
is its inverse.

- *(f is injective) Let $z_1^2 = z_2^2$, then $z_1 = \pm z_2$ and only one of $z_2, -z_2$ can be in U. Since $z_1, z_2 \in U$ we have that $z_1 = z_2$*

- *(f is surjective) Let $w = re^{i\theta}$ with $\theta \in (0, \pi)$, so $w \in \mathbb{H}$, then $z^2 = w$ has 2 solutions, namely*
$$z_{1,2} = \pm w^{\frac{1}{2}} = \pm r^{\frac{1}{2}} e^{\frac{i\theta}{2}}$$
and $z = r^{\frac{1}{2}} e^{\frac{i\theta}{2}}$ is in U

In general, let $n \in \mathbb{N}^*$ and let the sector $S_n = \{z \in \mathbb{C} : \mathrm{Arg}(z) \in \left(0, \frac{\pi}{n}\right)\}$, then the map

$$\begin{aligned} f : S_n &\to \mathbb{H} \\ z &\mapsto z^n \end{aligned}$$



with inverse

$$\begin{aligned} f^{-1} : \mathbb{H} &\to S_n \\ w &\mapsto w^{\frac{1}{n}} = \exp\left(\frac{1}{n}\,\mathrm{Log}(w)\right) \end{aligned}$$

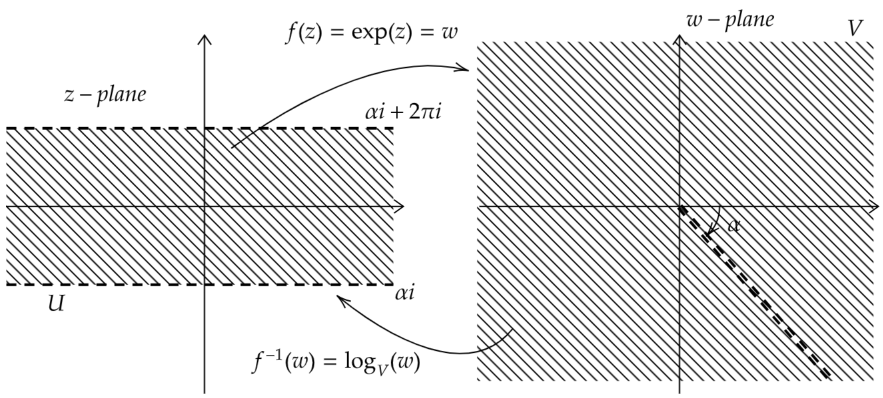
Example 3.27. *Any horizontal strip of length 2π is conformally equivalent to a cut plane (slit plane). The map*

$$\begin{aligned} f : \mathbb{H} &\to \mathbb{C}^- \\ z &\mapsto -z^2 \end{aligned}$$

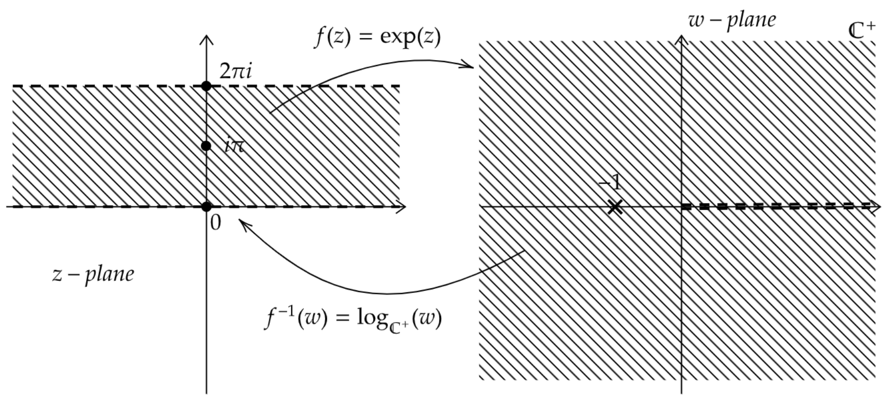
maps $\mathbb{H}$ to $\mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$ and the map

$$\begin{aligned} \tilde{f} : \mathbb{H} &\to \mathbb{C}^+ \\ z &\mapsto z^2 \end{aligned}$$

maps $\mathbb{H}$ to the slit plane cut of the positive reals $\mathbb{C}^+ := \mathbb{C} \setminus [0, +\infty)$.



In particular, for $V = \mathbb{C}^+$



with a fixed branch of logarithm on $\mathbb{C}^+$*, such that* $\log_{\mathbb{C}^+}(-1) = i\pi$

Remark 3.23. *It is to notice that conformal inequalities do not preserve the boundedness property of sets.*

Example 3.28 (Important non-example)**.** *Let* $U := \mathbb{C}$ *and* $V := \mathbb{D}$*, then there is no biholomorphic map between* U *and* V*, since if there were such a map*

$$f : \mathbb{C} \to \mathbb{D}$$

which is holomorphic. Then f *would be bounded, since* $\left|f(z)\right| < 1$*. Hence by Liouville's Theorem 2.8, it is constant, hence is not injective. Hence*

$$\mathbb{C} \not\sim_c \mathbb{D}$$

Riemann's Theorem 3.21 says that any simply connected domain U, which is a proper subset of $\mathbb{C}$, i.e. $U \neq \emptyset$ and $U \neq \mathbb{C}$, is conformally equivalent to $\mathbb{D}$.

This leads to the following important Theorem:

Theorem 3.21 (Riemann mapping Theorem). [SS10, Theorem VIII.3.1] Suppose $\Omega \subset \mathbb{C}$ is proper (i.e. $\emptyset \neq \Omega \neq \mathbb{C}$) and simply connected. Then

$$\Omega \sim_c \mathbb{C}$$

and if $z_0 \in \Omega$, then there is a unique conformal equivalence $F \in \mathscr{H}(\Omega, \mathbb{D})$, such that $F(z_0) = 0$ and $F'(z_0) \in (0, +\infty) \subset \mathbb{C}$

From which follows that

Corollary 3.5. Any two proper simply connected open subsets of $\mathbb{C}$ are conformally equivalent.

Remark 3.24. *Riemann's mapping Theorem is remarkable: it classifies all simply connected open subsets $\Omega \subseteq \mathbb{C}$, up to conformal equivalence. There are three of them, namely $\emptyset, \mathbb{C}$ and $\mathbb{D}$*

The proof is though not constructive, as we will see. In general, it is not easy to find an explicit map. During the rest of the course we will prove this Theorem. The strategy of he proof is as follows:

1. *(Uniqueness) This is going to be easy. It boils down to finding all automorphisms of the unit disc, since if we have two conformal equivalences*

$$f_1 : \Omega \to \mathbb{D}$$
$$f_2 : \Omega \to \mathbb{D}$$

then

$$f_2 \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}$$

is an automorphism of $\mathbb{D}$

2. *If $\emptyset \neq \Omega \neq \mathbb{C}$, we will show that there is a conformal map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$. Hence Ω is conformally equivalent to an open subset of $\mathbb{D}$, hence*

$$\Omega \sim_c f(\Omega) \subseteq \mathbb{D}$$

3. *The second step shows that the set $\mathcal{F} = \left\{ f \in \mathbb{D}^{\Omega} : f \text{ is conformal and } f(z_0) = 0 \right\} \neq \emptyset$. We will see that $s := \sup_{f \in \mathcal{F}} \left| f'(z_0) \right|$ exists and we will show that $\exists f \in \mathcal{F}$ such that $\left| f'(z_0) \right|$ is maximal, i.e. the supremum s is taken. This f has "maximal expansion speed".*

4. *The f we found in the third step is surjective.*
   If this is the case, writing $f'(z_0) = se^{i\theta}$ and $\boxed{g(z) = e^{-i\theta} f}$ gives the map we are looking for, namely $g \in \mathbb{D}^{\Omega}$ with $g(z_0) = e^{-i\theta} f(z_0) = 0$ and $g'(z_0) = s > 0$

Step 1: Automorphism and uniqueness
For the (conformal) automorphisms of $\mathbb{D}$ we have

Theorem 3.22. [SS10, Theorem VIII.2.2] If $f \in Aut_{\mathscr{H}}(\mathbb{D})$ is a (conformal) automorphism of $\mathbb{D}$, then

$$\exists \theta \in \mathbb{R} \exists \alpha \in \mathbb{D} : f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

satisfying

$$f(0) = e^{i\theta}\alpha$$
$$f'(0) = e^{i\theta}\big(|\alpha|^2 - 1\big)$$

Conversely, every map of this form is a (conformal) automorphisms of $\mathbb{D}$

Remark 3.25. 1. *Note that an immediate Corollary of Theorem 3.22 is that the only automorphisms of $\mathbb{D}$ that fix 0 are rotations, since*

$$f(0) = e^{i\theta}\alpha = 0 \Longrightarrow \alpha = 0$$

$$\Longrightarrow f(z) = -e^{i\theta} z = e^{i\tilde{\theta}} z$$

for some $\tilde{\theta} \in \mathbb{R}$

2. *This Theorem 3.22 is enough to prove the uniqueness of conformal equivalence $f \in \mathbb{D}^{\Omega}$*

Proof of the uniqueness of F in Theorem 3.21. If f_1, f_2 are two such maps with $f_1(z_0) = f_2(z_0) = 0$ and $f_1'(z_0), f_2'(z_0) > 0$, then $g := f_2 \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$. Hence, by Theorem 3.22

$$g(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Since $f_1(z_0) = 0, f_2(z_0) = 0$ we have $g(0) = \big(f_2 \circ f_1^{-1}\big)(0) = 0$, so $\alpha = 0$ and $g(z) = -e^{i\theta} z$ for $z \in \mathbb{D}$ and $g'(z) = -e^{i\theta}$. Then

$$-e^{i\theta} = g'(0) = f_2'\left(f_1^{-1}(0)\right) \cdot \left(f_1^{-1}\right)'(0) = f_2'(z_0) \frac{1}{f_1'(z_0)}$$

Hence

$$\frac{f_2'(z_0)}{f_1'(z_0)} = -e^{i\theta} > 0$$

since $f_k'(z_0) > 0$ for $k \in \{1, 2\}$. It follows that $-e^{i\theta} \in \mathbb{R}^{>0}$ and hence that $\theta = \pi + 2\pi k$ as $e^{i\theta} = -1$. Also, $\alpha = 0$, so we can conclude that

$$g(z) = z \Rightarrow f_1 = f_2$$

□

The proof of Theorem 3.21 uses a simple, but important Lemma:

Lemma 3.4 (Schwarz)**.** [SS10, Lemma VIII.2.1] Let $f \in \mathscr{H}(\mathbb{D}, \mathbb{D})$ with $f(0) = 0$. Then

(i) $\forall z \in \mathbb{D} : |f(z)| \leq |z|$

(ii) If for some $z_0 \neq 0$, we have $\left|f(z_0)\right| = |z_0|$, then f is a rotation.

(iii) $\left|f'(0)\right| \leq 1$ and equality holds if and only if f is a rotation, i.e. $\exists \theta \in \mathbb{R} : f(z) = e^{i\theta} z$

Remark 3.26. 1. *(ii) and (iii) give conditions on f, so that up to a rotation, f is the identity map. Since we assume that $f(0) = 0$, the condition in (ii) about the $\left|f(z_0)\right|$ is true ($\left|f(0)\right| = |0|$) for $z_0 = 0$, but we cannot conclude from it that f is a rotation. (iii) is the necessary condition at 0 to conclude that f is a rotation (i.e. $\left|f'(0)\right| = 1$).*

2. *This Lemma 3.4 is once again a statement for holomorphic functions. One cannot conclude for a real differentiable function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ any of (i),(ii),(iii).*

Proof of the Lemma 3.4. It is a consequence of the Maximum Modulus Principle (Theorem 3.13).

(i) The assumption $f(0) = 0$ implies that $\operatorname{ord}_0(f) \geq 1$, so we can define

$$g : \mathbb{C} \to \mathbb{C}, z \mapsto g(z) := \frac{f(z)}{z}$$

for $z \in \mathbb{D}$. Since $\operatorname{ord}_0(f) \geq 1$ and $\operatorname{ord}_0(z) = 1$, in fact g has a removable singularity at $z = 0$, so $g \in \mathscr{H}(\mathbb{D}, \mathbb{D})$

Fix $z \in \mathbb{D}$ and let $r \in \big(|z|, 1\big)$. For $w \in C_r(0)$ we have, since $\left|f(w)\right| < 1$ on $\mathbb{D}$, that

$$|g(z)| \leq \max_{w \in C_r(0)} \left|g(w)\right| = \frac{1}{r} \max_{w \in C_r(0)} \left|f(w)\right| \leq \frac{1}{r}$$

This holds for all $z \in D_r(0)$. By the Maximum Modulus Principle 3.13, we have

$$\forall z \in \overline{D}_r(0) : \left|g(z)\right| \leq \frac{1}{r}$$

(the holomorphic function g cannot attain a maximum in $D_r(0)$)

This is true for all $z \in \mathbb{D}$ such that $|z| < r < 1$, then by letting $r \to 1$ it follows that

$$|g(z)| \leq 1$$

and hence

$$\forall z \in \mathbb{D} : \left|f(z)\right| \leq |z|$$

(ii) We proved that (i) gives $\sup_{z\in\mathbb{D}} \left|g(z)\right| \leq 1$, but the assumption $\left|f\left(z_0\right)\right| = |z_0|$ for some $z_0 \in \mathbb{D} \setminus \{0\}$ implies that g has a local maximum at $z_0 \in \mathbb{D}$. By the Maximum Modulus Principle 3.13 this can only happen if g is constant, hence

$$\exists c \in \mathbb{C} \forall z \in \mathbb{D} : f(z) = zg(z) = cz$$

Since $\left|f\left(z_0\right)\right| = |z_0|$ for that $z_0 \in \mathbb{D} \setminus \{0\}$, it follows that $|c| = 1$. Hence, $c = e^i\theta$ for some $\theta \in \mathbb{R}$ and $f(z) = e^{i\theta}z$

(iii)

$$g(0) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$$

so $\left|f'(0)\right| = \left|g(0)\right| \leq 1$. If $\left|f'(0)\right| = 1$, then again 0 is a local maximum of g and we conclude as in (ii) that $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$, namely a rotation.

□

We can now give the proof of classification of $Aut_{\mathscr{H}}(\mathbb{D})$

Proof of Theorem 3.22. First note that any function $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ of the form

$$z \mapsto \varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

for $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, is an automorphism of $\mathbb{D}$. This since:

1. Since $|\alpha| < 1$, then $1 - \bar{a}z \neq 0$ for $z \in \mathbb{D}$ (as $|z| < 1$), so $\varphi_\alpha \in \mathscr{H}(\mathbb{D}, \mathbb{D})$

2. φ_α is injective:

$$\varphi_\alpha(z) = \varphi_\alpha(w)$$
$$\frac{\alpha - z}{1 - \bar{\alpha}z} = \frac{\alpha - w}{1 - \bar{\alpha}w}$$
$$\alpha - |\alpha|^2 w - z + \bar{\alpha}zw = \alpha - |\alpha|^2 z - w + \bar{\alpha}zw$$
$$\left(1 - |\alpha|^2\right) z = \left(1 - |\alpha|^2\right) w$$
$$z = w$$

Hence φ_α is a conformal map $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$

3. $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$: This point might look superfluous, but checking this condition guarantees that the map is well-defined on its domain of definition, as any argument will have an image in the codomain. So if $|z| = 1$, then $z = e^{i\theta}$ and

$$\varphi_\alpha\left(e^{i\theta}\right) = \frac{\alpha - e^{i\theta}}{e^{i\theta}\left(e^{-i\theta} - \bar{\alpha}\right)} = e^{-i\theta}\left(\frac{\alpha - e^{i\theta}}{e^{-i\theta} - \bar{\alpha}}\right) = e^{-i\theta}\frac{w}{-\bar{w}}$$

with $w := \alpha - e^{i\theta}$. Hence

$$\left|\varphi_\alpha\left(e^{i\theta}\right)\right| = \left|e^{-i\theta}\frac{w}{-\bar{w}}\right| = 1$$

By the Maximum Modulus Principle 3.13, we have that $\forall z \in \mathbb{D} : \left|\varphi_\alpha(z)\right| < 1$ (not being $\varphi_\alpha(z)$ a constant map, it cannot have a local maximum inside of $\mathbb{D}$).

4. We have that

$$\left(\varphi_\alpha \circ \varphi_\alpha\right)(z) = \frac{\alpha - \frac{\alpha - z}{1 - \bar{\alpha}z}}{1 - \bar{\alpha}\left(\frac{\alpha - z}{1 - \bar{\alpha}z}\right)} = \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \bar{\alpha}z - |\alpha|^2 + \bar{\alpha}z} = \frac{\left(1 - |\alpha|^2\right)z}{1 - |\alpha|^2} = z$$

Hence, φ_α is its own inverse.

Clearly any rotation $R : \mathbb{D} \to \mathbb{D}, z \mapsto R(z) = e^{i\theta}z$ is also an automorphism of $\mathbb{D}$

Hence

$$\left(R \circ \varphi_\alpha\right)(z) = e^{i\theta}\frac{\alpha - z}{1 - \bar{\alpha}z}$$

is on automorphism of $\mathbb{D}$

Now, let f be any (conformal) automorphism of $\mathbb{D}$, then $\exists!\alpha \in \mathbb{D} : f(\alpha) = 0$. Consider $g = f \circ \varphi_\alpha$ with $g \in \mathbb{D}^{\mathbb{D}}$, then $g(0) = f(\alpha) = 0$

Schwarz Lemma 3.4 (a) applied to g gives

$$\forall z \in \mathbb{D} : \left|g(z)\right| \leq |z|$$

Since $g^{-1}(0) = 0$, we can also apply the Schwarz Lemma 3.4 to g^{-1} and get

$$\forall w \in \mathbb{D} : \left|g^{-1}(w)\right| \leq |w|$$

Using this for $w = g(z)$ gives

$$\forall z \in \mathbb{D} : |z| = \left|g^{-1}\big(g(z)\big)\right| \leq \left|g(z)\right|$$

Combined with $\big|g(z)\big| \leq |z|$ we get that $|g(z)| = |z|$. Once again by Schwarz Lemma 3.4 (b), $g(z) = e^{i\theta}z$ is a rotation with some $\theta \in \mathbb{R}$. Hence $e^{i\theta}z = (f \circ \varphi_\alpha)(z) = g(z)$

Replacing z with $\varphi_\alpha(z)$ now gives

$$\begin{aligned} e^{i\theta}\varphi_\alpha(z) = g\big(\varphi_\alpha(z)\big) = (f \circ \varphi_\alpha)\left(\varphi_\alpha(z)\right) = \\ = (f \circ \varphi_\alpha \circ \varphi_\alpha)(z) = f\Big((\varphi_\alpha \circ \varphi_\alpha)(z)\Big) = f(z) \end{aligned}$$

using the fact that $\varphi_\alpha \circ \varphi_\alpha = id$

□

Remark 3.27. *Combining automorphisms of* $\mathbb{D}$ *together with the* ***Cayley map***

$$\begin{aligned} F : \mathbb{H} &\to \mathbb{D} \\ z &\mapsto \frac{z-i}{z+i} \end{aligned}$$

allows one to find all automorphisms of $\mathbb{H}$ *in the following manner.*

Theorem 3.23. [SS10, Theorem VIII.2.4] Every automorphism $g \in Aut_{\mathscr{H}}(\mathbb{H})$ is of the form

$$g(z) = \frac{az+b}{cz+d}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ such that $ad - bc > 0$

Remark 3.28. *It is of interest that one can see that* $Aut_{\mathscr{H}}(\mathbb{H})$ *can be described via the action of* $\mathrm{SL}_2(R) \curvearrowright \mathbb{H}$ *via fractional linear transformations.*

Remark 3.29. *Being these maps invariant under re-scaling by a real factor, we discover that these (conformal) automorphism are in fact represented in* $\mathrm{GL}_2(\mathbb{R})$ *quotient by* $\mathbb{R}^*$*, namely the* ***projective general linear group*** $\mathrm{PGL}_2(\mathbb{R})$*, but only considering those elements with positive determinant, hence a subgroup of it. These elements form* $\mathrm{PSL}_2(\mathbb{R})$*, the* ***projective special linear group****, this being the quotient of* $\mathrm{SL}_2(\mathbb{R})$ *and* $\{\pm 1\}$*, and can be seen as a subgroup of the former (of the orientation-preserving transformations).*
Over $\mathbb{R}$ *these two groups are different, in particular the second one being a strict subgroup of the first one, but for instance over* $\mathbb{C}$ *surprisingly*

$$\mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$$

This is going to have a greater relevance later in geometry, in particular in the context of ***Möbius Transformations****.*

Proof. Exercise: read in the book. □

Note that using the map

$$\gamma : Aut(\mathbb{D}) \to Aut(\mathbb{H})$$
$$\varphi \mapsto \gamma(\varphi) = F^{-1} \circ \varphi \circ F$$

any automorphism of $\mathbb{D}$ is lead to an automorphism of $\mathbb{H}$. Moreover, γ is an isomorphism with inverse

$$\gamma^{-1} : Aut(\mathbb{H}) \to Aut(\mathbb{D})$$
$$\beta \mapsto \gamma^{-1}(\beta) = F \circ \beta \circ F^{-1}$$

Using γ we can pull the automorphisms of $\mathbb{D}$ to automorphisms of $\mathbb{H}$ and show that they are of the above form.
Now, we move to step two in the proof of the Riemann mapping Theorem 3.21.

Step 2: There is a conformal map $f \in \mathbb{D}^{\Omega}$
(i.e. if Ω is a proper, simply connected of $\mathbb{C}$, then it is conformally equivalent to a subset of $\mathbb{D}$)

We have the following

Proposition 3.9. [SS10, Step 1 in Section VIII.3.3, p.228] Let $\Omega \subset \mathbb{C}$ such that $\emptyset \neq \Omega \neq \mathbb{C}$, open and simply connected. Then there exists a conformal map $f \in \mathbb{D}^{\Omega}$ such that $0 \in f(\Omega)$, i.e. Ω is conformally equivalent to a subset of $\mathbb{D}$, which contains the origin.

Proof. Without loss of generality we assume that $\Omega \subset \mathbb{C}^*$, hence that $0 \notin \Omega$. In fact, being Ω proper, $\exists \alpha \in \mathbb{C} : \alpha \notin \Omega$, by replacing Ω with $\Omega - \alpha := \{z - \alpha : z \in \Omega\}$, we can assume that $\alpha = 0 \notin \Omega$. Hence $\Omega \subset \mathbb{C}^*$; furthermore, since Ω is simply connected, there exists a branch of logarithm $\log_{\Omega} \in \mathscr{H}(\Omega)$

Note that $\log_{\Omega}$ is also injective, since if

$$\log_{\Omega}(z) = \log_{\Omega}(w)$$

then exponentiating both sides, we get that

$$z = \exp\big(\log_{\Omega}(z)\big) = \exp\big(\log_{\Omega}(w)\big) = w$$

and hence $\log_{\Omega}$ is a conformal map.

Now let $w \in \Omega$, then note that for any $z \in \Omega$

$$\log_\Omega(z) \neq \log_\Omega(w) + 2\pi i$$

otherwise, exponentiating we would get

$$z = \exp\big(\log_\Omega(z)\big) = \exp\big(\log_\Omega(w)\big)\exp(2\pi i) = w$$

Hence $z = w$, but then $\log_\Omega(z) = \log_\Omega(w)$ results in a contradiction. In fact, $\log_\Omega(z)$ stays away from $\log_\Omega(w) + 2\pi i$ in the sense that

$$\exists \delta > 0 : D_{2\delta}\big(\log(w) + 2\pi i\big) \cap \log_\Omega(\Omega) = \emptyset$$

Indeed otherwise, if for all $n \in \mathbb{N}^*$, say with $\delta_n = \frac{1}{n}$, we got a sequence $(z_n)_{n\in\mathbb{N}^*} \in \Omega^{\mathbb{N}^*}$ such that

$$\left|\log_\Omega(z_n) - \big(\log_\Omega(w) + 2\pi i\big)\right| < \frac{1}{n}$$

Hence

$$\log_\Omega(z_n) \xrightarrow{n\to\infty} \log_\Omega(w) + 2\pi i$$

and exponentiating and using the fact that exp is continuous, we would get that $z_n \xrightarrow{n\to\infty} w$ and hence $\log_\Omega(z_n) \xrightarrow{n\to\infty} \log_\Omega(w)$, using the continuity of $\log_\Omega$, which is a contradiction to $\log_\Omega(z_n) \xrightarrow{n\to\infty} \log_\Omega(w) + 2\pi i$

Now, we can consider the map (for the same w as before)

$$\begin{aligned} F : \Omega &\to \mathbb{C} \\ z &\mapsto \frac{1}{\log_\Omega(z) - \big(\log_\Omega(w) + 2\pi i\big)} \end{aligned}$$

Note that $F \in \mathscr{H}(\Omega)$, since

$$\forall z \in \Omega : \log_\Omega(z) \neq \log_\Omega(w) + 2\pi i$$

Since $\log_\Omega$ is injective, so is F and hence F is a conformal map. Furthermore, the above estimate gives

$$\forall z \in \Omega : \left|\log_\Omega(z) - \big(\log_\Omega(w) + 2\pi i\big)\right| \geq 2\delta$$

Hence

$$\forall z \in \Omega : \big|F(z) - 0\big| = \left|\frac{1}{\log_\Omega(z) - (\log_\Omega(w) + 2\pi i)}\right| \leq \frac{1}{2\delta} < \frac{1}{\delta}$$

and so $F(\Omega) \subset D_{\frac{1}{\delta}}(0)$. We can now translate and rescale F to obtain a function $f \in \mathbb{D}^\Omega$ which contains the origin in its image.
Let $f(z) := \frac{\delta}{4}\big(F(z) - F(w)\big)$, then $f \in \mathbb{C}^\Omega$ is conformal, as we have $f(w) = 0$ and

$$\forall z \in \Omega : |f(z)| \leq \frac{\delta}{4}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) \leq \frac{1}{2}$$

Therefore $f(\Omega) \subset \mathbb{D}$ for all $z \in \Omega$ and since $f(w) = 0$, i.e. $0 \in f(\Omega)$ $\square$

Step 3: An extremal problem

Let Ω be a proper, non-empty, simply connected subset of $\mathbb{C}$ and $z_0 \in \Omega$. By Step 2 we have at least one $f \in \mathbb{D}^\Omega$ such that $f(z_0) = 0$. Let

$$\mathcal{F} := \left\{ f \in \mathbb{D}^\Omega : f \text{ is conformal and } f(z_0) = 0 \right\}$$

Then $\mathcal{F} \neq \emptyset$. We start by the following (see Step 2 in Section 3.3 in Chapter 8 in [SS10]).

Lemma 3.5. The set of values $\left\{ \left| f'(z_0) \right| \in \mathbb{R}^{\geq 0} : f \in \mathcal{F} \right\}$ is bounded in $[0, +\infty)$. Therefore, it exists

$$s := \sup_{f \in \mathcal{F}} \left| f'(z_0) \right| < +\infty$$

Proof. Let $\delta > 0$, such that $\overline{D}_{2\delta}(z_0) \subseteq \Omega$ and let $f \in \mathcal{F}$. The Cauchy Integral Formula 2.6 gives

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_\delta(z_0)} \frac{f(z)}{(z - z_0)^2} dz$$

Hence, using the standard estimate one finds that

$$\left| f'(z_0) \right| \leq \frac{1}{2\pi} 2\pi\delta \max_{z \in C_\delta} \frac{\left| f(z) \right|}{\delta^2} \leq \frac{1}{\delta}$$

since $\left| f(z) \right| \leq 1$ for all $z \in \Omega$. Hence $\left| f'(z_0) \right|$ is bounded by $\frac{1}{\delta}$ for an arbitrary $f \in \mathcal{F}$, thus there exists an upper bound and by consequence of it also the supremum. □

The next Proposition is key and states that the supremum

$$s = \sup_{f \in \mathcal{F}} \left| f'(z_0) \right|$$

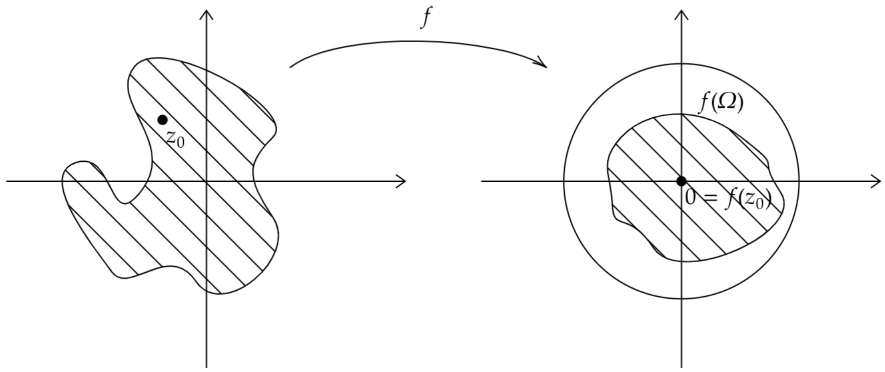
is taken.

Proposition 3.10. $\exists f \in \mathcal{F} : \left| f'(z_0) \right| = s$, hence $s = \max_{f \in \mathcal{F}} \left| f'(z_0) \right|$

The proof of this Proposition 3.10 uses a compactness argument which we will come back to, but we first see why this is key, in the sense that it gives the conformal equivalence that we are looking for, between Ω and $\mathbb{D}$ (see Step 3 in Section 3.3 in [SS10], p.231).

Remark 3.30. *The step 2 shows that Ω is conformally equivalent to an open subset of $\mathbb{D}$, which contains 0*

Why are we looking for an extremal function that realises the extremal value $s = \sup_{f \in \mathcal{F}} \left| f'(z_0) \right|$?



We can assure without loss of generality that Ω *is an open subset of* $\mathbb{D}$ *that contains* 0, *so we can assume that* $z_0 = 0$

We want to conformally stretch Ω *to fill* $\mathbb{D}$

$$\mathcal{F} = \left\{ f \in \mathbb{D}^{\Omega} : f \in \mathscr{H}(\Omega) \text{ is injective and } f(0) = 0 \right\}$$

We want to choose a function in $\mathcal{F}$ *with "maximal expansion", but what does "expanding" mean? Consider*

$$f(0) = 0 \Longrightarrow f(z) \sim f'(0)z$$

for z *near* 0, *so if* $\left|f'(0)\right| > 1$, *we say that* f *is* ***expanding***, *since the distances between nearby points are expanding*

$$\left|f(z_1) - f(z_2)\right| \approx \left|f'(0)\right|\left|z_1 - z_2\right| > |z_1 - z_2|$$

Step 4: f from the key Proposition in Step 3 is surjective

Proposition 3.11. Let $f \in \mathcal{F}$ be such that $\left|f'(z_0)\right| = s$, then $f \in \mathbb{D}^{\Omega}$ is a conformal equivalence (i.e. f is also onto $\mathbb{D}$).

Proof. We want to show that f is a surjection: we assume that it is not, then $\exists \alpha \in \mathbb{D}$. which is not in $f(\Omega)$. We are going to construct $g \in \mathcal{F}$ with $\left|g'(z_0)\right| > \left|f'(z_0)\right|$ which is going to be a contradiction to

$$\left|f'(z_0)\right| = s = \sup_{g \in \mathcal{F}} \left|g'(z_0)\right|$$

To do this we are going to use φ_α and the square root map. Let

$$\begin{aligned}\varphi = \varphi_\alpha : \mathbb{D} &\to \mathbb{D}\\ z &\mapsto \frac{\alpha - z}{1 - \bar{\alpha} z}\end{aligned}$$

be the automorphism of $\mathbb{D}$, with $\varphi_\alpha(0) = \alpha$ and $\varphi_\alpha(\alpha) = 0$

Then $\varphi_\alpha \circ f : \Omega \to \mathbb{D}$ is conformal and $0 \notin (\varphi_\alpha \circ f)(\Omega)$, since if for some $z \in \Omega$ held $\big(\varphi_\alpha \circ (f(z))\big) = 0$, then we would have $f(z) = \alpha$, which we assured is not the case.

Since $0 \notin (\varphi \circ f)(\Omega)$, and Ω is simply connected, a logarithm and a square root of of $\varphi \circ f$ exist, i.e.

$$\exists \tilde{f} \in \mathscr{H}(\Omega) \forall z \in \Omega : \tilde{f}^2(z) = (\varphi \circ f)(z)$$

One can simply take $\tilde{g}$ as primitive of $\frac{(\varphi_\alpha \circ f)'}{(\varphi_\alpha \circ f)}$, so that $\exp\big(\tilde{g}(z)\big) = (\varphi_\alpha \circ f)(z)$

Note that $\tilde{f}$ is also injective: if $\tilde{f}(z) = \tilde{f}(w)$, then $(\varphi \circ f)(z) = (\varphi \circ f)(w)$. Since $\varphi \circ f$ is conformal and f, φ are injective, we have that $z = w$

Now, $\tilde{f}$ is not yet the function we want, since $\tilde{f}(z_0) \neq 0$, as $\varphi\big(f(z_0)\big) \neq 0$, due to the fact that $0 \notin (\varphi \circ f)(\Omega)$

Let $\tilde{f}(z_0) = \beta$ and consider the (conformal) automorphism of $\mathbb{D}$

$$\begin{aligned}\varphi_\beta : \mathbb{D} &\to \mathbb{D}\\ z &\mapsto \frac{\beta - z}{1 - \bar{\beta} z}\end{aligned}$$

with $\varphi_\beta(\beta) = 0$. Finally, let $g(z) := \varphi_\beta \circ \tilde{f} \in \mathbb{D}^\Omega$. Then $g(z_0) = 0$ and $g \in \mathscr{H}(\Omega, \mathbb{D})$, since $\tilde{f} \in \mathscr{H}(\Omega, \mathbb{D})$ and $\varphi_\beta \in Aut_{\mathscr{H}}(\mathbb{D})$
Moreover, g is injective, since φ_β and $\tilde{f}$ are injective, also $g \in \mathcal{F}$ by definition.

Claim. $\left|g'(z_0)\right| > \left|f'(z_0)\right|$

This will give the contradiction that we are looking for.

Proof of the Claim. **Recall**: we first looked at $\varphi_\alpha \circ f$ as

$$\varphi_\alpha \circ f : \Omega \xrightarrow{f} \mathbb{D} \xrightarrow{\varphi_\alpha} \mathbb{D}$$

Then we took the $\sqrt{\cdot}$ function, call it h

$$\begin{aligned}h : (\varphi_\alpha \circ f)(\Omega) &\to \mathbb{D}\\ w &\mapsto \exp\left(\frac{1}{2} w\right)\end{aligned}$$

and composed it with $\varphi_\alpha \circ f$ as follows

$$\tilde{f} := h \circ \varphi_\alpha \circ f : \Omega \to \mathbb{D}$$

so that $\tilde{f}^2 = \varphi_\alpha \circ f$
Then we composed it with φ_β, to get $g \in \mathbb{D}^\Omega$ such that

$$g := \varphi_\beta \circ \tilde{f} = \varphi_\beta \circ \underbrace{h \circ \varphi_\alpha \circ f}_{\tilde{f}}$$

Being these conformal equivalences, we have that $\varphi_\beta^{-1} \circ g = \tilde{f}$

$$\begin{aligned} &\Rightarrow \left(\varphi_\beta^{-1} \circ g\right)^2 = \varphi_\alpha \circ f \\ &\Rightarrow \varphi_\alpha^{-1} \circ \left(\varphi_\beta^{-1} \circ g\right)^2 = f \end{aligned}$$

Let $s : \mathbb{D} \to \mathbb{D}, z \mapsto s(z) = z^2$ be the squaring map. Then define Φ such that

$$f = \underbrace{\varphi_\alpha^{-1} \circ s \circ \varphi_\beta^{-1}}_{=:\Phi} \circ g = \Phi \circ g$$

Note that Φ is not injective. Now $\Phi \in \mathscr{H}(\mathbb{D}, \mathbb{D})$ by composition and since $\varphi_\beta^{-1}(0) = \beta$

$$\Phi(0) = \left(\varphi_\alpha^{-1} \circ s \circ \varphi_\beta^{-1}\right)(0) = \varphi_\alpha^{-1}\left(\beta^2\right)$$

But recall that $\tilde{f}(z_0) = \beta$, it follows that $\beta^2 = \left(\tilde{f}(z_0)\right)^2 = (\varphi_\alpha \circ f)(z_0)$. Hence

$$\Phi(0) = \left(\varphi_\alpha^{-1} \circ \varphi_\alpha \circ f\right)(z_0) = f(z_0) = 0$$

Hence, we can apply Schwarz Lemma 3.4 (iii) to get $\left|\Phi'(0)\right| < 1$ (note that $|\Phi'(0)| \neq 1$, since if it were so, then $\Phi(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$ and it would mean that Φ is injective, but Φ cannot be such, since the squaring function is not injective). Using the chain rule applied to $f = \Phi \circ g$ we have

$$\begin{aligned} f'(z_0) &= \Phi'\big(g(z_0)\big) \cdot g'(z_0) = \Phi'(0) \cdot g'(z_0) \\ \left|f'(z_0)\right| &= \left|\Phi'\big(g(z_0)\big)\right| \cdot \left|g'(z_0)\right| = \left|\Phi'(0)\right| \cdot \left|g'(z_0)\right| < \left|g'(z_0)\right| \end{aligned}$$

Hence $\left|f'(z_0)\right| < \left|g'(z_0)\right|$, which is a contradiction. □

This concludes the proof. □

Proof of Proposition 3.10 in Step 3

1. **Existence of the maximum**: We want to prove the existence of $f \in \mathcal{F}$, such that $\left|f'(z_0)\right| = s = \sup\{g'(z_0) : g \in F\}$

   Recall: For any bounded subset $u \subseteq \mathbb{R}$ there is a non-decreasing sequence $(a_n)_{n\in\mathbb{N}^*} \in U^{\mathbb{N}^*}$ such that $\lim_{n\to\infty} a_n = \sup(U)$

   Recalling the definition of supremum, we take a sequence $(f_n)_{n\in\mathbb{N}^*} \in (\mathcal{F})^{\mathbb{N}^*}$ with $\left|f_n'(z_0)\right| \xrightarrow{n\to\infty} s$. We want to show that this sequence has a limit f in $\mathcal{F}$

   Note that the proof will not be constructive and will only guarantee the existence of a limit $f \in \mathcal{F}$

2. **Recall**: We have seen that a sequence of holomorphic functions that converge uniformly on compact sets has a holomorphic limit (Theorem 2.14), but we cannot expect that an arbitrary sequence $(f_n)_{n\in\mathbb{N}^*}$ to be uniformly convergent on compact sets. May it be that a subsequence has this property?

   Recall: In the finite dimensional vector space $\mathbb{R}^n$ with $n \in \mathbb{N}$, every bounded sequence has a convergent subsequence.

   So we are looking for an analogue of this for $\mathcal{F}$. This is provided in the following by Montel's Theorem.

   Theorem 3.24 (Montel's Theorem)**.** [SS10, Theorem VIII.3.3] Let $\Omega \subseteq \mathbb{C}$ open and $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathscr{H}(\Omega)\right)^{\mathbb{N}^*}$. Suppose that

   $$\forall K \subseteq \Omega \text{ compact } \exists M_K \geq 0 \forall n \in \mathbb{N}^* \forall z \in K : \left|f_n(z)\right| \leq M_K$$

   Then $\exists (f_{n_k})_{k\in\mathbb{N}^*} \in \left(\mathscr{H}(\Omega)\right)^{\mathbb{N}^*}$, a subsequence of $(f_n)_{n\in\mathbb{N}^*}$, which converges uniformly on compact subsets of Ω.

3. In application to Riemann's Theorem 3.21 we have a sequence $(f_n)_{n\in\mathbb{N}^*} \in \mathcal{F}^{\mathbb{N}^*}$, so that $\forall z \in \Omega \forall n \in \mathbb{N}^* : |f_n(z)| \leq 1$ (not only compact sets, as $\mathbb{D}$ is bounded).

   Hence we can apply Montel's Theorem 3.24 to find a sequence $(f_{n_k})_{k\in\mathbb{N}^*} \in \mathcal{F}^{\mathbb{N}^*}$ which converges uniformly on compact sets and this will give the $\lim_{k\to\infty} f_{n_k} = f$

(holomorphic) that we are looking for, provided that we can show that $f \in \mathcal{F}$

We now need to argue for the injectivity of such f, to this purpose we use the following result:

Proposition 3.12. Let $(f_n)_{n\in\mathbb{N}^*} \in \mathcal{F}^{\mathbb{N}^*}$ be a sequence in $\mathcal{F}$ and suppose that $f_n \to f$ uniformly on any compact set $K \subseteq \Omega$. Then, either f is constant or $f \in \mathcal{F}$. Moreover, for any $z \in \Omega$

$$\lim_{n\to\infty} f_n'(z) = f'(z)$$

Proof. Clearly, if $f_n \to f$ uniformly on compact sets, then $f \in \mathscr{H}(\Omega)$ and $\lim_{n\to\infty} f_n'(z) = f'(z)$ for $z \in \Omega$, by Theorem 2.15.

We need to show that $f(\Omega) \subseteq \mathbb{D}$ and that f is injective or constant.

Since for $z \in \Omega$ we have $\left|f_n(z)\right| < 1$, we deduce that $\left|f(z)\right| \le 1$; if $|f(z)| = 1$ for some $z \in \Omega$, then z would be a local maximum of $|f|$, which is impossible by the Maximum Modulus Principle 3.13, unless f is constant. Otherwise indeed $f(\Omega) \subseteq \mathbb{D}$

What is left to show is that f is injective or constant. For this we have

Lemma 3.6. [SS10, Lemma VIII.3.5] Let $\Omega \subseteq \mathbb{C}$ open and connected. $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{D}^\Omega\right)^{\mathbb{N}^*}$ conformal. If $f_n \to f \in \mathbb{C}^\Omega$ uniformly on compact sets, then f is either injective or constant.

Proof. We will suppose that f is not injective and show that then f is constant. Suppose that for $z_1 \ne z_2 \in \Omega : f(z_1) = f(z_2)$. If f is not constant, since the zeroes of holomorphic functions are isolated, we can find a disc $D_\delta(z_2) \subseteq \Omega$, so that $f(z) - f(z_2) \ne 0$ in $\dot{D}_\delta(z_2)$. Hence, in particular

$$\forall z \in C_{\frac{\delta}{2}}(z_2) : f(z) - f(z_2) \ne 0$$

Note that this also says that $z_1 \notin C_{\frac{\delta}{2}}(z_2)$, since we assumed that $f(z_1) = f(z_2)$

We apply the Argument Principle 3.10 to the function $f(z) - f(z_1)$ which has a zero, namely z_2, in $D_{\frac{\delta}{2}}(z_2)$ to get

$$\frac{1}{2\pi i}\int_{C_{\frac{\delta}{2}}(z_2)} \frac{f'(z)}{f(z) - f(z_1)} dz \ge 1$$

We have $f_n \to f$ uniformly on compact sets, hence also on $C_{\frac{\delta}{2}}(z_2)$, in particular $f_n(z) \neq f_n(z_1)$ for all $n \in \mathbb{N}^*$ and $z \in C_{\frac{\delta}{2}}(z_2)$, since the f_n 's are injective and $z_1 \notin C_{\frac{\delta}{2}}(z_2)$. Hence

$$\frac{f_n'(z)}{f_n(z) - f_n(z_1)} \xrightarrow{n\to\infty} \frac{f'(z)}{f(z) - f(z_1)}$$

uniformly on $C_{\frac{\delta}{2}}(z_2)$. Therefore we get

$$\frac{1}{2\pi i}\int_{C_{\frac{\delta}{2}}(z_2)} \frac{f'(z)}{f(z)-f(z_1)}dz = \lim_{n\to\infty} \underbrace{\frac{1}{2\pi i}\int_{C_{\frac{\delta}{2}}(z_2)} \frac{f_n'(z)}{f_n(z)-f_n(z_1)}dz}_{=0 \text{ , for all } n\in\mathbb{N}^*}$$

since the integrals on the right counts the zeroes of the holomorphic function $f_n(z) - f_n(z_1)$ in $C_{\frac{\delta}{2}}(z_2)$, so none by the injectivity of f_n, but the integer on the left is ≥ 1, which is a contradiction. Hence, f must be a constant. □

□

Remark 3.31. *Finally note that in the case $(f_n)_{n\in\mathbb{N}^*} \in \mathcal{F}^{\mathbb{N}^*}$ with $\lim_{n\to\infty} \left|f_n'(z_0)\right| = s$ and $\lim_{n\to\infty} \left|f_n'(z_0)\right| = \left|f(z_0)\right|$ we have that*

1. *$\forall n \in \mathbb{N}^* : f_n'(z_0) \neq 0$, using Proposition 3.8 and that f_n 's are conformal (hence $\forall z \in \Omega : f_n'(z) \neq 0$).*
2. *By the definition of supremum,there exists a non-decreasing sequence*

$$0 < \left|f_1'(z_0)\right| \leq ... \leq \left|f_n'(z_0)\right| \leq ...$$

such that $\lim_{n\to\infty} \left|f_n'(z_0)\right| = s > 0$. Hence $f'(z_0) \neq 0$ and f is not constant.

Proof of Montel's Theorem 3.24

Montel's Theorem 3.24 actually consists of 2 parts:

1. The first part is about the complex behaviour of sequences of holomorphic functions, which says
   (a) A sequence of holomorphic functions which is uniformly bounded on compact sets $K \subseteq \Omega$, i.e.

$$\forall K \subseteq \Omega \text{ compact } \exists M_k > 0 \forall z \in K \forall n \in \mathbb{N}^* : |f_n(z)| \leq M_k$$

is equicontinuous on compact sets.

(b) A sequence $(f_n)_{n\in\mathbb{N}^*} \in \left(\mathbb{C}^\Omega\right)^{\mathbb{N}^*}$ is **equicontinuous on a compact set** $K \subseteq \Omega$, if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z, w \in K : |z-w| < \delta \Longrightarrow \forall n \in \mathbb{N}^* : \left|f_n(z) - f_n(w)\right| < \varepsilon$$

This is a complex behaviour in the sense that it is not true for sequences of real functions. For example, $f_n(x) = \sin(nx)$ on $(0,1)$ is uniformly bounded on compact sets, but not equicontinuous.

Equicontinuity is a very strong condition and requires uniform continuity uniformly in the family.

The family $(f_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^{[0,1]}$ on $[0,1]$, given by $f_n(x) = x^n$, is not equicontinuous, even though each

$$\begin{aligned} f_n : [0,1] &\to \mathbb{R} \\ x &\mapsto x^n \end{aligned}$$

is uniformly continuous on $[0,1]$. The family $(f_n)_{n\in\mathbb{N}^*}$ is not equicontinuous. For example take any $w \in (0,1)$, then $\left|f_n(1) - f_n(w)\right| \xrightarrow{n\to\infty} 1$

2. The second part is known as Arzelà-Ascoli Theorem [EW22], which says that any family $\mathcal{F}$ of functions, which is uniformly bounded and equicontinuous on compact subsets of Ω, has a subsequence which converges uniformly on every compact subset of Ω (the limit need not be in $\mathcal{F}$).

   This part belongs to Topology/Functional Analysis, hence will be assumed without proof.

Theorem 3.25 (Arzelà-Ascoli)**.** [EW22] Let $K \subseteq \mathbb{R}^n$ be compact and $(f_n)_{n\in\mathbb{N}^*} \in \left(C^0(K;\mathbb{R}^m)\right)^{\mathbb{N}^*}$ a sequence of continuous functions on K. Suppose that

1. $\exists x_0 \in K \exists M > 0 \forall n \in \mathbb{N}^* : \left|f_n(x_0)\right| \leq M$ (i.e. $\left(\left|f_n(x_0)\right|\right)_{n\in\mathbb{N}^*}$ is bounded in $\mathbb{R}^m$).
2. $(f_n)_{n\in\mathbb{N}^*}$ is equicontinuous.

Then $\exists (f_{n_k})_{k\in\mathbb{N}^*} \in \left(C^0(K;\mathbb{R}^m)\right)^{\mathbb{N}^*}$, which converges uniformly on K to some continuous function $f \in C^0(K;\mathbb{R}^m)$

Assuming the Arzelà-Ascoli Theorem 3.25, the proof of Montel's Theorem 3.24 reduces to proving that every sequence of holomorphic functions, which is uniformly bounded on compact sets, is equicontinuous on compact sets.

This uses Cauchy's Integral Formula 2.6 together with the following Lemma:

Lemma 3.7. [SS10, Lemma VIII.3.4] Let $\Omega \subseteq \mathbb{C}$ be open. Then $\exists$ compact sets $\{K_l\}_{l\in\mathbb{N}^*}$ such that

1. $\forall l \in \mathbb{N}^* : K_l \subseteq \operatorname{int}(K_{l+1})$
2. Any compact set $K \subseteq \Omega$ is contained in K_l for some $l \in \mathbb{N}^*$. In particular

$$\Omega = \bigcup_{l=1}^{\infty} K_l$$

(Such a sequence $\{K_e\}_{l=1}^{\infty}$ of compact subsets of Ω is called an **exhaustion**)

Proof. Exercise. □

Now, assuming the Arzelà-Ascoli Theorem 3.25 and Lemma 3.7 we can give the proof of Montel's Theorem 3.24.

Proof of Theorem 3.24. Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence of holomorphic functions, which are uniformly bounded on compact sets. We want to show that $(f_n)_{n\in\mathbb{N}^*}$ is equicontinuous.

1. Let $K \subseteq \Omega$ be compact, let $r > 0$ such that $D_{3r}(z) \subseteq \Omega$ for $z \in K$ (we can choose r so that $3r$ is less than the distance from K to the boundary of Ω, i.e. $3r < d(K, \partial\Omega)$).

   Let $z, w \in K$ with $|z - w| < r$. The Cauchy Integral Formula 2.6 gives

   $$f_n(z) - f_n(w) = \frac{1}{2\pi i}\int_\gamma f_n(\xi)\left(\frac{1}{(\xi - z)} - \frac{1}{(g - w)}\right) d\xi$$

   where $im(\gamma) = C_{2r}(w)$. On $C_{2r}(w)$ we have

   $$\left|\frac{1}{\xi - z} - \frac{1}{\xi - w}\right| = \frac{|z - w|}{|\xi - w||\xi - z|} \leq \frac{|z - w|}{2r \cdot r}$$

   since $|\xi - z| \geq |\xi - w| - |w - z| \geq 2r - r = r$ and $|\xi - w| = 2r$. Hence using the standard estimate for the integral

   $$\left|f_n(z) - f_n(w)\right| \leq \frac{1}{2\pi} 2\pi(2r)|z - w|\frac{1}{2r^2}M_k \leq \frac{|z - w|}{r}M$$

since $\forall \xi \in C_{2r}(w) \forall n \in \mathbb{N}^* : \left|f_n(\xi)\right| \leq M$, where M in the uniform bound for all $f_n \in F$ in $\overline{D}_{2\delta}(w)$
So, for any $\varepsilon > 0$ we get

$$\forall n \in \mathbb{N}^* \forall z, w \in K : \left|f_n(z) - f_n(w)\right| < \varepsilon$$

as soon as

$$|z - w| \leq \min\left\{r, \frac{\varepsilon r}{M}\right\}$$

2. To extract a subsequence which converges uniformly on all compact sets we use a standard trick, called the “diagonal argument”.

Let $\{K_l\}_{l \in \mathbb{N}^*}$ be the sequence of compact sets given by the last Lemma 3.7.
By the first step and the Arzelà-Ascoli Theorem 3.25, there is a subsequence of $(f_n)_{n \in \mathbb{N}^*}$ converging uniformly on K_1, say

$$(f_n)_{n \in L_1}$$

Then there exists a subsequence of $(f_n)_{n \in L_1}$, with $L_1 \subset \mathbb{N}$ infinite, converging uniformly on K_2, hence on K_1 and K_2, say $(f_n)_{n \in L_2}$ $L_2 \subset L_1 \subset \mathbb{N}$ infinite. Inductively, we get a subsequence $(f_n)_{n \in L_k}$ converging uniformly on $K_1, K_2, ..., K_k$ with $L_k \subset L_{k-1} \subset ... \subset L_1 \subset \mathbb{N}$

Now let

$$\begin{aligned}
n_1 &:= \min\{n \in \mathbb{N}^* : n \in L_1\} \\
n_2 &:= \min\{n \in \mathbb{N}^* : n \in L_2 \setminus \{n_1\}\} \in L_2 \subset L_1 \\
n_3 &:= \min\{n \in \mathbb{N}^* : n \in L_3 \setminus \{n_1, n_2\}\} \in L_3 \subset L_2 \subset L_1 \\
&\vdots
\end{aligned}$$

We get $n_1 < n_2 < ...$ with $n_k \in L_k \subset L_{k-1} \subset ... \subset L_1$

Note that $L := \{n_1, n_2, ...\}$ has the property that $L \setminus L_k$ is finite for each k

For each $k \in \mathbb{N}^*$, $(f_n)_{n \in L} = (f_{n_j})_{j \in \mathbb{N}}$ is up to finitely many terms (which has no effect on convergence) a subsequence of $(f_n)_{n \in L_k}$

Hence $(f_{n_j})_{j \in \mathbb{N}}$ converges uniformly on every K_k, since any compact set K is contained in K_k for some $k \in \mathbb{N}^*$

□

Appendix A

The analytic continuation of the Riemann Zeta Function

So far we have seen the following results:

1. We have seen

   Proposition A.1. [SS10, Proposition VI.2.1] The series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely and uniformly on every half plane

   $$\forall \delta > 0 : U_\delta := \{s \in \mathbb{C} : Re(s) \geq 1 + \delta\}$$

   and is holomorphic in $\{s \in \mathbb{C} : Re(S) > 1\}$

2. We have also seen

   Proposition A.2. $\forall z \in \mathbb{H} : \theta(z) := \sum_{n=0}^{\infty} e^{i\pi n^2 z}$ converges and defines a holomorphic function in $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$

   We have seen that

3. For a function $f \in \mathbb{C}^{\mathbb{R}}$, which is Riemann integrable on every $[a, b]$ and for which $\int_{-\infty}^{\infty} \left|f(t)\right| dt$ converges, its Fourier transform is defined as

   $$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

We also have shown that $f(x) = e^{-\pi x^2}$ has $\hat{f}(\xi) = e^{-\pi \xi^2}$. What we have not seen, but can be proved, are the following results about Fourier Transform, which can be found in Chapter IV in [SS10].

For $a > 0$, denote by $\mathcal{F}_a$ the class of all functions f that satisfy the following two conditions:

1. f is holomorphic in the horizontal strip $S_a := \left\{z \in \mathbb{C} : \left|Im(z)\right| < a\right\}$
2. It exists a constant $A > 0$ such that

$$\forall x \in \mathbb{R} \forall y \in (-a, a) : \left|f(x+iy)\right| \leq \frac{A}{1+x^2}$$

   i.e. f is of moderate decay on each horizontal line $Im(z) = y$ uniformly in $y \in (-a, a)$

Example A.1. $f(z) = e^{-\pi z^2} \in \mathcal{F}_a$ *for all* $a > 0$*. Let* $\mathcal{F} := \left\{f \in \mathbb{C}^{\mathbb{C}} : \exists a > 0 : f \in \mathcal{F}_a\right\}$

The **Fourier Inversion** says

Theorem A.1 (Fourier Inversion)**.** [SS10, Proposition VI.2.2] If $f \in \mathcal{F}$, then

$$\forall x \in \mathbb{R} : f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

and $\widehat{(\hat{f})}(x) = f(-x)$

The **Poisson Summation** says

Theorem A.2 (Poisson Summation Formula)**.** [SS10, Proposition VI.2.4] If $f \in \mathcal{F}$, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Corollary A.1. The Poisson Summation Formula applied to $f_t(x) = e^{-\pi t x^2}$, for $t \in \mathbb{R}^{>0}$ gives

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \sum_{n=-\infty}^{\infty} t^{\frac{-1}{2}} e^{\frac{-\pi n^2}{t}}$$

Hence

$$\vartheta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi t n^2} = t^{\frac{-1}{2}} \sum_{n\in\mathbb{Z}} e^{\frac{-\pi n^2}{t}} = \frac{1}{\sqrt{t}} \vartheta\left(\frac{1}{t}\right)$$

So

$$\boxed{\vartheta(t) = \frac{1}{\sqrt{t}}\vartheta\left(\frac{1}{t}\right)}$$

and note that $\vartheta(t) = \theta(it)$

We can now use this transformation of the θ-function to give analytic continuation and functional equation of $\zeta(s)$, namely

Theorem A.3. [SS10, Theorem VI.2.3] Let for $Re(s) > 1$ and

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) g(s)$$

Then $\Lambda(s)$ has a meromorphic continuation to all of s-plane, with simple poles at $s = 0, 1$ and satisfies the functional equation

$$\Lambda(1-s) = \Lambda(s)$$

Recall: For $Re(s) > 0$ we have

$$\Gamma(s) := \int_0^\infty e^{-t}t^s \frac{dt}{t}$$

Theorem A.4. Γ has analytic continuation to a meromorphic function on $\mathbb{C}$, with simple poles at $s = 0, -1, -2, \ldots$ and residue at $s = -n$ equal to

$$\mathrm{Res}_{-n}(\Gamma) = \frac{(-1)^n}{n!}$$

Proof of Theorem A.3. **Idea**: to relate $\Lambda(s)$ and $\vartheta(t) = \theta(it)$ via an integral transform and use the transformation property of $\vartheta(t) = \frac{1}{\sqrt{t}}\vartheta\left(\frac{1}{t}\right)$ inside the integral to analytically continue $\Lambda(s)$. □

We start by collecting growth and decay property of $\vartheta(t)$, for example

$$\vartheta(t) \leq Ct^{\frac{-1}{2}} \xrightarrow{t\to 0} 0$$

(follows from the functional equation) and $\left|v(t) - 1\right| \leq Ce^{-\pi t}$ for some $C > 0$ and for all $t \geq 1$. Since for $t \geq 1$, we get

$$2\sum_{n\in\mathbb{N}^*} e^{-\pi n^2 t} \leq 2\sum_{n\in\mathbb{N}^*} e^{-\pi n t} \leq Ce^{-\pi t}$$

The relation between Λ and ϑ is now given by the fact that for any $s \in \mathbb{C}$ with $Re(s) > 1$ we have

$$\boxed{\Lambda(s) = \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \big(\vartheta(t)-1\big)t^{\frac{s}{2}}\frac{dt}{t}}$$

This is based on the simple observations that

1. $\int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}} \frac{dt}{t} = (\pi n^2)^{\frac{-s}{2}}\Gamma(s) = \pi^{\frac{-s}{2}}\Gamma(s) n^{-s}$
2. $\vartheta(t) - 1 = 2\sum_{n=1}^\infty e^{-\pi n^2 t}$

Using the estimates on $\vartheta(t)$ as $t \to 0$ and as $t \to \infty$, one can justify the change of $\int$ and $\sum$ to get for $Re(s) > 1$

$$\frac{1}{2}\int_0^\infty \big(\vartheta(t)-1\big)t^{\frac{s}{2}}\frac{dt}{t} = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}}\frac{dt}{t} = \pi^{\frac{-s}{2}}\Gamma(s)\sum_{n=1}^\infty \frac{1}{n^s} = \Lambda(s)$$

$$\boxed{\Lambda(s) = \frac{1}{2}\int_0^\infty \big(\vartheta(t)-1\big)t^{\frac{s}{2}}\frac{dt}{t}}$$

Now, we will see that we can make sense of the right hand side for $s \in \mathbb{C}$. Now

$$\begin{aligned} &\forall t \geq 1 : \big|\vartheta(t) - 1\big| < e^{-\pi t} \\ \Longrightarrow &\forall s \in \mathbb{C} : \frac{1}{2}\int_1^\infty (\vartheta(t)-1)t^{s/2}\frac{dt}{t} \text{ converges} \end{aligned}$$

Hence, it defines an analytic function for all $s \in \mathbb{C}$. On the other hand, for $\frac{1}{2}\int_0^1 \big(\vartheta(t) - 1\big)t^{\frac{s}{2}}\frac{dt}{t}$ we use the functional equation $\vartheta(t) = t^{\frac{-1}{2}}\vartheta\left(\frac{1}{t}\right)$. This gives $\vartheta(t) - 1 = t^{\frac{-1}{2}}\vartheta\left(\frac{1}{t}\right) - 1 = t^{\frac{-1}{2}}\left(\vartheta\left(\frac{1}{t}\right) - 1\right) + t^{\frac{-1}{2}} - 1$. Hence

$$\begin{aligned} \frac{1}{2}\int_0^1 (\vartheta(t)-1)t^{\frac{s}{2}}\frac{dt}{t} &= \frac{1}{2}\int_0^1 \left(t^{\frac{-1}{2}}\left(\vartheta\left(\frac{1}{t}\right)-1\right) + t^{\frac{-1}{2}} - 1\right)t^{\frac{s}{2}}\frac{dt}{t} = \\ &= \frac{1}{2}\int_0^1 \left(\vartheta\left(\frac{1}{t}\right)-1\right)t^{\frac{s-1}{2}}\frac{dt}{t} + \frac{1}{2}\int_0^1 t^{\frac{s-1}{2}}\frac{dt}{t} - \frac{1}{2}\int_0^1 t^{\frac{s}{2}}\frac{dt}{t} = \\ &= \frac{1}{2}\int_0^1 \left(\vartheta\left(\frac{1}{t}\right)-1\right)t^{\frac{s-1}{2}}\frac{dt}{t} + \frac{1}{2}\left[\frac{t^{\frac{s-1}{2}}}{\frac{s-1}{2}}\right]_0^1 - \left[\frac{1}{2}\frac{t^{\frac{s}{2}}}{\frac{s}{2}}\right]_0^1 = \\ &= \frac{1}{2}\int_0^1 \left(\vartheta\left(\frac{1}{t}\right)-1\right)t^{\frac{s-1}{2}}\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s} \end{aligned}$$

Now we make the change of variables $u = \frac{1}{t}$ with $\frac{du}{u} = -\frac{dt}{t}$ we get

$$\frac{1}{2}\int_0^1 \left(\vartheta\left(\frac{1}{t}\right)-1\right)t^{\frac{s-1}{2}}\frac{dt}{t} = \int_0^\infty (\vartheta(u)-1)u^{\frac{1-s}{2}}\frac{du}{u}$$

Hence, overall we have

$$\Lambda(s) = \frac{1}{2}\int_0^\infty (\vartheta(t)-1)\frac{dt}{t} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (\vartheta(t)-1)\left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right)\frac{dt}{t}\right)$$

Note that the integral on the right defines an entire function for all $s \in \mathbb{C}$ due to the exponential decay of

$$\forall t \geq 1 : \left|\vartheta(t) - 1\right| < Ce^{-\pi t}$$

Hence $\Lambda(s)$ has a continuation to all of the s-plane, which is holomorphic except for the simple poles at $s = 1, 0$ with resides 1 and -1 respectively.

Both $\frac{1}{s-1} - \frac{1}{s}$ and $\int_1^\infty (\vartheta(t)-1)\left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right)\frac{dt}{t}$ are invariant under $s \mapsto 1-s$, hence

$$\Lambda(s) = \Lambda(1-s)$$

$\Lambda(s) = \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ has analytic continuation to all $s \in \mathbb{C}$ except for the simple poles at $s = 0$ and $s = 1$

Since $\Gamma\left(\frac{s}{2}\right)$ has poles at $\frac{s}{2} = 0, -1, -2, ...$; $\zeta(s)$ does not have a pole at $s = 0$ and must vanish at $s = -2, -4, -6, ...$; since $\Lambda(s)$ does not have poles at $s = -2, -4, ...$

These are called the **trivial zeroes of** $\zeta(s)$

For $Re(s) > 1$, we have that $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ also has an infinite product expansion, called **Euler product**

$$\boxed{\zeta(s) = \prod_{p \text{ primes}} \frac{1}{1 - \frac{1}{p^s}}}$$

which is an analytic statement of the **Fundamental Theorem of Arithmetic**: every positive integer is a unique product of prime powers.

Due to the Euler product, we have that $\zeta(s) \neq 0$ for $Re(s) > 1$ and by the functional equation, that $\zeta(s) \neq 0$ for $Re(s) < 0$

Definition A.1 (Riemann Hypothesis)**.** If $s \neq -2, -4, ..., s \in \mathbb{C}$ with $\zeta(s) = 0$, then $Re(s) = \frac{1}{2}$

Theorem A.5 (Prime Number Theorem (PNT))**.** Let

$$\pi(x) := \#\{p \text{ prime } : p < x\}$$

Then

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1$$

The Prime Number Theorem is a consequence of

Theorem A.6. If $Re(s) = 1$, then $\zeta(s) \neq 0$

Appendix B

Other results

B.1 Substitution rule in complex line integration

A straightforward fact that is quite different from the case of real analysis is that, once the reader shall have gained some experience in the practice of the calculation of complex line integrals, it will be evident that the integration by substitution does not work as easily as in the real case.

Example B.1. *Consider the integral*

$$\int_{\ell_{[0,2]}} e^{iz} dz = e^{2i} - 1$$

on another hand one tries to approach it by naively setting $w = iz$ and consequently $\frac{dw}{dz} = i$, the result would turn out to be

$$\int_{\ell_{[0,2i]}} e^{w} i dw = i \int_{\ell_{[0,2i]}} e^{w} dw = i(e^{2i} - 1)$$

which leads to a different incorrect result.

Part of the reason for this limit to integration is linkable to the Homotopy Theorem 3.14, as a change of variable, once composed with the path, deforms it in possibly non-viable ways. For instance, exiting the domain of the integrand function, crossing a singularity, etc. To name one more, also changes in the "simplicity" of the curve can have effects.

Theorem B.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $\gamma \in C^1\big([a, b]; \Omega\big)$ be a path in Ω. If $\phi : \Omega \to \phi(\Omega)$ is biholomorphic, i.e. is a bijection that is holomorphic and with holomorphic inverse, then

$$\int_{\gamma} f(z)dz = \int_{\phi^{-1}\circ\gamma} f\big(\phi(w)\big)\phi'(w)dw$$

A useful result to accompany this Theorem is the following Lemma, which states the form that biholomorphisms need to assume from $\mathbb{C}$ to $\mathbb{C}$, namely **Affine maps**, denoted by $Aut_{\mathscr{H}}(\mathbb{C})$

Lemma B.1. All biholomorphisms from $\mathbb{C}$ to $\mathbb{C}$ are affine maps, namely

$$\forall f \in Aut\mathscr{H}(\mathbb{C})\exists a, b \in \mathbb{C}\forall z \in \mathbb{C} : f(z) = az + b$$

Proof. We want to show this in two steps, first showing that such a function is a polynomial and second that its degree is 1
Let $f \in Aut_{\mathscr{H}}(\mathbb{C})$. First, the series expansion of f at 0 is

$$f(z) = \sum_{n=0}^{\infty} a_\ell z^\ell$$

and converges over all of $\mathbb{C}$, since f is entire. Composing f with the inversion $\eta : \mathbb{C}^* \to \mathbb{C}, z \mapsto \frac{1}{z}$ it results that for

$$g := f \circ \eta : \mathbb{C}^* \to \mathbb{C}, z \mapsto \sum_{n=0}^{\infty} a_\ell z^{-\ell}$$

we have that $\overline{g(\mathbb{C}^*)} = \overline{f(\mathbb{C}^*)} = \mathbb{C} \setminus \{f(0)\}$. Now, if 0 were an essential singularity of g, then $\overline{g\big(\dot{D}_1(0)\big)} = \overline{f\big(\mathbb{C} \setminus D_1(0)\big)} = \mathbb{C}$ and since f is non-constant we have that $f\big(\mathbb{C} \setminus D_1(0)\big)$ is open in $\mathbb{C}$ and so

$$f\big(\mathbb{C} \setminus D_1(0)\big) \cap f\big(D_1(0)\big) \neq \emptyset$$

Hence, a contradiction to the bijectivity of f. So, 0 is either a pole of order $N \in \mathbb{N}^*$ or removable. In the latter case, $g(z) = a_0 = f(z)$ for all $z \in \mathbb{C}$, otherwise

$$f(z) = \sum_{n=0}^{N} a_\ell z^\ell$$

for N as above. To prove that $N = 1$, assume that $N \geq 2$, thus $f(z) - a_0$ either has multiple zeroes (infraction to the injectivity, as the preimage of 0 has more than one element) of one with higher multiplicity. In this case, we again conclude that the injectivity of f produces a contradiction. Therefore $N = 1$ and the first direction is shown.
Conversely, every affine function is bijective and biholomorphic on $\mathbb{C}$ □

There are many more (conformal) automorphism between two proper subset of $\mathbb{C}$ with specific characteristics, these can be obtained using the Riemann mapping Theorem 3.21.

B.2 L'Hôpital's Rule in $\mathbb{C}$

Theorem B.2 (L'Hôpital's Rule in $\mathbb{C}$)**.** Let $a \in \mathbb{C}$ and $f, g \in \mathscr{H}(U_a)$ with $U_a \in \mathcal{O}_{\mathbb{C}}$ and $a \in U_a$, such that $f(a) = g(a) = 0$, then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}$$

Proof. Let $f \in \mathscr{H}(U_a)$ and $f'(a) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a} = \lim_{z \to a} \frac{f(z)}{z-a}$, since $f(a) = 0$. Hence, f' is continuous and therefore $\lim_{z \to a} f'(z) = f'(a)$, from which follows that

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{f(z)}{z-a} \cdot \frac{z-a}{g(z)} = \left(\lim_{z \to a} \frac{f(z)}{z-a} \right) \cdot \left(\lim_{z \to a} \frac{z-a}{g(z)} \right) = \lim_{z \to a} \frac{f'(z)}{g'(z)}$$

□

B.3 Properties of the Residue

Proposition B.1. Let $\dot{D}_r(z_0) \in \mathcal{O}_{\mathbb{C}}$ for some $r > 0$ and $f, g \in \mathscr{H}\big(\dot{D}_r(z_0)\big)$ with $z_0 \in \mathbb{C}$ being a pole of both, then

(i) $\operatorname{Res}_{z_0}(f+g) = \operatorname{Res}_{z_0}(f) + \operatorname{Res}_{z_0}(g)$

(ii) $\operatorname{Res}_{z_0}(fg)$ has a finitary expression.

(iii) If the extension $\tilde{f} \in \mathscr{H}\big(D_r(z_0)\big)$, then $\operatorname{Res}_{z_0}(\tilde{f}') = 0$

(iv) If a sequence $(f_n)_{n \in \mathbb{N}^*} \in \Big(\mathscr{H}\big(\dot{D}_r(z_0)\big)\Big)^{\mathbb{N}^*}$ is such that $f_n \underset{n \to \infty}{\rightrightarrows} f$, then

$$\operatorname{Res}_{z_0}\left(\sum_{n=0}^{\infty} f_n\right) = \sum_{n=0}^{\infty} \operatorname{Res}_{z_0}(f_n)$$

Proof. (i) Assume that f and g are analytic in a punctured neighbourhood of z_0. Then by definition of the residue one has

$$\operatorname{Res}_{z_0}(f+g) = \frac{1}{2\pi i} \int_{\gamma} \big(f(z) + g(z)\big)\, dz$$

where γ is a sufficiently small circle with center z_0. By linearity of integrals, it follows that

$$\mathrm{Res}_{z_0}(f+g) = \mathrm{Res}_{z_0}(f) + \mathrm{Res}_{z_0}(g)$$

(ii) There is no "nice" formula for $\mathrm{Res}_{z_0}(fg)$. However, when f and g just have poles at z_0 it is possible to compute $\mathrm{Res}_{z_0}(fg)$ in a "finitary" fashion from the Laurent expansions of f and g as follows: assuming $z_0 = 0$ for simplicity we have

$$f(z) = \sum_{k=-m}^{\infty} a_k z^k$$
$$g(z) = \sum_{\ell=-n}^{\infty} b_\ell z^\ell$$

for $n, m \in \mathbb{N}$; therefore

$$f(z)g(z) = \left(\sum_{k=-m}^{n-1} a_k z^k\right)\left(\sum_{\ell=-n}^{m-1} b_\ell z^\ell\right) + h(z)$$

where h is holomorphic at 0. The residue of fg at 0 can now be extracted:

$$\mathrm{Res}_0(fg) = \sum_{k=-m}^{n-1} a_k b_{-k-1}$$

(iii) We have that for z in a punctured neighbourhood of z_0

$$\tilde{f}(z) = \frac{h(z)}{z}$$

for some holomorphic h in the same neighbourhood. Then

$$\tilde{f}'(z) = \frac{h'(z)}{z} - \frac{h(z)}{z^2} = \frac{zh'(z) - h(z)}{z^2}$$

(iv) Let $\varepsilon > 0$, when the series $\sum_{n=0}^{\infty} f_n$ converges uniformly in annuli $\mathcal{A}(z_0, \varepsilon, 2\varepsilon)$, then one has

$$\mathrm{Res}_{z_0}\left(\sum_{n=0}^{\infty} f_n\right) = \frac{1}{2\pi i}\int_\gamma \left(\sum_{n=0}^{\infty} f_n\right) dz = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_\gamma f_n(z)\, dz = \sum_{n=0}^{\infty} \mathrm{Res}_{z_0}(f_n)$$

for any curve γ with $im(\gamma) \subseteq \mathcal{A}(z_0, \varepsilon, 2\varepsilon)$

□

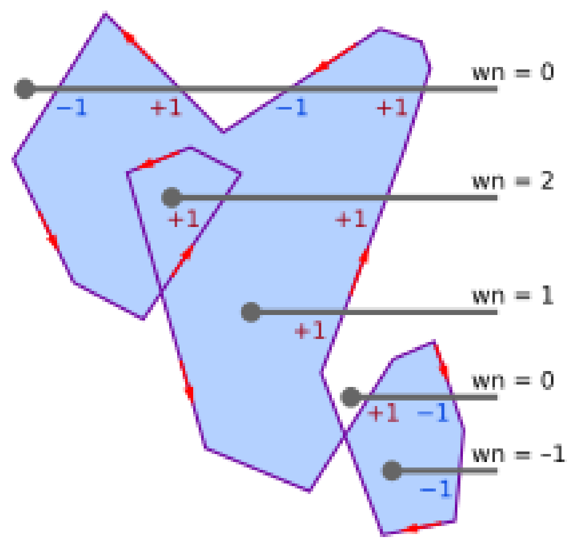
B.4 A winding number algorithm

To check if a given point lies inside or outside a polygon:

1. Draw a horizontal line to the right of each point and extend it to infinity.

2. Count the number of times the line intersects with polygon edges.

3. A point is inside the polygon if either count of intersections is odd or point lies on an edge of the polygon. If none of the conditions are true, then point lies outside.



Furthermore, this algorithm can be generalized to an arbitrary direction of the ray, preserving the sign of every crossing direction, hence left to right or right to left (in the direction of the ray pointing to infinity). An improved version of this is already far beyond the scope of this section and is known as "Dan Sunday's RAY Algorithm".

Change Log

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