Eidgenössische Technische Hochschule Zürich

Complex Analysis

Lectures by Professor Ö. Imamoglu – Autumn Semester 2024

Ruocheng Wang

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Preface

These lecture notes are based on the lecture Complex Analysis/Funktionentheorie given by Prof. Dr. Özlem Imamoglu in Autumn Semester 2024 at ETH Zürich. I am deeply grateful for Prof. Imamoglu's exceptional teaching and guidance throughout this course. I also want to thank Noah Larsson for his crucial contribution to these notes. The lectures partially followed the book [4], but only the most essential topics were discussed in detail. While I have made every effort to improve the accuracy and clarity of the content, these notes are the product of my personal understanding. As such, they are still subject to errors and omissions, and in case of any discrepancy, please refer to the original lecture notes on the course webpage[1].

Ruocheng Wang

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[1] `https://metaphor.ethz.ch/x/2024/hs/401-2303-00L/`

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Introduction

Lecture 1

Our goal this semester is to study functions $f : \mathbb{C} \to \mathbb{C}$ defined on the complex plane $\mathbb{C}$, or on an open subset of $\mathbb{C}$.
We will see that the study of complex function theory is not simply the study of functions on $\mathbb{R}^2$. We will see that in many ways the theory of one real variable is more complicated than the theory of functions of complex variable. To give an idea of what we mean, let us try to compare and contrast:

1. It is not too difficult to write down a function of a real variable that is n times differentiable but not infinitely differentiable, for example

$$f(x) := \begin{cases} x^2 \sin\left(1/x^2\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   The derivative of f exists for every x including $x = 0$, but the derivative is not continuous. By integrating f as many times as you like, you get a function that is n times differentiable, but not infinitely many times differentiable.
   In contrast we will see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable once, it is differentiable infinitely many times.

2. There are functions $f : \mathbb{R} \to \mathbb{R}$ that are infinitely many times differentiable, but whose Taylor seres does not represent f i.e. f is not analytic. For example consider

$$f(x) := \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Then f is infinitely many times differentiable. Unfortunately at $x = 0$ all derivatives are zero. Hence its Taylor series is identically zero and cannot represent f.

In contrast, if $f : \mathbb{C} \to \mathbb{C}$ is a function of complex variable which is differentiable, then f is analytic, i.e. it can be represented by a power series (in $\mathbb{C}$ differentiable = analytic).

3. There are plenty of C^∞ functions of a real variable that are bounded, for example $\sin x, \cos x$.

   In contrast we will see that if a function $f : \mathbb{C} \to \mathbb{C}$ is differentiable and bounded then it is constant (Liouville's Theorem 2.4.6).

4. For two functions of a real variable f, g f and g can agree on an open set without being equal.

   In contrast, if $f, g : \mathbb{C} \to \mathbb{C}$ are two differentiable functions which coincide on an arbitrary small disc (or even on a convergent sequence (z_n)), then $f = g$ (Analytic continuation principle 2.4.16).

Remark. The power of complex function theory comes from this "robustness" or rigidity. It is a subject where in some sense analysis, geometry and algebra come together. This, we will see, allows one to prove theorems that a priori have nothing to do with complex numbers.

1. With complex analysis one can show the Fresnel integrals (example 2.3.2)

$$\int_0^\infty \cos\bigl(t^2\bigr)\mathrm{d}t = \int_0^\infty \sin\bigl(t^2\bigr)\mathrm{d}t = \frac{\sqrt{2\pi}}{4}.$$

2. Let $\pi(x) := \#\{p \text{ prime} | p \leq x\}$. Then with complex analysis one can show

$$\pi(x) \underset{x\to\infty}{\sim} \frac{x}{\log x} \quad \text{(Prime Number Theorem)}.$$

3. If $f \in \mathbb{C}[x]$ a non zero polynomial, then f has a zero in $\mathbb{C}$ (Fundamental Theorem of Algebra 2.4.7 or exercise 3.4.5).

Chapter 1

Preliminaries to Complex Analysis

1.1 Complex numbers and complex plane

Definition 1.1.1: Complex Numbers

The complex numbers are defined by

$$\mathbb{C} := \{x + iy | x, y \in \mathbb{R}, i^2 = -1\}.$$

For $z = x + iy \in \mathbb{C}$ we define the real part of z

$$\operatorname{Re}(z) := x$$

and the imaginary part

$$\operatorname{Im}(z) := y.$$

Definition 1.1.2: Complex Conjugate

The complex conjugate of a complex number $z = x + iy$ is defined by

$$\overline{z} := x - iy.$$

One can also show that

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \overline{z}}{2i}.$$

Also

$$z \in \mathbb{R} \Leftrightarrow z = \overline{z},$$
$$z \in i\mathbb{R} \Leftrightarrow z = -\overline{z}.$$

Complex numbers can also be represented as ordered pairs of real numbers in $\mathbb{R}^2$ with $z = (x, y)$ where we have $z = w$ with $w = (u, v)$ if and only if $x = u$ and $y = v$. Let $z = x + iy$ and $w = u + iv$, then the addition of $z + w$ in $\mathbb{C}$ is given by

$$z + w := (x + u) + i(v + y)$$

and the multiplication

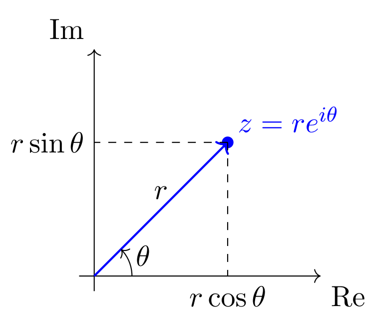
$$z \cdot w := (xu - vy) + i(xv + yu).$$

Definition 1.1.3: Absolute Value of a complex number

The norm/absolute value of a complex number is given by

$$|z| := \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

We can also represent the complex numbers in **polar coordinates** with $x = r\cos\theta$ and $y = r\sin\theta$. We write $z = re^{i\theta}$.



The polar representation is not unique unless we restrict $\theta \in (-\pi, \pi]$ or any other interval of length 2π.

Definition 1.1.4: Argument

The angle θ is called the argument of z. It is defined uniquely up to a multiple of 2π and is denoted by $\arg z$

$$\arg z = \{\theta \in \mathbb{R} \mid z = |z|e^{i\theta}\}.$$

The argument chosen in the interval $(\pi, \pi]$ is called the principal argument and denoted by $\operatorname{Arg} z$.

Remark. No assignment of argument is made to $0 \in \mathbb{C}$.
For $z = x + iy \neq 0$ we have

$$\operatorname{Arg} z = \begin{cases} \arcsin(y/|z|) & \text{if } x \geq 0, \\ \pi - \arcsin(y/|z|) & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi - \arcsin(y/|z|) & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

Observe that $\arg(z^{-1}) = -\arg z$ and $\arg(zw) = \arg z + \arg w$. But it is *not* always the case that $\operatorname{Arg}(z^{-1}) = -\operatorname{Arg}(z)$ or $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$. For example $\operatorname{Arg}(-1/2) = \pi \neq -\operatorname{Arg}(-2) = -\pi$ and $\pi = \operatorname{Arg}(-1) = \operatorname{Arg}((-i)(-i)) \neq \operatorname{Arg}(-i) + \operatorname{Arg}(-i) = -\pi$.
Also recall the following:

- $|z| = 0 \iff z = 0 \quad \forall\, z \in \mathbb{C}$
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \quad \forall\, z_1, z_2 \in \mathbb{C}$
- $|z_1 z_2| = |z_1||z_2| \quad \forall\, z_1, z_2 \in \mathbb{C}$
- $|\overline{z}| = |z|$
- $|\operatorname{Re}(z)| \leq |z|,\ |\operatorname{Im}(z)| \leq z \quad \forall\, z \in \mathbb{C}$

We can also represent the complex numbers as 2×2 matrices. For $z = a + ib, w = c + id$ let

$$Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } W = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

then

$$ZW = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix}.$$

On the other hand $zw = ac - bd + i(bc + ad)$, hence the multiplication in $\mathbb{C}$ corresponds to multiplication of corresponding matrices in $M_2(\mathbb{R})$. We can represent any $z \in C$

with the matrix

$$\begin{pmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{pmatrix}.$$

1.2 Topology and convergence

Definition 1.2.1: Open disk

The open disc of radius r centred at z_0 is denoted by $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$. The corresponding closed disc is $\overline{D_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$. The boundary of $D_r(z_0)$ is the circle $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$.

Definition 1.2.2: Connected and disconnected

A subset $A \subseteq \mathbb{C}$ is called **disconnected** if there exist open sets $U, V \subseteq \mathbb{C}$ such that $U \cap V = \varnothing$, both U and V have non-empty intersections with A, and $A \subseteq U \cup V$. A subset of $\mathbb{C}$ is **connected** if it is not disconnected, and is called a region or a **domain**.

Remark. Any two points $z_1, z_2 \in A$ where $A \subseteq \mathbb{C}$ is open and connected can be joined by a polygonal path, hence A is automatically path-connected.

For a more in depth recap consider the Analysis I and Analysis II scripts of Prof. Alessio Figalli (Analysis I) [2] and Prof. Joaquim Serra (Analysis II) [3].

1.3 Holomorphic functions

Definition 1.3.1: Holomorphic Function

Let $\Omega \subseteq \mathbb{C}$ be an open set, $f : \Omega \to \mathbb{C}$. f is called **holomorphic** (or complex differentiable) at $z_0 \in \Omega$ if the limit

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, where $h \in \mathbb{C}$ is chosen such that $z_0 + h \in \Omega$. We call the limit the **derivative** of f at z_0 and denote it by $f'(z_0)$. f is called holomorphic on Ω if it is complex differentiable at every $z_0 \in \Omega$. f is called entire if it is differentiable on $\mathbb{C}$.

Example 1.3.2

Let $f : \mathbb{C} \to \mathbb{C}, z \mapsto z$. Then

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{z_0+h-z_0}{h} = 1.$$

Proposition 1.3.3: Basic properties

Let $\mathcal{H}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic on } \Omega\}$, then $\mathcal{H}(\Omega)$ is a $\mathbb{C}$-vector space. Moreover, if $f, g \in \mathcal{H}(\Omega)$, the following holds:

1. *$\alpha f + \beta g$ is holomorphic on Ω for any $\alpha, \beta \in \mathbb{C}$, and $(\alpha f + \beta g)' = \alpha f' + \beta g'$.*
2. *$fg \in \mathcal{H}(\Omega)$ and $(fg)' = f'g + fg'$.*
3. *If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and*

$$(f/g)' = \frac{f'g - g'f}{g^2}.$$

Further, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \text{for all } z \in \Omega.$$

Lecture 2

Remark. If $f : \Omega \to \mathbb{C}$ is differentiable at $z_0 \in \mathbb{C}$, then there exists a complex number $c \in \mathbb{C}$ such that

$$f(z) = f(z_0) + c(z - z_0) + E(z, z_0)$$

with $E : \Omega \to \mathbb{C}$ satisfying

$$\lim_{z\to z_0} \left| \frac{E(z, z_0)}{z - z_0} \right| = 0.$$

Here $c = f'(z_0)$.

Example 1.3.4

$f(z) = z$ if differentiable anywhere. This together with the previous proposition gives us that every polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is differentiable for all $z \in \mathbb{C}$, and for example, $(z^n)' = nz^{n-1}$.

Here is an important non-example.

Example 1.3.5

$f(z) = \overline{z}$, let $z_0 \in \mathbb{C}$. Then we have

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{\overline{z_0+h} - \overline{z_0}}{h} = \frac{\overline{h}}{h}.$$

If we choose $h = t$, $t \in \mathbb{R}$ with $t \to 0$, then this limit is 1. If we choose $h = it$, then the quotient evaluates to -1, hence the limit does not exist as $h \to 0$. Hence $f(z) = \overline{z}$ is not complex differentiable.
If we view f as a function of 2 real variables $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x, -y)$, or

$$\tilde{f}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\tilde{f}$ is linear and hence differentiable anywhere in $\mathbb{R}^2$, with Jacobian

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall: $F : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $p_0 = (x_0, y_0)$ if there exists a linear map $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{p \to p_0} \frac{\|F(p) - F(p_0) - J(p - p_0)\|}{\|p - p_0\|} = 0.$$

Recall from Linear Algebra we can view $\mathbb{C}$ as a 1-dimensional vector space over $\mathbb{C}$, with basis $\{1\}$ or as a 2-dimensional real vector space, with basis $\{1, i\}$. A map $T : \mathbb{C} \to \mathbb{C}$ is a $\mathbb{C}$ linear if $T(z) = \lambda z$ for some $\lambda \in \mathbb{C}$. On the other hand $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$ linear if $T(z) = T(x + iy) = xT(1) + yT(i) = \lambda z + \mu \overline{z}$, where

$$\lambda = \frac{1}{2}(T(1) - iT(i)), \mu = \frac{1}{2}(T(1) + iT(i))$$

which can be seen using

$$x = \frac{z + \overline{z}}{2}, y = \frac{z - \overline{z}}{2i}.$$

Hence any $\mathbb{C}$-linear map is also $\mathbb{R}$-linear (with $\mu = 0$), but not every $\mathbb{R}$-linear map is $\mathbb{C}$-linear. It is $\mathbb{C}$-linear if and only if $\mu = 0$, which is equivalent to $T(i) = iT(1)$. If $T(1) = a + bi$ and $T(i) = c + di$, then $T(i) = iT(1)$ if and only if $b = -c, a = d$.

If we identify $\mathbb{C}$ with $\mathbb{R}^2$, a $\mathbb{R}$-linear map is also $\mathbb{C}$-linear if its matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}.$$

Note the matrix corresponding to $(x, y) \mapsto (x, -y)$ was

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and does not have the desired form.
How does this Linear Algebra fact reflect on the complex and real differentiability of a function $f : \mathbb{C} \to \mathbb{C}$?

1.4 Cauchy-Riemann equations

Let $f : \mathbb{C} \to \mathbb{C}$ holomorphic at z_0, $f(x + iy) = u + iv$. We can also view f as a function on $\mathbb{R}^2$ defined by $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (u(x, y), v(x, y))$. Then the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. First approach z_0 along $z = x + iy_0, x \to x_0$:

$$\begin{aligned} f'(z_0) &= \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0). \end{aligned} \tag{1.1}$$

In the similar manner we can approach z_0 with $z = x_0 + iy, y \to y_0$, then there is

$$f'(z_0) = v_y(z_0) - iu_y(z_0) \tag{1.2}$$

The two equations (1.1) and (1.2) must be equal, hence the **Cauchy-Riemann equations** hold

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{1.3}$$

In terms of the real function $\tilde{f}$, we have

$$J\tilde{f} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

where one can also see that the Cauchy-Riemann equations must be satisfied for the matrix to represent a $\mathbb{C}$-linear map. We also define the differential operator Lecture 3

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Proposition 1.4.1: ∂z and $\partial \overline{z}$

If f is holomorphic at z_0 and $f(x) = u + iv$. Then,

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0 \quad \textit{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

If we write

$$\begin{aligned} \tilde{f} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (u(x, y), v(x, y)) \end{aligned}$$

so that $f(z) = \tilde{f}(x, y)$, then $\tilde{f}$ is real differentiable with Jacobian

$$J\tilde{f} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix},$$

and $\det J\tilde{f} = |f'(z_0)|^2 = u_x^2 + u_y^2 = u_x^2 + v_x^2$.

Proof.

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \\ &= \frac{1}{2}(u_x + iv_x + iu_y - v_y) \\ &= \frac{1}{2}\big((u_y - v_x) + i(v_x + u_y)\big) = 0, \end{aligned}$$

where we applied Cauchy-Riemann equations (1.3). Adding (1.1) and (1.2) and dividing by 2 we get

$$\begin{aligned}
f'(z_0) &= \frac{1}{2}\big(u_x(z_0) + iv_x(z_0) + v_y(z_0) - iu_y(z_0)\big) \\
&= \frac{1}{2}\big((u_x(z_0) + iv_x(z_0)) - i\,(u_y(z_0) + iv_y(z_0)\big) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}f(z_0) - i\frac{\partial}{\partial y}f(z_0)\right) \\
&= \frac{\partial}{\partial z}f(z_0).
\end{aligned}$$

Also we can see that

$$f'(z_0) = u_x + iv_x = u_x - iu_y = 2\frac{\partial u}{\partial z}.$$

Now we show the last part of the proposition. If $z_0 = x_0 - iy_0$ and $h = h_1 + h_2 i$. Since f is holomorphic at z_0, it means

$$f(z_0 + h) - f(z_0) = f'(z_0)h + \varepsilon(h)h$$

with $\lim_{h\to 0} \varepsilon(h) \to 0$. If $f'(z_0) = a + ib$,

$$\begin{aligned}
f'(z_0)h &= (a + ib)(h_1 + h_2 i) = ah_1 - bh_2 + i(bh_1 + ah_2) \\
&= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
\end{aligned}$$

If we write $\tilde{f}(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$, then

$$\frac{\left|\tilde{f}((x_0, y_0) + (h_1, h_2)) - \tilde{f}(x_0, y_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right|}{|(h_1, h_2)|} \to 0$$

as $|h| \to 0$, which means $\tilde{f}$ is differentiable with a $\mathbb{C}$-linear differential. A quick computation shows that the Jacobian determinant takes the given value. □

Remark. Recall

1. $\mathbb{C}$ can ba identified with 2×2 real matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

2. $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (u(x, y), v(x, y))$,

$$J\tilde{f} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Now if $f'(z_0) = a + bi$, then the Jacobian of the corresponding $\tilde{f}$ should be in the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Hence we can see the Cauchy-Riemann equations.

We have seen that f being differentiable at z_0 implies that f satisfies the Cauchy-Riemann equations. There is a partial converse.

Theorem 1.4.2: Cauchy-Riemann equations and differentiability

Suppose $\Omega \subset \mathbb{C}$ is an open subset and $f : \Omega \to \mathbb{C}$ with $f = u + iv$. If u, v are continuously differentiable and f satisfies the Cauchy-Riemann equations in Ω, then f if differentiable on Ω and $f'(z_0) = \frac{\partial}{\partial z} f(z_0)$.

Proof. Let $z_0 = (x_0, y_0) \in \Omega$ and $h = (h_1, h_2)$. Since u and v are differentiable, we have

$$u(z_0 + h) - u(z_0) = u_x(z_0)h_1 + u_y(z_0)h_2 + |h|\varepsilon_1(h),$$

where $\varepsilon_1(h) \to 0$ as $h \to 0$, and

$$v(z_0 + h) - v(z_0) = v_x(z_0)h_1 + v_y(z_0)h_2 + |h|\varepsilon_2(h),$$

where $\varepsilon_2(h) \to 0$ as $h \to 0$. Hence we have

$$\begin{aligned} f(z_0 + h) - f(z_0) &= (u + iv)(z_0 + h) - (u + iv)(z_0) \\ &= (\partial_x u - i\partial_y u)h_1 + (\partial_y u + i\partial_x u)h_2 + |h|\varepsilon(h) \qquad (1.4) \\ &= (\partial_x u - i\partial_y u)(h_1 + h_2 i) + |h|\varepsilon(h), \end{aligned}$$

where $\varepsilon(h) = \varepsilon_1(h) + i\varepsilon_2(h)$ and we used the Cauchy-Riemann equations (1.3) in

(1.4). This implies

$$\begin{aligned} &= \left| \frac{f(z_0+h)-f(z_0)}{h} - (\partial_x u(z_0) - i\partial_y u(z_0)) \right| \\ &= \left| \frac{f(z_0+h)-f(z_0) - (\partial_x u(z_0) - i\partial_y u(z_0))(h_1+h_2 i)}{h} \right| \\ &= \frac{|h||\varepsilon(h)|}{|h|} = |\varepsilon(h)|. \end{aligned}$$

Since $\lim_{h\to 0} |\varepsilon(h)| = \lim_{h\to 0} \sqrt{\varepsilon_1(h)^2 + \varepsilon_2(h)^2} = 0$, using the definition of the limit we conclude that

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \partial_x u(z_0) - i\partial_y u(z_0).$$

In particular the limit of the differential quotient exists and f is differentiable. Using Proposition 1.4.1 we finally get

$$f'(z_0) = \partial_x u(z_0) - i\partial_y u(z_0) = 2\frac{\partial u}{\partial z}(z_0) = \frac{\partial f}{\partial z}(z_0).$$

□

Example 1.4.3

$f(z) + x^2 + y^2 + i2xy$, then

$$\partial_x u = 2x \qquad \partial_2 v = 2y$$

$$\partial_y u = 2y \qquad \partial_y v = 2x$$

Hence f is only differentiable at $y = 0$, that is, for any $z = x$.

1.5 Power series

Recall that a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n \qquad z \in \mathbb{C}.$$

Theorem 1.5.1: Power series and Radius of Convergence

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series. Then there exists $R \in \mathbb{R}$ *with* $0 \leq R \leq \infty$ *such that*

1. *if* $|z| < R$, *then the series converges absolutely,*
2. *if* $|z| > R$, *then it diverges.*

Moreover, with the convention that $1/0 = \infty$ *and* $1/\infty = 0$, R *is given by*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

R *is called the* ***radius of convergence*** *and* $D_R(0) = \{z \in \mathbb{C} : |z| < R\}$ *is called the disc of convergence.*

Example 1.5.2

The exponential function is given by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

One can check that

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty,$$

so the power series converges for all $z \in \mathbb{C}$.

Theorem 1.5.3: Radius of convergence of the derivative

The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *defines a holomorphic function in its disc of convergence and moreover,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

and f' *has the same radius of convergence as* f.

Proof. Let R be the radius on convergence of f. Since $\lim n^{\frac{1}{n}} = 1$, $\limsup |a_n n|^{1/n} = \limsup |a_n|^{1/n} = R$, hence $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence. Now let $z \in \mathbb{C}$ with $|z| < R$. Choose $\delta > 0$ such that $|z| + \delta < R$ (for instance take

$\delta = (R - |z|)/2)$. Let $h \in \mathbb{C}$, $|h| < \delta$, we want to show

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| \to 0$$

as $h \to 0$. We have

$$\begin{aligned}
\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| &= \left| \sum_{n=0}^{\infty} \frac{a_n (z+h)^n - a_n z^n}{h} - n a_n z^{n-1} \right| \\
&\leq \sum_{n=0}^{\infty} |a_n| \left| \frac{1}{h} \left(\sum_{k=0}^{n} \binom{n}{k} h^k z^{n-k} - z^n \right) - n z^{n-1} \right| \\
&= \sum_{n=1}^{\infty} |a_n| \left| \sum_{k=1}^{n} \binom{n}{k} h^{k-1} z^{n-k} - n z^{n-1} \right| \\
&= \sum_{n=2}^{\infty} |a_n| \left| \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k} \right| \\
&\leq \sum_{n=2}^{\infty} |a_n| n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} |h|^{k-2} |z|^{n-k} |h|,
\end{aligned} \tag{1.5}$$

where for (1.5) we used that for $k \geq 2$,

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} = \frac{n(n-1)}{k(k-1)} \binom{n-2}{k-2} \leq n(n-1) \binom{n-2}{k-2}.$$

It follows that

$$\begin{aligned}
\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| &\leq |h| \sum_{n=2}^{\infty} |a_n| n(n-1) (|z| + |h|)^{n-2} \\
&\leq |h| \sum_{n=2}^{\infty} |a_n| n(n-1) \left(\frac{R + |z|}{2} \right)^{n-2}.
\end{aligned} \tag{1.6}$$

In (1.6) we used that

$$|h| + |z| < \delta + |z| < \frac{R - |z|}{2} + |z| = \frac{R + |z|}{2}.$$

Note that

$$\frac{R + |z|}{2} < R,$$

hence the power series converges for all $|z| \leq R$. Taking the limit as $h \to 0$ gives that for all $|z| \leq R$

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}\right| = 0 \qquad \Longleftrightarrow \qquad f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

This concludes the proof. □

Lecture 4

Example 1.5.4

1. Exponential function:
$$\sum_{n=0}^{\infty}\frac{z^n}{n!} = e^z,$$
$$(e^z)' = e^z.$$

2. Trignometric functions:
$$\cos z = \frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!},$$
$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}.$$
The trignometric functions are *not bounded* on $\mathbb{C}$.

3. The geometric series
$$\sum_{n=0}^{\infty} z^n$$
converges in $D_1(0)$ (for $|z| < 1$).

4. The alternating harmonic series
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$$
converges for $|z| < 1$. It also converges for $z = 1$ (by Leibniz Criterion) but diverges for $z = -1$, where it becomes the harmonic series.

1.6 Complex line integrals

Definition 1.6.1: Some basic definitions

A **parametrised curve** in $\mathbb{C}$ is a continuous function

$$\gamma : [a, b] \longrightarrow \mathbb{C}$$

where $[a, b] \subset \mathbb{R}$ is a closed interval.

A **smooth curve** is a curve $\gamma : [a, b] \to \mathbb{C}$ with $\gamma(t) = x(t) + iy(t)$, for which $\gamma'(t) = x'(t) + iy'(t)$ exists for all $t \in [a, b]$ and $\gamma'(t)$ is continuous with $\gamma'(t) \neq 0$ for all $t \in [a, b]$. Here

$$\gamma'(a) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{\gamma(a + h) - \gamma(a)}{h}.$$

A **piecewise smooth curve** is a curve $\gamma : [a, b] \to \mathbb{C}$ which is continuous and there exist points $a = a_0 < a_1 < \cdots < a_n = b$ such that γ is smooth on each $[a_i, a_{i+1}]$.

A curve is called **closed** if $\gamma(a) = \gamma(b)$.

A curve is called **simple** if it is not self intersecting i.e. $\gamma(t) \neq \gamma(s)$ unless $s = t$ or $s = a, t = b$.

$\tilde{\gamma} : [c, d] \to \mathbb{C}$ is called a **reparametrisation** of $\gamma : [a, b] \to \mathbb{C}$ if there exists a function $\sigma : [c, d] \to [a, b]$ which is bijective and $\sigma'(t) > 0$ (orientation preserving) for all $t \in [c, d]$ and $\tilde{\gamma} = \gamma \circ \sigma$.

There are some elementary methods to change or combine curves

1. Change of orientation: If $\gamma : [a, b] \to \mathbb{C}$. The reverse path is given by $-\gamma$ or γ^-, where $-\gamma : [a, b] \to \mathbb{C}, t \mapsto \gamma(b + a - t)$.

2. If $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ two paths such that $\gamma_1(b_1) = \gamma_2(a_1)$, then the **concatenation** or the sum of the two paths γ_1, γ_2 is a path

$$\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \longrightarrow \mathbb{C},$$

$$(\gamma_1 + \gamma_2) = \begin{cases} \gamma_1(t) & \text{if } a_1 \leq t \leq b_1 \\ \gamma_2(t - b_1 + a_2) & \text{if } b_1 \leq t \leq b_1 + b_2 - a_2. \end{cases}$$

Example 1.6.2

1. Given $z_1, z_2 \in \mathbb{C}$, then the line segment between them is

$$\begin{aligned} \gamma : [a,b] &\longrightarrow \mathbb{C} \\ t &\longmapsto (1-t)z_1 + tz_2. \end{aligned}$$

2. Let $\gamma : [0,4] \to \mathbb{C}$ be defined by

$$\gamma(t) := \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 + i(t-1) & \text{if } 1 \le t \le 2 \\ (3-t) + i & \text{if } 2 \le t \le 3 \\ i(4-t) & \text{if } 3 \le 4. \end{cases}$$

   This is a concatenation of 4 curves to form a square.

3. A circle with centre z_0 and radius r has the paramatrisation

$$\begin{aligned} \gamma : [0,2\pi] &\longrightarrow \mathbb{C} \\ t &\longmapsto z_0 + re^{it}. \end{aligned}$$

Next we want to define complex path (line) integrals. Recall, if $g : [a,b] \to \mathbb{R}$ continuous then it is Riemann-integrable:

$$\int_a^b g(t)\mathrm{d}t.$$

If $h : [a,b] \to \mathbb{C}$, we can define the integral as the integral of the real and imaginary parts:

$$\int_a^b h(t)\mathrm{d}t = \int_a^b h_1(t)\mathrm{d}t + i\int_a^b h_2(t)\mathrm{d}t$$

where $h(t) = h_1(t) + ih_2(t)$.

Definition 1.6.3: Path Integral

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path and $f : \Omega \to \mathbb{C}$ is a complex valued function which is defined and continuous on γ. We define the **integral of f along γ** by

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

As long as we can show that the integral is independent of the parametrisation of

γ, this gives a well-defined definition for the path integral. Indeed, one can prove that for $\tilde{\gamma} : [c, d] \to \mathbb{C}$ where $\tilde{\gamma} = (\gamma \circ \sigma)$ for some $\sigma : [c, d] \to [a, b]$, then

$$\int_{\tilde{\gamma}} f(z)\mathrm{d}z = \int_{\gamma} f(z)\mathrm{d}z.$$

Proposition 1.6.4: Basic properties of path integrals

Let $f, g : \Omega \to \mathbb{C}$ continuous functions on $\gamma, \gamma_1, \gamma_2$, where γ, γ_1 and γ_2 are piecewise smooth curves $\gamma, \gamma_1, \gamma_2 : [a, b] \to \mathbb{C}$. Then,

1. *The line integral is linear, that is, if $a, b \in \mathbb{C}$, then*

$$\int_{\gamma} (af + bg)\mathrm{d}z = a \int_{\gamma} f \,\mathrm{d}z + b \int_{\gamma} g \,\mathrm{d}z.$$

2. *If γ^- is γ with the reverse orientation, then*

$$\int_{\gamma} f(z)\mathrm{d}z = -\int_{\gamma^-} f(z)\mathrm{d}z.$$

3. $$\int_{\gamma_1+\gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z.$$

4. *We have*

$$\left| \int_{\gamma} f(z)\mathrm{d}z \right| \leq \sup_{z \in \gamma} |f(z)| \mathrm{length}(\gamma)$$

where $\sup_{z\in\gamma} |f(z)| = \sup_{t\in[a,b]} |f(\gamma(t))|$ and

$$\mathrm{length}(\gamma) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |\gamma'(t)|\mathrm{d}t.$$

Proof. 1. Follows directly from the linearity of the Riemann integral.
2. If $\gamma : [a, b] \to \mathbb{C}$, then we define the reverse path $\gamma^- : [a, b] \to \mathbb{C}$ by

$$\gamma^-(t) = \gamma(a + b - t),$$

and its derivative is

$$\gamma^{-'}(t) = -\gamma'(a + b - t).$$

This gives

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_a^b f(\gamma(a + b - t))\gamma'(a + b - t)\,\mathrm{d}t.$$

Now, let $u = a + b - t$, so $\mathrm{d}u = -\mathrm{d}t$. Then we have

$$\begin{aligned}\int_b^a f(\gamma(u))\gamma'(u)\,\mathrm{d}u &= -\int_a^b f(\gamma(u))\gamma'(u)\,\mathrm{d}u \\ &= -\int_\gamma f\,\mathrm{d}z.\end{aligned}$$

3. Follows directly if one writes it out.
4. We have

$$\begin{aligned}\left|\int_\gamma f(z)\mathrm{d}z\right| &= \left|\sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \\ &\le \sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} |f(\gamma(t))||\gamma'(t)|\mathrm{d}t \\ &\le \sup_{t\in[a,b]} |f(\gamma(t))| \sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} |\gamma'(t)|\mathrm{d}t.\end{aligned}$$

□ Lecture 5

Definition 1.6.5: Primitive

A **primitive** of f on Ω is a function F that is holomorphic on Ω and such that $F' = f$.

Theorem 1.6.6: Path integrals and primitives

Let $f : \Omega \to \mathbb{C}$ continuous on $\Omega \subset \mathbb{C}$ (open). If f has a primitive F in Ω and γ is a curve which begins at z_1 and ends at z_2 ($\gamma : [a, b] \to \Omega, \gamma(a) = z_1, \gamma(b) = z_2$).
Then

$$\int_\gamma f(z)\mathrm{d}t = F(z_2) - F(z_1).$$

Proof. Let $F = u(x, y) + iv(x, y)$. First, assume γ is smooth. Define $G : [a, b] \to \mathbb{C}$, $t \mapsto F(\gamma(t)) = F(x(t), y(t))$, where $\gamma(t) = x(t) + iy(t)$. We compute

$$G'(t) = u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) + i(v_x(x(t), y(t)) + v_y(x(t), y(t))y'(t)).$$

The Cauchy Riemann equations say that

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x,$$

hence,

$$
\begin{aligned}
G'(t) &= u_x(x(t),y(t))x'(t) + u_y(x(t),y(t))y'(t) + i(v_x(x(t),y(t)) + v_y(x(t),y(t))y'(t)) \\
&= (u_x(x,y)x' - v_x(x,y)y') + i(v_x(x,y)x' + u_x(x,y)y') \\
&= (u_x(x,y) + iv_x(x,y))x'(t) + (-v_x(x,y) + iv_x(x,y))y'(t) \\
&= (u_x(x,y) + iv_x(x,y))(x'(t) + iy'(t)) \\
&= F_x(x(t),y(t))\gamma'(t) \\
&= F'(x(t),y(t))\gamma'(t) \\
&= f(x(t),y(t))\gamma'(t)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t \\
&= \int_a^b G'(t)\mathrm{d}t \\
&= G(b) - G(a) \\
&= f(\gamma(b)) - f(\gamma(b)) \\
&= f(z_2) - f(z_1)
\end{aligned}
$$

If γ is piecewise smooth, then $[a,b] = [a_1,a_1] \cup \cdots \cup [a_{n-1},a_n]$ $(a = a_0, b = a_n)$ and $\gamma = \gamma_1 + \cdots + \gamma_{n-1}$ each $\gamma_i = \gamma_{[a_i,a_{i+1}]}$. Then,

$$
\begin{aligned}
\int_\gamma f(z)\mathrm{d}z = \int_{\gamma_1} &\mathrm{d}z + \cdots + \int_{\gamma_{n-1}} \mathrm{d}z \\
&= F(\gamma(a_1)) - F(\gamma(a_1) + F(\gamma(a_2)) - F(\gamma(a_1)) + \cdots + F(\gamma(a_n)) - F(\gamma_{a_{n-1}}) \\
&= F(\gamma(b)) - F(\gamma(a)).
\end{aligned}
$$

□

Corollary 1.6.7: Closed curve vs. primitive

If γ is a closed $(\gamma(a) = \gamma(b))$ curve on an open set Ω, f continuous on Ω and has a primitive in Ω, then

$$\int_\gamma f(x)\mathrm{d}z = 0.$$

Corollary 1.6.8: $f' = 0$

If $f : \Omega \to \mathbb{C}$ is holomorphic where Ω is open and connected. If $f' = 0$ then f is constant.

Proof. We want to show that for any two points $z, w \in \Omega, f(z) = f(w)$. Since Ω is open and connected, there is a (polygonal) path $\gamma : [0,1] \to \Omega$ connecting z to w such that $\gamma(0) = z, \gamma(1) = w$. Now since f is holomorphic, f is clearly a primitive of f', hence

$$\begin{aligned}\int_\gamma f' \mathrm{d}z &= f(\gamma(1)) - f(\gamma(0)) \\ &= f(z) - f(w).\end{aligned}$$

But since $f' = 0$ the integral on the left is zero. Hence $f(z) = f(w)$. □

Example 1.6.9

Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, z \mapsto 1/z$. Claim: f has no primitive on $\mathbb{C} \setminus \{0\}$. Indeed, let $\gamma : [0, 2\pi] \to \mathbb{C}, \gamma(t) = \exp(it)$.

$$\int_\gamma f(z)\mathrm{d}z = \int_0^{2\pi} f(e^{it}) \cdot ie^{it}\mathrm{d}t = i\int_0^{2\pi} \mathrm{d}t = 2\pi i \neq 0.$$

This shows that f cannot have a primitive.

Example 1.6.10

What is

$$\int_\gamma z^2 \mathrm{d}z,$$

where $\gamma : [0,1] \to \mathbb{C}, t \mapsto t + \pi i t^2$. $F(z) = z^3/3$ is a primitive of f, hence,

$$\int_\gamma z^2 \mathrm{d}z = F(\gamma(1)) - F(\gamma(0)) = \frac{(1+\pi i)^3}{3}.$$

An alternative solution is to directly use the definition of the line integral:

$$\begin{aligned}\int_\gamma z^2 \mathrm{d}z &= \int_0^1 \left(t + \pi i t^2\right)^2 (1 + 2\pi i t)\mathrm{d}t \\ &= \cdots = \frac{(1+\pi i)^3}{3}.\end{aligned}$$

Chapter 2

Cauchy's Theorem and Its Applications

If f is holomorphic in $\Omega \subset \mathbb{C}$ an open subset and $\gamma \subset \Omega$ a closed curve whose interior is contained in Ω, then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

2.1 Goursat's Theorem

Theorem 2.1.1: Goursat's Theorem

Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \mathbb{C}$ a triangular with $\partial T = \gamma$, whose interior is contained in Ω. Let $f : \Omega \to \mathbb{C}$ a holomorphic function. Then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Proof. Note that a triangle T is a closed curve, which is the union of 3 line segments. If T has corners at $z_1, z_2, z_3 \in \Omega$, with line segments $\alpha_1, \alpha_2, \alpha_3$ such that $T = \alpha_1 + \alpha_2 + \alpha_3 \underset{\text{def}}{=} \langle z_1, z_2, z_3 \rangle$, where

$$\begin{aligned}
\alpha_1(t) &= z_1 + (t-0)(z_2 - z_1), \quad 0 \le t \le 1,\\
\alpha_2(t) &= z_2 + (t-1)(z_3 - z_2), \quad 1 \le t \le 2,\\
\alpha_3(t) &= z_3 + (t-2)(z_1 - z_3), \quad 2 \le t \le 3.
\end{aligned}$$

$\Delta := \{z \in \mathbb{C} : z = t_1 z_1 + t_2 z_2 + t_3 z_3, 0 \le t_1, t_2, t_3, t_1 + t_2 + t_3 = 1\}$, is the smallest convex set containing z_i, where $i \in \{1, 2, 3\}$. Note that $\operatorname{im}(T) = \partial\Delta$. Define

$T^{(n)} = \left\langle z_1^{(n)}, z_2^{(n)}, z_1^{(n)} \right\rangle$ as follows:

1. $T^{(0)} = T$

2. When $T^{(n)}$ is defined as $\left\langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \right\rangle$, then $T^{(n+1)}$ is one of the following 4 (counterclockwise) triangular paths formed by joining the midpoints of the sides of $T^{(n)}$:

$$\begin{aligned}
T_1^{(n+1)} &= \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, z_2^{(n)}, \frac{z_2^{(n)} + z_3^{(n)}}{2} \right\rangle, \\
T_2^{(n+1)} &= \left\langle \frac{z_2^{(n)} + z_3^{(n)}}{2}, z_3^{(n)}, \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle, \\
T_3^{(n+1)} &= \left\langle \frac{z_1^{(n)} + z_3^{(n)}}{2}, z_1^{(n)}, \frac{z_1^{(n)} + z_2^{(n)}}{2} \right\rangle, \\
T_4^{(n+1)} &= \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, \frac{z_2^{(n)} + z_3^{(n)}}{2}, \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle.
\end{aligned}$$

We notice that there will be some cancellations of line integrals along these paths, this means that

$$\int_{T^{(n)}} f \mathrm{d}z = \int_{T_1^{(n+1)}} f \mathrm{d}z + \int_{T_2^{(n+1)}} f \mathrm{d}z + \int_{T_3^{(n+1)}} f \mathrm{d}z + \int_{T_3^{(n+1)}} f \mathrm{d}z.$$

Hence,

$$\left| \int_{T^{(n)}} f(z) \mathrm{d}z \right| \leq 4 \max_{1 \leq i \leq 4} \left| \int_{T_i^{(n+1)}} f(z) \mathrm{d}z \right|.$$

We choose $T^{(n)}$ as one of $T_i^{(n)}$, $1 \leq i \leq 4$, so that

$$\left| \int_{T^{(n)}} f(z) \mathrm{d}z \right| \leq 4 \left| \int_{T^{(n+1)}} f(z) \mathrm{d}z \right|.$$

Let $\Delta^{(n)}$ be the field triangle with $\partial \Delta^{(n)} = T^{(n)}$. By construction,

$$\Delta = \Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \cdots.$$

Put $d_n := \operatorname{diam}\left(\Delta^{(n)}\right) = \sup_{z,w \in \Delta^{(n)}} |z - w|$ and $p_n =$ perimeter of $\Delta^{(n)}$. By construction one can see that

$$d_n = \frac{d_0}{2^n} \qquad \text{and} \qquad p_n = \frac{p_n}{2^n}.$$

Hence $d_n \to 0$ as $n \to 0$. Define

$$I := \left| \int_T f \mathrm{d}z \right| \leq 4^n \left| \int_{T^{(n)}} f \mathrm{d}z \right|.$$

Let z_0 be the unique point such that $z_0 \in \Delta^{(n)}$ for all n (which exists due to nesting principle (Proposition 2.1.2)). Since f is holomorphic at z_0, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z - z_0),$$

with

$$\lim_{z \to z_0} \frac{|E(z)|}{z - z_0} = 0.$$

Clearly $E(z)$ is continuous at z_0 and $\lim_{z \to z_0} E(z) = 0$. We compute

$$\int_{T^{(n)}} f(z) \mathrm{d}z = \underbrace{\int_{T^{(n)}} f(z_0) + f'(z_0)(z - z_0) \, \mathrm{d}z}_{=0} + \int_{T^{(n)}} E(z) \mathrm{d}z.$$

The first term vanishes because

$$g(z) = f(z_0)z + f'(z_0)(z - z_0)^2/2$$

is a primitive of the integrand. Hence we can reduce the estimate to

$$I = \left| \int_T f(z) \mathrm{d}z \right| \leq 4^n \left| \int_{T^{(n)}} f(z) \mathrm{d}z \right| = 4^n \left| \int_{T^{(n)}} E(z) \mathrm{d}z \right|.$$

Not let

$$I_n = \left| \int_{T^{(n)}} E(z) \mathrm{d}z \right|.$$

For given $\varepsilon > 0$, choose $D_\delta(z_0) \subset \Omega$ ($\delta > 0$ small) such that

$$|E(z)| \leq \varepsilon |z - z_0|$$

for all $z \in D_\delta(z_0)$. Because $d_n = \operatorname{diam}\left(\Delta^{(n)}\right) \to 0$, there exists $N \in \mathbb{N}$ such that

$d_n < \delta$ for all $n \geq N$. This implies that $\Delta^{(n)} \subset D_\delta(z_0)$ for all $n \geq N$ and we get

$$\begin{aligned} |I| &= 4^n |I_n| \\ &= 4^n \left| \int_{T^{(n)}} E(z) \mathrm{d}z \right| \\ &\leq 4^n \int_{T^{(n)}} |E(z)||\mathrm{d}z| \\ &\leq 4^n \varepsilon \int_{T^{(n)}} |z - z_0||\mathrm{d}z| \\ &\leq 4^n \varepsilon d_n \int_{T^{(n)}} |\mathrm{d}z| \\ &= 4^n \varepsilon d_n p_n \\ &= 4^n \varepsilon \frac{d_0}{2^n} \frac{p_0}{2^n} = \varepsilon d_0 p_0. \end{aligned}$$

This implies $I = 0$ as $\varepsilon > 0$ is arbitrary. □

Proposition 2.1.2: Nesting principle

If $\Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \cdots$ a sequence of non-empty compact sets in $\mathbb{C}$ with the property diam $\left(\Delta^{(n)}\right) \to 0$ as $n \to \infty$. Then there exists a unique point $z_0 \in \mathbb{C}$ such that $z_0 \in \Delta^{(n)}$ for all $n \in \mathbb{N}$.

Lecture 6

Corollary 2.1.3: Integral over a rectangle

If f is holomorphic in an open set Ω which contains a solid rectangle $\mathcal{R}$ and its interior with $\partial\mathcal{R} = R$. Then

$$\int_R f(z) \mathrm{d}z = 0.$$

Theorem 2.1.4: Goursat 2nd version

If f is continuous on $\Omega \subset \mathbb{C}$ and holomorphic on $\Omega \setminus \{z_0\}$ for some $z_0 \in \Omega$, then

$$\int_{\partial R} f(z) \mathrm{d}z = 0,$$

where R is any rectangle whose interior is also in Ω.

Proof. Assume without loss of generality that $z_0 \in \mathcal{R}$. Let n be a positive integer.

Divide the big rectangle into n^2 congruent rectangles. Once again we have

$$\int_R f(z)\mathrm{d}z = \sum_{k=1}^{n}\sum_{\ell=1}^{n}\int_{R_{k\ell}} f(z)\mathrm{d}z.$$

If $z_0 \notin R_{k\ell}$ then $\int_{R_{k\ell}} f(z)\mathrm{d}z = 0$ by Corollary 2.1.3. If $z_0 \in \mathcal{R}_{k\ell}$, then

$$\left|\int_{R_{k\ell}} f\mathrm{d}z\right| \leq M \cdot (\text{perimeter of } R_{k\ell}) = M\frac{L}{n},$$

where $M = \max_{z\in\mathcal{R}} |f|$ and $L =$ perimeter of R. The point z_0 can belong to at most four rectangles, hence

$$\left|\int_R f(z)\mathrm{d}z\right| \leq \frac{4ML}{n},$$

let $n \to \infty$ we get

$$\int_R f(z)\mathrm{d}z = 0.$$

□

2.2 Cauchy's Theorem in a disk

Theorem 2.2.1: Holomorphic functions vs. primitive

A holomorphic function in an open disc D has a primitive in that disc.

Theorem 2.2.2: Integral over rectangles and primitives

Let D be an open disc in $\mathbb{C}$ and f a continuous function on D with the property that

$$\int_{\partial\mathcal{R}} f(z)\mathrm{d}z = 0$$

for every closed rectangle $\mathcal{R} \subset D$, whose sides are parallel to the coordinate axis. Then f has a primitive in D.

We will use this theorem to prove Cauchy's Theorem.

Theorem 2.2.3: Cauchy's Theorem in a disc

Suppose $D \subset \mathbb{C}$ is an open disc, f holomorphic in D (or more generally, continuous in D and holomorphic in $D \setminus \{z_0\}$ for some $z_0 \in D$). Then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

for every closed (piecewise smooth) curve in D.

Proof. Suppose f is continuous in D and holomorphic in $D \backslash \{z_0\}$. Then by Goursat's Theorem 2.1.4,

$$\int_R f(z)\mathrm{d}z = 0$$

for every rectangle, in particular also the ones whose sides are parallel to the coordinate axis. Then by Theorem 2.2.2, f has a primitive in D, hence, by Corollary 1.6.7

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

□

Proof of Theorem 2.2.2. Let $z_0 = x_0 + iy_0$ be the centre of the disc D. Let $z \in D$ with $z = x + iy \neq z_0$. Let $z_1 = x + iy_0$ and $z_2 = x_0 + iy$. By assumption,

$$\int_R f(z)\mathrm{d}z = 0,$$

so we have

$$0 = \int_{z_0}^{z_2} f(z)\mathrm{d}z + \int_{z_2}^{z} f(z)\mathrm{d}z + \int_{z}^{z_1} f(z)\mathrm{d}z + \int_{z_1}^{z_0} f(z)\mathrm{d}z,$$

where the upper and lower limits are to be understood as begin- and endpoints of the line integral along the edges of the triangle. We define $F : D \to \mathbb{C}$ as follows. For $z \in D$

$$F(z) := \int_{z_0}^{z_2} f(w)\mathrm{d}w + \int_{z_2}^{z} f(w)\mathrm{d}w.$$

Using the previous equation we have

$$F(z) = \int_{z_0}^{z_1} f(w)\mathrm{d}w + \int_{z_1}^{z} f(w)\mathrm{d}w.$$

We parametrise the line segments to get

$$F(z) = i\int_{y_0}^{y} f(x_0 + it)\mathrm{d}t + \int_{x_0}^{x} f(t + iy)\mathrm{d}t \tag{2.1}$$

or

$$F(z) = \int_{x_0}^{x} f(t + iy_0)\mathrm{d}t + i\int_{y_0}^{y} f(x + it)\mathrm{d}t. \tag{2.2}$$

The Fundamental Theorem of Calculus states that if $g : (a-r, a+r) \to \mathbb{C}$ continuous, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x g(t)\mathrm{d}t = g(x).$$

We now use this result and differentiate (2.1) and (2.2) with respect to x and y respectively to get

$$\frac{\partial F}{\partial x}(z) = f(x + iy)$$

and

$$\frac{\partial F}{\partial y}(z) = if(x + iy),$$

both of which are continuous by assumption, hence F is C^1 as a real function. If we write $F = U + iV$, we get

$$\begin{aligned} f(z) &= \frac{\partial F}{\partial x} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} \\ &= -i\frac{\partial F}{\partial y} = -i\left(\frac{\partial U}{\partial y} + i\frac{\partial V}{\partial y}\right) = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}. \end{aligned}$$

This shows that F satisfies the Cauchy Riemann equations. We can apply Theorem 1.4.2 to obtain that F is holomorphic and $F'(z) = \frac{\partial}{\partial x}F(z) = f(z)$. Therefore, we conclude that f has a primitive F. □

Lecture 7

Corollary 2.2.4: Integral over a circle

f is holomorphic in an open set containing a circle and its interior, then

$$\int_C f(z)\mathrm{d}z = 0.$$

Theorem 2.2.5: Cauchy's Theorem

If f is holomorphic in Ω, γ is a closed curve whose interior is also in Ω, then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Remark. Corollary 2.2.4 is in fact valid whenever we can defined the interior of a contour unambiguously and construct polygonal paths in an open neighbourhood of the contour and its interior. Any closed curve where the notion of interior is obvious is called a "toy contour". They are useful because we can deform every loop into a circle which is homotopic to it, the domains are indeed simply connected.

Remark. Note that Cauchy's Theorem does not say anything about the integral of f over arbitrary closed curves. Indeed for $f(z) = 1/z$, $\Omega = \mathbb{C}\setminus\{0\}$, and $\gamma = C_1(0)$, we have

$$\int_{C_1(0)} \frac{1}{z}\mathrm{d}z = 2\pi i \neq 0,$$

because $\mathbb{C} \setminus \{0\}$ is not simply connected.

2.3 Applications of Cauchy's Theorem

Now we will see some concrete examples of how to apply Cauchy's Theorem on integrals.

Example 2.3.1

Using Cauchy's Theorem on a disc one can show

$$\int_R \frac{1}{z - z_0}\mathrm{d}z = 2\pi i,$$

where R is any rectangle with centre at z_0.

Example 2.3.2

The Fresnel integrals are given by:

$$\int_0^\infty \cos\left(x^2\right)\mathrm{d}x = \int_0^\infty \sin\left(x^2\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

Proof. Note that $\exp\left(ix^2\right)$ has real and imaginary parts $\cos x^2$ and $\sin x^2$. If we can prove

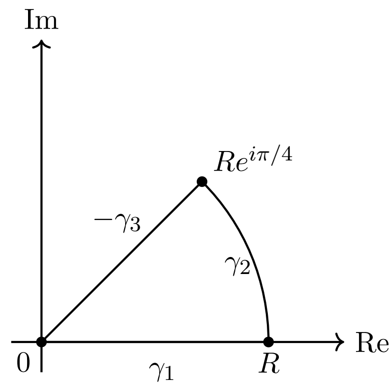
$$\int_0^\infty \exp\left(ix^2\right)\mathrm{d}x = (1+i)\frac{\sqrt{2\pi}}{4},$$

then we are done. In this way, we are led naturally to

$$f(z) = \exp\left(iz^2\right),$$

which is holomorphic in $\mathbb{C}$. The contour we choose is the one along the sides of a sector of $\pi/4$: first through radius γ_1, then through the arc γ_2, and through the radius again γ_3:

$$\begin{aligned}
\gamma_1 &= t && 0 \le t \le R, \\
\gamma_2 &= Re^{it} && 0 \le \frac{\pi}{4}, \\
\gamma_3 &= te^{i\pi/4} && 0 \le t \le R.
\end{aligned}$$



Let γ be the concatenation of the three paths. By Theorem 2.2.3 (Cauchy's Theorem in a disc) we get

$$\int_\gamma \exp(iz^2)\mathrm{d}z = 0,$$

which translates into

$$\begin{aligned}
\int_0^R e^{it^2}\mathrm{d}t &= -\int_{\gamma_2} e^{iz^2}\mathrm{d}z + \int_{\gamma_3} e^{iz^2}\mathrm{d}z \\
&= -\int_{\gamma_2} e^{iz^2}\mathrm{d}z + e^{i\pi/4}\int_0^R e^{-t^2}\mathrm{d}t.
\end{aligned}$$

We claim that

$$\left|\int_{\gamma_2} e^{iz^2}\mathrm{d}z\right| \le \frac{\pi(1-e^{-R^2})}{4R}.$$

Indeed, we notice that the sine function is concave for $x \in [0, \pi/2]$:

$$(\sin x)'' = -\sin x \le 0 \qquad \forall x \in [0, \pi/2],$$

which means that the graph of sine lies above the line joining $(0, 1)$ and $(\pi/2, 1)$. In other words

$$\sin(x) \ge \frac{2}{\pi}x \qquad \forall x \in [0, \pi/2],$$

which is equivalent to

$$\sin(2t) \geq \frac{4}{\pi}t \qquad \forall t \in [0, \pi/4].$$

This implies that

$$\begin{aligned}
\left|\int_{\gamma_2} e^{iz^2}\mathrm{d}z\right| &= \left|\int_0^{\frac{\pi}{4}} \exp\big(i(Re^{it})^2\big)iRe^{it}\mathrm{d}t\right| \\
&\leq \int_0^{\frac{\pi}{4}} \left|\exp\big(iR^2(\cos(2t) + i\sin(2t))\big)\right| R|ie^{it}|\mathrm{d}t \\
&= R\int_0^{\frac{\pi}{4}} \exp\big(-R^2\sin(2t)\big)\left|\exp\big(iR^2\cos(2t)\big)\right|\mathrm{d}t \\
&= R\int_0^{\frac{\pi}{4}} \exp\big(-R^2\sin(2t)\big)\mathrm{d}t \\
&\leq R\int_0^{\frac{\pi}{4}} \exp\left(-R^2 \cdot \frac{4}{\pi}t\right)\mathrm{d}t \\
&= -\frac{\pi}{4R}\exp\left(-R^2 \cdot \frac{4}{\pi}t\right)\Bigg|_0^{\pi/4} \\
&= \frac{\pi(1 - e^{-R^2})}{4R}.
\end{aligned}$$

Hence the claim is valid and this integral converges to 0 as $R \to \infty$. As $R \to 0$, what we have left is only

$$\int_0^\infty e^{ix^2}\mathrm{d}x = \lim_{R\to 0} e^{i\pi/4}\int_0^R e^{-it^2}\mathrm{d}t = \frac{(1+i)\sqrt{2}}{2}\int_0^\infty e^{-t^2}\mathrm{d}t = (1+i)\frac{\sqrt{2\pi}}{4}.$$

□

2.4 Cauchy Integral Formulas

Theorem 2.4.1: Cauchy Integral Formula

Suppose f is holomorphic in an open set $\Omega \subset \mathbb{C}$ that contains the closure of a disc D. If C denotes the boundary of the circle on D with positive orientation, then

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\mathrm{d}w = f(z)$$

for all $z \in D$.

Proof. Let $z \in D_r(z_0)$, $\overline{D_r(z_0)} \subset \Omega$. There exists $\varepsilon > 0$ such that $D_{r+\varepsilon}(z_0) \subset \Omega$.

For $w \in D_{r+\varepsilon}(z_0)$, we define

$$g(w) := \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z. \end{cases}$$

$g : D_{r+\varepsilon}(z_0) \to \mathbb{C}$ is continuous and away from z it is holomorphic. By Theorem 2.2.3 (Cauchy's Theorem in a disk) applied to g we have

$$\int_{C_r(z_0)} g(w)\mathrm{d}w = 0,$$

i.e.

$$\int_{C_r(z_0)} \frac{f(w)-f(z)}{w-z}\mathrm{d}w = 0,$$

which implies that

$$\int_{C_r(z_0)} \frac{f(w)}{w-z}\mathrm{d}w = f(z)\int_{C_r(z_0)} \frac{\mathrm{d}w}{w-z}.$$

We will be done if we can show

$$\int_{C_r(z_0)} \frac{\mathrm{d}w}{w-z} = 2\pi i.$$

We define γ to paramterize the circle in a special way: First let $\gamma(t) := z_0 + re^{it}$. Another parametrisation is $\tilde{\gamma}(s) = z + \rho(s)e^{is}$. Here t changes with s and $\rho : [0, 2\pi] \to \mathbb{R}, \rho(s) = |\gamma(t(s)) - z| = |\tilde{\gamma}(s) - z|$. The derivative of $\tilde{\gamma}$ is

$$\tilde{\gamma}'(s) = \rho'(s)e^{is} + i\rho(s)e^{is}.$$

Hence,

$$\begin{aligned}\int_{C_r(z_0)} \frac{\mathrm{d}w}{w-z} &= \int_0^{2\pi} \frac{\rho'(s)e^{is} + i\rho(s)e^{is}}{\rho(s)e^{is}}\mathrm{d}s \\ &= \underbrace{\int_0^{2\pi} \frac{\rho'(s)}{\rho(s)}\mathrm{d}s}_{\ln|\rho(s)|\big|_0^{2\pi}=0} + i\underbrace{\int_0^{2\pi} \mathrm{d}s}_{2\pi i} \\ &= 2\pi i,\end{aligned}$$

since $\rho(2\pi) = 0$. $\square$

Remark. Cauchy Integral Formula says that the value of f in D is only determined by its values on the boundary C.

Example 2.4.2

We will show that $e^{-\pi x^2}$ is its own Fourier transform. For a function $f : \mathbb{R} \to \mathbb{C}$ which is Riemann integrable on every $[a, b]$ and

$$\int_{-\infty}^{\infty} |f(t)| \mathrm{d}t < +\infty,$$

i.e. converges, the Fourier transform is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \mathrm{d}x.$$

We claim that

$$\hat{f}(\xi) = e^{-\pi \xi^2},$$

for $f(x) = e^{-\pi x^2}$. We want to show that $e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} \mathrm{d}x$, which is equivalent to

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x^2} e^{\pi \xi^2} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} \mathrm{d}x.$$

We know that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \mathrm{d}x = 1.$$

We want to consider a rectangle with vertices at $(-R, 0)$, $(R, 0)$, $(-R, \xi) = -R + i\xi$ and $(R, \xi) = R + i\xi$, and let γ_R the closed path along its sides. It is clear that

$$\int_{\gamma_R} f(z) \mathrm{d}z = 0.$$

Note that on γ_1, which is the horizontal side on the real axis,

$$\int_{-R}^{R} e^{-\pi x^2} \mathrm{d}x \to 1 \qquad \text{as } R \to \infty.$$

The same happens for γ_3. Consider γ_2,

$$\int_{\gamma_2} f(z) \mathrm{d}z = \int_0^{\xi} f(R + iy) \mathrm{d}y,$$

hence,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &= \left|\int_0^{\xi} e^{-\pi(R^2+2iRy-y^2}\mathrm{d}y\right| \\ &\le \xi \sup_{0\le y\le\xi}\left|e^{-\pi R^2}\cdot e^{-2\pi iRy}\cdot e^{\pi y^2}\right| \\ &\le Ce^{-\pi R^2}.\end{aligned}$$

As $R \to \infty$ the two vertical integrals go to 0.

Lecture 8

Theorem 2.4.3: Condition power series expansion

Suppose f is holomorphic in an open set Ω. Let $z_0 \in \Omega$ and $r > 0$, such that $\overline{D_r(z_0)} \subset \Omega$. Then f has a power series expansion at z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D_r(z_0)$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w.$$

Proof. Fix $s \in (0, r)$, let $C_s(z_0)$ be the circle of radius s, centred at z_0. Then by Theorem 2.4.1 (Cauchy Integral Formula),

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}\mathrm{d}w.$$

for all $z \in D_s(z_0)$ and γ is the positive oriented parametrisation of $C_s(z_0)$. We rewrite the expression above,

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left(\frac{1}{1-\frac{z-z_0}{w-z_0}}\right).$$

We are integrating on γ for some $w \in \gamma$, so

$$\left|\frac{z-z_0}{w-z_0}\right| = \frac{|z-z_0|}{s} < 1.$$

Hence we can write the expression as a geometric series

$$\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n$$

for $w \in \gamma$ and $z \in D_s(z_0)$. The convergence is uniform due to the bound

$$\frac{|z-z_0|}{s} < 1,$$

which does not depend on w. Substituting this expression into the equation we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_\gamma f(w) \sum_{n=0}^{\infty} (z-z_0)^n (w-z_0)^{-(n+1)} \mathrm{d}w \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \int_\gamma f(w)(w-z_0)^{-(n+1)} \mathrm{d}w \\
&= \sum_{n=0}^{\infty} a_n (z-z_0)^n,
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w.$$

Hence f is the sum of the power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z \in D_s(z_0).$$

The derivative can be given by term-wise differentiation,

$$\begin{aligned}
f'(z) &= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-z_0)^n
\end{aligned}$$

Evaluating at $z = z_0$ gives

$$\begin{aligned}
a_0 &= f(z_0) \\
1a_1 &= f'(z_0) \\
&\vdots \\
n!a_n &= f^{(n)}(z_0).
\end{aligned}$$

The coefficients are independent of s, hence the series converges in the whole disc.

□

Remark. Note this shows that if f is differentiable at z_0, in fact it is differentiable infinitely many times at z_0.

Corollary 2.4.4: Cauchy Integral Formula for derivatives

If $f \in \mathcal{H}(\Omega)$, then f is infinitely many times differentiable in Ω. If $z_0 \in \Omega$ such that for $r > 0$, $\overline{D_r(z_0)} \subset \Omega$, then for all $z \in D_r(z_0)$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w.$$

This is the Cauchy Integral Formula for derivatives.

Proof. We use induction on n.

$n = 0$: this is the Cauchy Integral Formula, which we proved.

Assume $n-1$:

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n}} \mathrm{d}w$$

Choose h small such that z and $z+h$ stays away from the boundary circle and we have

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{h} \left(\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n} \right) \mathrm{d}w.$$

Now if you have $a^n - b^n$, then

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}).$$

with

$$a = \frac{1}{w-z-h} \quad \text{and} \quad b = \frac{1}{w-z},$$

then

$$\frac{a-b}{h} \to \frac{1}{(w-z)^2}$$

and

$$a^{n-1} + a^{n-2}b + \cdots + b^{n-1} \to \frac{n}{(w-z)^{n-1}}$$

for $h \to 0$ and the convergence is indeed uniform. Hence we get

$$\lim_{h \to 0} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_\gamma f(w) \left(\frac{1}{(w-z)^2} \frac{n}{(w-z)^{n-1}} \right) \mathrm{d}w,$$

which is exactly equal to

$$\frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w.$$

Again, here γ denotes $C_r(z_0)$. □

Corollary 2.4.5: Cauchy Inequality

If $f \in \mathcal{H}(\Omega)$, then f is infinitely many times differentiable in Ω. If $z_0 \in \Omega$ such that for $r > 0$, $\overline{D_r(z_0)} \subset \Omega$, then for all $z \in D_r(z_0)$ we have

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \|f\|_{C_r(z_0)}}{r^n},$$

where $\|f\|_{C_r(z_0)} = \sup_{|w-z_0|=r} |f(w)|$.

Proof. This follows from

$$|a_n| = \left| \frac{f^n(z_0)}{n!} \right| = \left| \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w \right|$$

and then

$$\begin{aligned} |n! a_n| = |f^n(z_0)| &= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} ire^{i\theta} \mathrm{d}\theta \right. \\ &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + re^{i\theta})|}{r^n} \mathrm{d}\theta \\ &\leq \frac{n!}{2\pi} \frac{||f||_C}{r^n}. \end{aligned}$$

□

An immediate corollary from these results is the remarkable Liouville's theorem.

Theorem 2.4.6: Liouville's Theorem

If $f \in \mathcal{H}(\mathbb{C})$ and bounded, then f is constant.

Proof. Since $\mathbb{C}$ is connected, to show that f is constant, it suffices to show that $f' \equiv 0$. Since f is holomorphic everywhere, by Corollary 2.4.5 (Cauchy Inequality) we have

$$|f'(z_0)| \leq \frac{\|f\|_{C_r(z_0)}}{r}.$$

But f is bounded, say $f(z) < M$, then

$$|f'(z_0)| < \frac{M}{r}$$

for all $r > 0$. Let $r \to \infty$ to get $f'(z) = 0$. □

Corollary 2.4.7: Fundamental Theorem of Algebra

Every polynomial $P(z) = a_n z^n + \cdots + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$, counting multiplicities, and

$$P(z) = a_n(z - w_1) \cdots (z - w_n).$$

Proof. Idea: We first show P has a root in $\mathbb{C}$. Suppose not, let $Q(z) = 1/P(z)$, then $Q(z) \in \mathcal{H}(\mathbb{C})$. The idea is to show that Q is bounded in $\mathbb{C}$, which will imply $Q(z)$ is a constant, hence $P(z)$ is constant, contradiction. □

Lecture 9

Remark. In Liouville's Theorem, the assumption that f is holomorphic on all of $\mathbb{C}$ is essential: Let $\Omega = \{z \in \mathbb{C} | \operatorname{Re}(z) > 0\}$ and $f : \Omega \to \mathbb{C}$,

$$f(z) = \frac{1}{z+1}$$

is bounded but not constant.

Our next goal is to show that if Ω is open, connected $f \in \mathcal{H}(\Omega)$ and f vanishes on an infinite set $\mathcal{Z}$ of distinct points, with a limit point $z_0 \in \Omega \setminus \mathcal{Z}$ then $f \equiv 0$.

Remark. Holomorphic functions can have infinitely many zeros. For example $f(z) = \cos(z)$, which has zeroes at $z = (2k+1)\pi/2$. But we will see that the zeroes are isolated, which means that we can find at each zero a neighbourhood of it such that there are no other zeroes in this neighbourhood.

It can also happen that f has no zeroes, like the exponential function $f(z) = e^z$.

Definition 2.4.8: Limit Point

$z_0 \in \mathbb{C}$ is a **limit point** of a set Ω, if there exists a sequence $(z_n)_{n=0}^{\infty} \subset \Omega \setminus \{z_0\}$ such that $\lim_{n\to\infty} z_n = z_0$. Hence for $r > 0$, $\Omega \cap (D_r(z_0) \setminus \{z_0\}) \neq \varnothing$.

Definition 2.4.9: Order of null

Let Ω be open, $z_0 \in \Omega$ and $f \in \mathcal{H}(\Omega)$. The **order of zero of f at** z_0, denoted by $\operatorname{ord}_{z_0} f$, $n_{z_0}(f)$, or $\nu_{z_0}(f)$, is either ∞ if $f^{(k)}(z_0) = 0$ for all $k \geq 0$, or the smallest positive integer k such that $f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$, but $f^{(k)}(z_0) \neq 0$. If $f(z_0) \neq 0$, then $k = 0$. In symbols,

$$\operatorname{ord}_{z_0} f = \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}.$$

Proposition 2.4.10:

Let Ω open, $z_0 \in \Omega$ and $f \in \mathcal{H}(\Omega)$. Then

1. *If $\operatorname{ord}_{z_0} f = \infty$ then $f(z) = 0$ for any $z \in D_r(z_0)$ and $D_r(z_0) \subset \Omega$ is (i.e. f is locally zero).*
2. *If $\operatorname{ord}_{z_0} f \neq \infty$, then there exist a disk $D_r(z_0) \subset \Omega$, a unique $h \in \mathcal{H}(D_r(z_0))$, and a unique $n \in \mathbb{N}$ such that*

$$f(z) = (z - z_0)^n h(z)$$

   for all $z \in D_r(z_0)$, where $h(z_0) \neq 0$, $n = \operatorname{ord}_{z_0} f$.
3. *For any $f, g \in \mathcal{H}(\Omega)$ there is $\operatorname{ord}_{z_0}(f + g) \geq \min\{\operatorname{ord}_{z_0} f, \operatorname{ord}_{z_0} g\}$, $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g$.*

Proof.

Proof of 1. By a previous theorem, since holomorphic functions are analytic, there exists $r > 0$ such that for all $z \in D_r(z_0) \subset \Omega$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for all $z \in D_r(z_0)$. Hence if $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$ then $f \equiv 0$ in $D_r(z_0)$.

Proof of 2. If $\operatorname{ord}_{z_0} f \neq \infty$, then by definition there exists $k \geq 0$ such that $f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$. Again as above we can use the power series expansion of f to claim that there exists $r \geq 0$ with $D_r(z_0) \subset \Omega$, such that for all $z \in D_r(z_0)$,

$$f(z) = \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \sum_{n=k+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

which leads to

$$f(z) = (z - z_0)^k \underbrace{\left(\underbrace{\frac{f^{(k)}(z_0)}{k!}}_{\neq 0} + \sum_{m=1}^{\infty} \frac{f^{(m+k)}(z_0)}{(m+k)!} (z - z_0)^m \right)}_{\neq 0 \text{ at } z = z_0}.$$

Now let

$$h(z) := \sum_{m=0}^{\infty} \frac{f^{(m+k)}}{(m+k)!} (z - z_0)^m$$

for all $z \in D_r(z_0)$, where $h \in \mathcal{H}(D_r(z_0))$ and $h(z_0) \neq 0$. Note as $h \in \mathcal{H}(D_r(z_0))$, it is also continuous. $h(z_0) \neq 0, h$ together with continuity implies the existence of ε with $0 < \varepsilon < r$ such that $h(z) \neq 0$ in $D_\varepsilon(z_0)$.

Moreover, h and $\operatorname{ord}_{z_0} f = 0$ is unique. Assume not, then

$$f(z) = (z - z_0)^n h(z) = (z - z_0)^m g(z),$$

with h, g holomorphic and non-zero at z_0. If $m > n$, we get $h(z) = (z - z_0)^{m-n} g(z)$ for $z \neq z_0$. But now taking limit $z \to z_0$ we get $h(z_0) = 0$ unless $m - n = 0$. Hence $m = n$ and $g(z) = h(z)$.

Proof of 3. For the third part, note that for any k, $(f+g)^{(k)}(z_0) = f^{(k)}(z_0) + g^{(k)}(z_0)$, so $\operatorname{ord}_{z_0}(f+g) \geq \min\{\operatorname{ord}_{z_0} f, \operatorname{ord}_{z_0} g\}$.

For the order of fg at z_0, we write $f(z) = (z - z_0)^{\operatorname{ord}_{z_0} f} h_1(z)$ and $g(z) = (z - z_0)^{\operatorname{ord}_{z_0} g} h_2(z)$, hence

$$fg = (z - z_0)^{\operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g} h_1(z) h_2(z),$$

where $h_1(z)h_2(z) \neq 0$ at z_0. This shows that $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g$. □

Theorem 2.4.11:

Let $\Omega \subset \mathbb{C}$ open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$. Assume $f(z_0) = 0$, $\operatorname{ord}_{z_0} f \geq 1$. If $\operatorname{ord}_{z_0} f \neq \infty$, then there exists $\delta > 0$ such that $f(z) \neq 0$ if $z \in D_\delta(z_0) \setminus \{z_0\}$.

Proof. By previous proposition there exists $r > 0$ such that for all $z \in D_r(z_0)$, $f(z) = (z - z_0)^n h(z)$ with $h(z_0) \neq 0$. Using the continuity if necessary we can go to a similar disk $D_\delta(z_0)$, $0 < \delta < r$, such that $h(z) \neq 0$ on $D_\delta(z_0)$, which means $(z - z_0)^n \neq 0$ on $D^*_\delta(z_0)$. Hence $f(z) \neq 0$ on $D^*_\delta(z_0)$. □

Theorem 2.4.12:

Let Ω open and connected, $f \in \mathcal{H}(\Omega)$. Let $\mathcal{Z}$ be an infinite set with a limit point $z_0 \in \Omega$, $z_0 \notin \mathcal{Z}$. If $f(z) = 0$ for all $z \in \mathcal{Z}$, then $f \equiv 0$.

Corollary 2.4.13: Principle of analytic continuation

Suppose $f, g \in \mathcal{H}(\Omega)$, Ω open and connected and $f(z) = g(z)$ for all $z \in U \subseteq \Omega$, U open, nonempty (or more generally for all sequence of distinct points in $\mathcal{Z}$), with limit point in Ω, then $f = g$.

Proof. We apply Theorem 2.4.12 to $f - g$. □

Note if $U \subset \Omega$ open and $U \neq \varnothing$, then for $z_0 \in U$, there exists $D_r(z_0)$ such that $D_r(z_0) \subset U$. The sequence $\left(z_0 + \frac{r}{n+1}\right)_{n=1}^{\infty} \subset D_r(z_0) \subset U$ has a limit point $z_0 \in \Omega \setminus \left(z_0 + \frac{r}{n+1}\right)_{n=0}^{\infty}$. Now by assumption f agrees with g on U and in particular, on this sequence of distinct points, hence we can apply Theorem 2.4.12 to obtain $f - g \equiv 0$.

Remark.

1. Corollary 2.4.13 is is called principle of analytic continuation because if $f \in \mathcal{H}(\Omega)$, Ω open and connected and $\Omega \subset \tilde{\Omega}$ open and connected, then there is at most one $\tilde{f} \in \mathcal{H}(\tilde{\Omega})$ such that $f(z) = \tilde{f}(z)$ for all $z \in \Omega$. $\tilde{f}$ is called the analytic continuation of f and is unique if it exists.

2. The assumption that Ω is connected is essential. If $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_i \neq \varnothing$ open, $\Omega_1 \cap \Omega_2 = \varnothing$, one can defined $f, g : \Omega \to \mathbb{C}$ with $f|_{\Omega_1} = 1$, $f|_{\Omega_2} = 0$, $g = 0$. f and g coincide on Ω_2 but not on Ω_1.

3. The condition that limit point of zeroes is in Ω is also crucial. Consider

$$\begin{aligned} f : \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sin\left(\frac{\pi}{z}\right). \end{aligned}$$

   We know that

$$\sin\frac{\pi}{z} = \frac{\exp\left(\frac{i\pi}{z}\right) - \exp\left(\frac{-i\pi}{z}\right)}{2i}$$

   and

$$f(i) = \frac{e^{\pi} - e^{-\pi}}{2i} \neq 0.$$

f has a sequence $f\left(\frac{1}{n}\right) = \sin(\pi n) = 0$ for all $n \geq 1$, but $\lim \frac{1}{n} = 0 \notin \Omega$.

Example 2.4.14

Let $f(z) = \sum_{n=0}^{\infty} z^n$ on $D_1(0) = \Omega$, let $\tilde{f}(z) = \frac{1}{1-z}$ with $\tilde{\Omega} = \mathbb{C} \setminus \{0\}$, but the two functions do not agree everywhere.

We will prove the following theorem which proves Theorem 2.4.12.

Theorem 2.4.15:

Let Ω be open connected, $f \in \mathcal{H}(\Omega)$. Then the following are equivalent

1. $f \equiv 0$.
2. *There exists a point $a \in \Omega$ such that $f^{(n)}(a) = 0$ for all $n \geq 0$.*
3. $\{z \in \Omega : f(z) = 0\}$ *has a limit point in* Ω.

Proof. Clearly 1. $\implies$ 3. We will show 3. $\implies$ 2. $\implies$ 1.

3. $\implies$ 2. Define $\mathcal{Z} := \{z \in \Omega : f(z) = 0\}$ which has a limit point $a \in \Omega$. Let $r > 0$ such that $D_r(a) \subset \Omega$, where a is a limit point of $\mathcal{Z}$. By definition there exists $(z_n)_{n=0}^{\infty} \in \mathcal{Z} \setminus \{a\}$ such that $\lim_{n\to\infty} z_n = a$. But then $0 = \lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(a)$. We claim that $f^{(n)}(a) = 0$ for all $n \geq 0$. We suppose this is not the case, then there exists $n \geq 0$ such that $f(a) = \cdots = f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$. f is analytic, it means that there exists $D_r(a) \subset \Omega$ where f is equal to its power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Similar to the proof of Proposition 2.4.10, it follows from the power series representation that $f(z) = (z-a)^n g(z)$ for a function $g(a) \neq 0$ and analytic in $D_r(a)$. Again using continuity we have $g(z) \neq 0$ for all $z \in D_\varepsilon(a)$ for some $\varepsilon > 0$. Then, $f(z) \neq 0$ for all $z \in D_z^*(a)$, so $\mathcal{Z} \cap D_\varepsilon^*(a) = \varnothing$, which contradicts that a is a limit point.

2. $\implies$ 1. Let $A = \{z \in \Omega : f^{(n)}(z) = 0 \,\forall\, n \in \mathbb{N}_0\}$ be the set of points such that derivatives of all orders are zero. By assumption this set is non-empty. We will show that $A = \Omega$ which will show $f \equiv 0$ and hence 1. holds. Recall, for an open set Ω connected means that the only both open and closed sets are $\varnothing$ and Ω. Hence to show that $A = \Omega$, since it is not empty, we need to show A is open and closed.

Claim 1: A is open.

Let $c \in A$ and $r > 0$ such that $D_r(c) \subset \Omega$, and

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

for all $z \in D_r(c)$, with

$$a_n = \frac{f^{(n)}(c)}{n!} = 0 \quad \text{since } c \in A.$$

Hence, $f(z) = 0$ for every $z \in D_r(c)$ and $D_r(c) \subset A$, thus A is open.

Claim 2: A is closed.

We want to show that if $(z_k)_{n=0}^{\infty}$ is a sequence of points in a and $\lim_{k\to\infty} z_k = c \in \Omega$, then $c \in A$, i.e. A contains all its limit points[1]. We have that $f^{(n)}(z_k) = 0$ since $z_k \in A$, and since $f^{(n)}$ are all continuous,

$$0 = \lim_{k\to\infty} f^{(n)}(z_k) = f^{(n)}\left(\lim_{k\to\infty} z_k\right) = f^{(n)}(c).$$

Hence $c \in A$. □

Corollary 2.4.16: Identity Theorem

$f, g \in \mathcal{H}(\Omega)$ open and connected, $\Omega \neq \varnothing$. Then the following are equivalent.

1. *$f = g$.*
2. *There exists $a \in \Omega$ such that $f^{(n)}(a) = g^{(n)}(a)$ for all $n \geq 0$.*
3. *$\{z \in \Omega : f(z) = g(z)\}$ has a limit point in Ω.*

Lecture 10

Remark. The Identity Theorem makes it clear that the identities we have for $\sin(x)$, $\cos(x)$ and e^x can be extended to the complex values. For example, $\sin(z)$, $\cos(x)$ are entire and for $z = x \in \mathbb{R}$ we have

$$\sin^2(x) + \cos^2(x) = 1.$$

Let $f(z) = \sin^2(z) + \cos^2(z)$ and $g(z) = 1$. Since f and g agree on the real line, they have to agree on all of $\mathbb{C}$, so we have

$$\sin^2(z) + \cos^2(z) = 1 \quad \forall z \in \mathbb{C}.$$

[1] Strictly speaking, c is not a limit point because it may not be distinct from z_k for all $k \in \mathbb{N}$. However, on the one hand, we know that A is closed if and only if all convergent sequences in A converge in A, which is shown here; on the other hand, A containing all its limit points also implies it is closed.

Remark. The Identity Theorem has a natural extension to function of 2 variables. We first consider the following example.

$$\exp(x+y) = \exp(x)\exp(y)$$

for all $x, y \in \mathbb{R}$. We first conclude that

$$\exp(z+y) = \exp(z)\exp(y)$$

$\forall\, z \in \mathbb{C}$, fixed $y \in \mathbb{R}$, y arbitrary. Applying the Identity Theorem once more we get

$$\exp(z+w) = \exp(z)\exp(w)$$

for all $z, w \in \mathbb{C}$. Now we have a look at the general result.

Let $\Omega \in \mathbb{C}$ be open and connected, containing a set $U \subset \Omega$ which itself contains a sequence of points with limit point also in U. Let $F(z, w)$ be a function defined for $z, w \in \Omega$ such that $F(z, w)$ is analytic in z for fixed w and vice versa. If $F(z, w) = 0$ whenever $z, w \in U$, then $F(z, w) = 0$ for all $z, w \in \Omega$.

Theorem 2.4.17: $fg = 0$

Let $f, g \in \mathcal{H}(\Omega)$, where Ω is open and connected. If $fg = 0$ then either $f \equiv 0$ or $g \equiv 0$.

Proof. Suppose $fg = 0$ and $f \neq 0$. We want to show $g \equiv 0$. Since $f \neq 0$, there exists $a \in \Omega$ such that $f(a) \neq 0$. By continuity of f there exists $\varepsilon > 0$ such that $D_\varepsilon(a) \subset \Omega$ and $f(z) \neq 0$ in $D_\varepsilon(a)$. By assumption, $f(z)g(z) = 0$ for all $z \in D_\varepsilon(a) \subset \Omega$. Since $f \neq 0$, it implies $g = 0$ for all $z \in D_\varepsilon(a)$. We can now apply the Identity Theorem to g and conclude $g \equiv 0$. □

Remark. Let $\Omega \subset \mathbb{C}$ be an open set, then analytic functions on Ω, $\mathcal{H}(\Omega)$ is a ring. What Theorem 2.4.17 says is: if Ω is connected then $\mathcal{H}(\Omega)$ is an integral domain. Lecture 11

Recall that Goursat's Theorem 2.1.1 states that if $f : \Omega \to \mathbb{C}$ is a holomorphic function and $T \subset \Omega$ is a triangle whose interior is contained in Ω, then

$$\int_T f(z)\mathrm{d}z = 0.$$

A converse to this is the following theorem.

Theorem 2.4.18: Morera's Theorem

Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ a continuous function. Assume that for any open disc $D \subset \Omega$ and any triangle T whose interior is also in D we have

$$\int_T f(z)\mathrm{d}z = 0,$$

then f is holomorphic on Ω.

Proof. Let $z_0 \in \Omega$ and $D_r(z_0) \subset \Omega$. Let $z \in D_r(z_0)$, we define

$$F(z) = \int_{\gamma_z} f(w)\mathrm{d}w,$$

where

$$\begin{aligned} \gamma_z : [0,1] &\longrightarrow \mathbb{C} \\ t &\longmapsto z_0(1-t) + z(t), \end{aligned}$$

denoted by $\gamma_z = [z_0, z]$. Then for small h with $z + h \in D_r(z_0)$ there is

$$F(z+h) - F(z) = \int_{\sigma_z} f(w)\mathrm{d}w,$$

where $\sigma_z = [z, z+h]$, since by assumption

$$\int_{T(z_0,z,z+h)} f(w)\mathrm{d}w = 0.$$

Using continuity of f we can show

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

More specifically, we proceed as follows.

$$\begin{aligned} F(z+h) - F(z) &= \int_{[z,z+h]} f(w) + f(z) - f(z)\mathrm{d}w \\ &= f(z) \underbrace{\int_{[z,z+h]} \mathrm{d}w}_{=h} + \underbrace{\int_{[z,z+h]} f(z) - f(w)\mathrm{d}w}_{\leq h \sup_{w\in[z,z+h]} |f(z)-f(w)|} \end{aligned}$$

Since f is continuous, $\sup_{w\in[z,z+h]} |f(z) - f(w)| \to 0$ as $h \to 0$. This gives

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \to 0 \quad \text{as} \quad h \to 0.$$

Hence F is holomorphic and $F' = f$ for $z \in D_r(z_0)$. It follows that f is also holomorphic being the derivative of the holomorphic function F. □

2.5 Sequences of holomorphic functions

In Real Analysis one sees that if f_n converges pointwise to f and f_n is continuous, then f is not necessarily continuous. This can be corrected if we have uniform convergence: the uniform limit of a sequence of continuous functions $(f_n)_{n=0}^{\infty}$ is also continuous. Also the line integrals

$$\int_\gamma f_n(x)\mathrm{d}s \to \int_\gamma f(x)\mathrm{d}s.$$

But even if f_n 's are differentiable, f does not need to be.

Recall: A sequence $f_1, f_2, \cdots : \Omega \to \mathbb{C}$ is called uniformly convergent to f if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $z \in \Omega$

$$|f_n(z) - f(z)| < \varepsilon.$$

Definition 2.5.1: Locally Uniformly Convergent

Let $\Omega \subset \mathbb{C}$ open and $f_n : \Omega \to \mathbb{C}$ a sequence of functions. $(f_n)_{n=0}^{\infty}$ is called **locally uniformly convergent** or uniformly convergent on compact sets if one of the following equivalent conditions are satisfied.

1. For all $a \in \Omega$ there exists $\varepsilon > 0$ such that $D_\varepsilon(a) \subset \Omega$ such that $f_n|_{D_\varepsilon(a)}$ converges uniformly.
2. For every compact set $K \subset \Omega$, $f_n|_K$ converges uniformly.

Proposition 2.5.2: Uniform convergence and continuity

Let $(f_n)_{n=0}^{\infty}$ be a sequence of continuous functions $f_n : \Omega \to \mathbb{C}$. If $(f_n)_{n=0}^{\infty}$ converges uniformly on a compact set to f, then f is continuous.

Proof. Exercise. □

Theorem 2.5.3: Uniform convergence and holomorphicity

Let $(f_n)_{n=0}^{\infty}$ be a sequence of holomorphic functions on $\Omega \subset \mathbb{C}$, Ω open. If $(f_n)_{n=0}^{\infty}$ converges uniformly in every compact subset of Ω to f, then f is also holomorphic.

Proof. By Proposition 2.5.2, f is continuous and we will show that it is also holomorphic by using Morera's Theorem 2.4.18, i.e., it is enough to show that

$$\int_T f(w)\mathrm{d}w = 0$$

for any disc D and triangle T, whose interior is contained in D. Let $D = D_r(z_0) \subset \Omega$ and T any triangle contained in D. We have that

$$\int_T f_n(w)\mathrm{d}w = 0$$

for all $n \in \mathbb{N}$ by Goursat's Theorem 2.1.1. Since $(f_n)_{n=0}^{\infty}$ converges to f on compact sets,

$$\begin{aligned}\left|\int_T f_n(z)\mathrm{d}z - \int_T f(z)\mathrm{d}z\right| &\leq \int_T |f_n(z) - f(z)|\mathrm{d}z \\ &\leq \sup_{z\in T} |f_n(z) - f(z)| \cdot \text{length}(T).\end{aligned}$$

Since $f_n \to f$ uniformly on compact sets and in particular on T, we have

$$\lim_{n\to\infty} \int_T f_n(z)\mathrm{d}z = \int_T f(z)\mathrm{d}z = 0,$$

and by Morera's Theorem 2.4.18, f is holomorphic. □

Theorem 2.5.4: Uniform convergence of the derivative

Let $(f_n)_{n=0}^{\infty}$ be a sequence of holomorphic functions on $\Omega \subset \mathbb{C}$ such that $f_n \to f$ uniformly on every compact set of Ω. Then $f_n' \to f'$ uniformly on compact sets of Ω.

Proof. Let $z_0 \in \Omega$ and $\overline{D_r(z_0)} \subset \Omega$. By assumption $f_n \to f$ uniformly on $\overline{D_r(z_0)}$. For $s > r$ such that $D_s(z_0) \subset \Omega$ and let $\sigma := (r+s)/2$. For $z \in \overline{D_r(z_0)} \subset D_\sigma(z_0)$, by Cauchy Integral Formula for derivatives 2.4.4 we have

$$f'(z) = \frac{1}{2\pi i}\int_{C_\sigma(z_0)} \frac{f(w)}{(w-z)^2}\mathrm{d}w.$$

Similarly we have

$$f_n'(z) = \frac{1}{2\pi i}\int_{C_\sigma(z_0)} \frac{f_n(w)}{(w-z)^2}\mathrm{d}w.$$

We want to show that the latter integral converges to the first one. It holds that

$$\begin{aligned}\left|f_n'(z) - f(z)\right| &= \left|\frac{1}{2\pi i}\int_{C_\sigma(z_0)} \frac{f_n(w)-f(w)}{(w-z)^2}\mathrm{d}w\right| \\ &\leq \frac{1}{2\pi}\cdot 2\pi\sigma \sup_{w\in C_\sigma(z_0)}\left|\frac{f_n(w)-f(w)}{(w-z)^2}\right|.\end{aligned}$$

In the integral above, $w \in C_\sigma(z_0)$ implies $|w - z_0| = \sigma$, and $z \in D_r(z_0)$ implies $|z - z_0| \leq r$. This leads to

$$|w-z| = |w - z_0 + z_0 - z| \geq \big||w-z_0| - |z-z_0|\big| \geq \sigma - r.$$

So from the inequality above we get

$$\begin{aligned}\left|f_n'(z) - f(z)\right| &\leq \sigma \sup_{w\in C_\sigma(z_0)}\left|\frac{f_n(w)-f(w)}{(w-z)^2}\right| \\ &\leq \underbrace{\frac{\sigma}{(\sigma-r)^2}\sup_{w\in C_\sigma(z_0)}\left|f_n(w)-f(w)\right|}_{\to 0 \text{ as } n\to\infty}.\end{aligned}$$

This shows that $f_n'(z) \to f'(z)$ uniformly for $z \in \overline{D_r(z_0)}$. Since any compact set is contained in a finite union of such discs we get the result. □

Remark. These theorems are often used to prove holomorphicity of functions defined by infinite series. Let $(f_n)_{n=0}^\infty$ be a sequence of holomorphic functions. Define

$$F(z) = \sum_{n=0}^\infty f_n(z)$$

and

$$S_N = \sum_{n=0}^N f_n(z)$$

the sequence partial sums. If $(S_N)_{n=0}^\infty$ converges uniformly on compact sets, then $\lim_{N\to\infty} S_N = F$ is also holomorphic.

The following theorem is useful.

Theorem 2.5.5: Weierstrass M. Test

Let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions and $U \subset \Omega$ a non-empty set. Suppose there exists a sequence of real numbers $M_n \geq 0$ such that

$$|f_n(z)| \leq M_n$$

for all $n \in \mathbb{N}$ and $z \in U$, and that

$$\sum_{n=0}^{\infty} M_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} f_n$$

converges absolutely and uniformly on U.

Proof. For each fixed $z \in U$, we have the inequality

$$|f_n(z)| \leq M_n.$$

Since the series $\sum_{n=1}^{\infty} M_n$ converges, by the comparison test, it follows that the series

$$\sum_{n=1}^{\infty} |f_n(z)|$$

also converges. Therefore, the series $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely for each $z \in U$. Next, we show that the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on U. To do this, we use the Cauchy criterion for uniform convergence. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, there exists an integer $N \geq 1$ such that for all $p, q \geq N$,

$$\sum_{n=p}^{q} M_n < \epsilon.$$

Now, for all $z \in U$ and for all $p, q \geq N$, we have

$$\left| \sum_{n=p}^{q} f_n(z) \right| \leq \sum_{n=p}^{q} |f_n(z)| \leq \sum_{n=p}^{q} M_n.$$

Thus for all $z \in U$, we get

$$\left| \sum_{n=p}^{q} f_n(z) \right| < \epsilon.$$

This shows that the sequence of partial sums of $\sum_{n=1}^{\infty} f_n(z)$ satisfies the Cauchy criterion uniformly on U. Therefore, the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on U. □

Corollary 2.5.6: Uniform convergence and series

Let

$$S_N = \sum_{n=1}^{\infty} f_n(z).$$

If $(S_N(z))_{N=1}^{\infty}$ *converges uniformly on a compact set then* $\lim_{n\to\infty} S_N = \sum_{n=1}^{\infty} f_N(z)$ *is also holomorphic.*

Example 2.5.7

For $s \in \mathbb{C}$ with $s = \sigma + it$ and $n \in \mathbb{N}$ the function $s \mapsto n^s := \exp(s \log n)$ is an analytic function on $\mathbb{C}$. It holds that

$$|n^s| = |e^{(\sigma+it)\log n}| = e^{\sigma \log(n)} = n^{\sigma}.$$

This leads to the Riemann ζ-function which we investigate below.

Proposition 2.5.8: Riemann ζ-function

The series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely and uniformly on every half plane $U_\delta := \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 1 + \delta\}$ *with* $\delta > 0$ *and is holomorphic in* $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.

Proof. For each $\delta > 0$, if $\mathrm{Re}(s) = \sigma \geq 1+\delta > 1$ then the series is uniformly bounded by

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty.$$

Hence,

$$\begin{aligned} |\zeta(s)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty. \end{aligned}$$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges uniformly on every half plane $\operatorname{Re} s \geq 1 + \delta > 1$ for all $\delta > 0$. Now, let

$$S_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}.$$

For each $N \in \mathbb{N}$, S_N defines a holomorphic function on $\operatorname{Re}(s) > 1$. Also, every compact subset of $\operatorname{Re}(s) > 1$ is contained in one of the half-planes U_δ. Combining these two with the result that S_N converges to $\zeta(s)$ uniformly on every U_δ (in particular on every compact subset), we conclude that the limit ζ is also holomorphic. □

Remark. The Riemann ζ-function given as an infinite sum converges for $\operatorname{Re}(s) > 1$. The Riemann Hypothesis says that if $\zeta(s) = 0$, then $\operatorname{Re}(s) = 1/2$, given it is not a trivial zero.

Example 2.5.9

For $z \in \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ we define the (single-variable) Θ-function

$$\begin{aligned}\Theta(z) &:= \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} \\ &= 1 + 2\sum_{n=1}^{\infty} e^{2\pi i n^2 z}.\end{aligned}$$

We can consider

$$\begin{aligned}\Theta^4(z) &= \sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4} e^{2\pi i (n_1^2+n_2^2+n_3^2+n_4^2) z} \\ &= \sum_{m=1}^{\infty} a(m) e^{2\pi i m z},\end{aligned}$$

where $a(m) = \#\{(n_1, \dots, n_4) \in \mathbb{Z}^4 : n_1^2 + n_2^2 + n_3^2 + n_4^2 = m\}$. This can be used to prove Lagrange's Four-square Theorem.

We want to show that the Θ-function converges uniformly on every set of the from $\mathbb{H}_\delta := \{z \in \mathbb{C} : \operatorname{Im}(z) \geq \delta\}$ for $\delta > 0$ and defines a holomorphic function on $\mathbb{H}$.

Proof. Let $z \in \mathbb{H}_\delta, z = x + iy, y \geq \delta > 0$. We then have

$$\begin{aligned} \left|e^{2\pi i n^2 z}\right| &= \left|e^{2\pi i n^2 x}\right| \left|e^{-2\pi n^2 y}\right| \\ &= e^{-2\pi n^2 y} \leq e^{-2\pi n y} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $y \geq \delta$,

$$e^{-2\pi y} \leq e^{-2\pi\delta} < 1,$$

we have

$$\left|\sum e^{2\pi i n^2 z}\right| \leq \sum e^{-2\pi n\delta} < \infty.$$

Hence $\sum e^{2\pi i n^2 z}$ converges uniformly on $\mathbb{H}_\delta$. Since every compact set of $\mathbb{H}$ of contained in $\mathbb{H}_\delta$ for some $\delta > 0$, the sum converges uniformly on compact sets and hence $\Theta(z)$ is a holomorphic functoin on $\mathbb{H}$. □

There is a relation between

$$\zeta(s) \longleftrightarrow \Theta(z)$$

which is given by

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty (\Theta(it) - 1)t^{s/2}\frac{dt}{t} \quad \text{for } \operatorname{Re}(s) > 1.$$

One can also show that in fact

$$\int_0^\infty (\Theta(it) - 1)t^{s/2}\frac{dt}{t}$$

makes sense also for every $s \in \mathbb{C} \setminus \{0, 1\}$. This gives the analytic continuation of $\zeta(s)$. The trivial zeros of ζ are at the negative integers, which correspond to places where the Γ-function blows up but the integral has no singularity. Lecture 12

Many special functions of mathematics are defined using integrals of the form

$$f(z) = \int_a^b F(z, t)\mathrm{d}t.$$

For example the Γ-function

$$\Gamma(z) := \lim_{M\to\infty}\int_{1/M}^M e^{-t}t^z\frac{\mathrm{d}t}{t} = \lim_{M\to\infty}\int_{1/M}^M t^{z-1}e^{-t}\mathrm{d}t.$$

Theorem 2.5.10: Integral and holomorphicity

Let $\Omega \subset \mathbb{C}$ be open and $I = [a, b] \subset \mathbb{R}$, closed and bounded interval. Let $F : \Omega \times I \to \mathbb{C}$ be a function with the following properties:

1. *F is continuous on $\Omega \times I$,*
2. *For each $t_0 \in I$, the function $f_{t_0}(z) = F(z, t_0)$ is holomorphic.*

Then the function

$$f(z) := \int_a^b F(z, t)\mathrm{d}t$$

is holomorphic on Ω.

Proof. The idea is to use Riemann sums to approximate the integral. Let's consider the standard Riemann sum

$$f_n(z) = \left(\frac{b-a}{n}\right) \sum_{j=0}^{n-1} F\left(z, a + \left(\frac{b-a}{n}\right) j\right).$$

We now have that $f_n(z)$ is a finite sum of holomorphic functions and hence holomorphic. We want to show that $f_n(z)$ converges uniformly on compact subsets of Ω and then use Theorem 2.5.3. Let K be a compact subset, then $F : K \times I \to \mathbb{C}$ is uniformly continuous, which means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(z_i, t_i) \in K \times I$, if $|z_1 - z_2| < \delta$ and $|t_1 - t_2| < \delta$ then

$$|F(z_1, t_1) - F(z_2, t_2)| < \frac{\varepsilon}{b-a}.$$

Now let N be large enough so that $\frac{b-a}{n} < \delta$. We claim that for all $z \in K$ we have

$$f_n(z) - f(z) = \sum_{j=0}^{n-1} \int_{a+j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} \left[F\left(z, a + j\left(\frac{b-a}{n}\right)\right) - F(z, t)\right] \mathrm{d}t.$$

This is because by definition

$$\begin{aligned} f(z) &= \int_a^b F(z, t)\mathrm{d}t \\ &= \int_a^{a+\frac{b-a}{n}} F(z, t)\mathrm{d}t + \int_{a+\frac{b-a}{n}}^{a+2\frac{b-a}{n}} F(z, t)\mathrm{d}t + \cdots + \int_{a+(n-1)\frac{b-a}{n}}^{b} F(z, t)\mathrm{d}t, \end{aligned}$$

and f_n is defined by

$$f_n(z) = \left(\frac{b-a}{n}\right)\sum_{j=0}^{n-1} F\left(z, a + j\left(\frac{b-a}{n}\right)\right)$$
$$= \sum_{j=0}^{n-1} \underbrace{\int_{a+j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} F\left(z, a + j\left(\frac{b-a}{n}\right)\right) \mathrm{d}t}_{\text{independent of } t}$$

We see that the integrals are independent from t. For $t \in \left[a + j\frac{b-a}{n}, a + (j+1)\frac{b-a}{n}\right]$.

$$\left| t - \left(a + j\left(\frac{b-a}{n}\right)\right)\right| \leq \frac{b-a}{n} < \delta$$

The other variable stays the same and satisfies $0 = |z - z| < \delta$. Hence

$$\left| F\left(z, a + j\left(\frac{b-a}{n}\right)\right) - F(z,t)\right| > \frac{\varepsilon}{b-a}.$$

Using uniform continuity of F it follows that

$$|f_n(z) - f(z)| \leq \frac{\varepsilon}{b-a}\sum_{j=0}^{n-1}\frac{b-a}{n} = \varepsilon.$$

for all $z \in K$. Hence $f_n \to f$ uniformly on K and f is holomorphic. □

Remark. With more work, one can also show that

$$f'(z) = \int_a^b F'(z,t)\mathrm{d}t \quad \forall\, z \in \Omega.$$

Example 2.5.11

The J-Bessel function

$$J_n(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin(t)}e^{-int}\mathrm{d}t$$

is a holomorphic function.

Chapter 3

Meromorphic Functions and the Logarithm

Lecture 13

The goal is to extend Cauchy's Theorem to functions with singularities. We will first talk about singularities of a function that are isolated. We will see that there are three types of singularities: we consider the functions

$$\frac{\sin z}{z}, \ \frac{1}{z}, \ \text{and } e^{1/z},$$

all of which have singularities. In the first case, the limit $z \to 0$ is finite, so we can extend the function analytically at 0, this is called a removable singularity. For the second function, the limit $z \to 0$ is unbounded, we say that is has a pole at 0. In the third case, the behaviour of the function is more complicated and 0 is called an essential singularity.

We will also see that if $z = z_0$ is a pole, so a singularity of type

$$\frac{1}{(z - z_0)^k},$$

then in fact in a neighbourhood of z_0 f will look like

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z),$$

where G is a holomorphic function in $D_r^*(z_0)$. a_{-1} is called the residue of f at $z = z_0$. This will lead us eventually to the Residue Theorem.

3.1 Singularities and poles

Definition 3.1.1: Isolated Singularity

Let $z_0 \in \mathbb{C}$, z_0 is called a (possible) isolated singularity of a function f if there exists $r > 0$ such that in the punctured disc $D_r^*(z_0) = D_r(z_0) \setminus \{z_0\}$, f is holomorphic.

Example 3.1.2

Let $f(z) = \tan \frac{1}{z}$ has singularities at $\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \ldots, 0$. Note that 0 is not an isolated singularity of f, the others are isolated. We will be only interested in isolated singularities.

Definition 3.1.3: Removable Singularity

An isolated singularity z_0 of a function $f \in \mathcal{H}(\Omega \setminus \{z_0\}$ is called removable if f is holomorphically extendable to all of Ω, i.e., there exists $F : \Omega \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in \Omega \setminus \{z_0\}$.

We have the following theorem.

Theorem 3.1.4: Riemann's Continuation Theorem

Let $z_0 \in \Omega \subset \mathbb{C}$, then the following for a function $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ are equivalent.

1. *f is holomorphically extendable to Ω.*
2. *f is continuously extendable to Ω.*
3. *f is bounded in a deleted neighbourhood of z_0, that is, there exists $r > 0$ so that f is bounded in $D_r^*(z_0)$.*
4. $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

Proof. Notice that the implications $1 \implies 2 \implies 3 \implies 4$ are elementary. We are left to show that $4 \implies 1$. Introduce the function

$$h(z) := \begin{cases} zf(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

and set $k(z) = zh(z)$. By assumption 4. h and k are holomorphic in $\mathbb{C} \setminus \{0\}$ and continuous in the whole complex plane $\mathbb{C}$. Since $k(z) = k(0) + zh(z)$ we deduce that k is complex differentiable in zero and hence holomorphic in $\mathbb{C}$. By Taylor representation of holomorphic functions, $k(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ for coefficients

$a_0, a_1, \dots \in C$. Since $k(0) = 0$ and $k'(0) = 0$ we deduce that $k(z) = a_2 z^2 + a_3 z^3 + \dots = z^2(a_2 + a_3 z + a_4 z^2 + \dots)$. Now, recalling that $k(z) = z^2 f(z)$ for $z \neq 0$ we deduce that $g(z) := a_2 + a_3 z + a_4 z^2 + \dots$ is indeed an holomorphic extension of f in $\mathbb{C}$. □

Remark. This means that when we assumed in Cauchy's Theorem that the function was holomorphic except at one point and continuous everywhere is not really a weaker assumption, because we can extend the function analytically at this point and integrate the new holomorphic function.

As a corollary we have:

Theorem 3.1.5: Riemann's Theorem on removable singularities

Suppose f is holomorphic in an open set Ω except possibly at $z_0 \in \Omega$. If f is bounded in $D_r^*(z_0)$ for some $D_r(z_0) \subset \Omega$, then z_0 is a removable singularity of f.

Proof. This is just 3. $\implies$ 1. of Theorem 3.1.4 (Riemann's Continuation Theorem). □

Example 3.1.6

$f(z) = \frac{\sin z}{z}$ with $z_0 = 0$. Using 4. from Theorem 3.1.4 (Riemann's Continuation Theorem) we have

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \sin z = 0.$$

Also note $\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1$. Or, note that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$ and hence

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

The power series above gives the holomorphic extension of $\frac{\sin z}{z}$.

If f does not have a removable singularity at z_0, then f is not bounded near z_0. We can ask whether its unboundedness is similar to $\frac{1}{(z-z_0)^n}$ for some n? In other words, we can ask whether

$$(z-z_0)^n f(z)$$

is bounded near z_0 for sufficiently large n?

Definition 3.1.7: Pole, order of pole and simple poles

If such an $n \in \mathbb{N}$ exists, then z_0 is called a pole of f and the natural number $m = \min\{n \in \mathbb{N} : (z-z_0)^n f(z) \text{ is bounded near } z_0\}$ is called the order of the pole of f at z_0. Poles of first order are called simple poles.

Example 3.1.8

For example

$$\frac{1}{(z-z_0)^m}$$

has a pole of order m at $z = z_0$.

Recall from proposition 2.4.10, f is holomorphic in a connected open set Ω, f has a zero at $z_0 \in \Omega$. Then there exists $r > 0$ such that $D_r(z_0) \subset \Omega$, $g \in \mathcal{H}(D_r(z_0))$, and a unique n such that

$$f(z) = (z-z_0)^n g(z)$$

for all $z \in D_r(z_0)$ with $g(z_0) \neq 0$.

Theorem 3.1.9:

For $m \in \mathbb{N}$ with $m \geq 1$ and $f \in \mathcal{H}(\Omega \setminus \{z_0\})$, the following are equivalent.

1. *f has a pole of order m at z_0.*
2. *There exists $r > 0$ and $g \in \mathcal{H}(D_r(z_0))$ such that $g(z_0) \neq 0$ and $f(z) = (z-z_0)^{-m} g(z)$ for all $z \in D_r^*(z_0)$.*
3. *There exists $r > 0$ such that $D_r(z_0) \subset \Omega$ and $h \in \mathcal{H}(D_r(z_0))$ such that $h(z) \neq 0$ for all $z \in D_r^*(z_0)$, h has a zero of order m at z_0 with*

$$f(z) = \frac{1}{h(z)}$$

for all $z \in D_r^*(z_0)$.

Proof. $\boxed{1. \implies 2.}$ f having a pole of order m at z_0 means that $(z-z_0)^m f(z)$ is bounded near z_0, and m is minimal. Theorem 3.1.4 (Riemann's Theorem) says that there exists $g \in \mathcal{H}(D_r(z_0))$ such that $g(z) = (z-z_0)^m f(z)$ whenever $z \neq z_0$. If $g(z_0)$ were zero, then it would imply by the Proposition 17 that $g(z) = (z-z_0)h(z)$ where

h is holomorphic in $D_r(z)$. Consequently this will give

$$h(z) = (z - z_0)^{m-1} f(z)$$

is bounded near z_0, which will contradict the minimality of m. Hence $g(z_0) \neq 0$. It follows that

$$f(z) = (z - z_0)^{-m} g(z)$$

for $z \in D_r^*(z_0)$ and $g \in \mathcal{H}(D_r(z_0))$ with $g(z_0) \neq 0$.

$\boxed{2. \implies 3.}$ Suppose there exists $g \in \mathcal{H}(D_r(z_0))$ such that $g(z_0) \neq 0$, for which $f(z) = (z - z_0)^{-m} g(z)$ for all $z \in D_r^*(z_0)$. Let

$$h(z) = \frac{(z - z_0)^m}{g(z)}$$

for all $z \in D_r(z_0)$. If necessary, we move to a smaller disk in which g does not vanish. Then, $h(z) \neq 0$ for all $z \in D_r^*(z_0)$ and $h \in \mathcal{H}(D_r(z_0))$. We also get that

$$\frac{1}{h(z)} = g(z)(z - z_0)^{-m} = f(z)$$

for all $z \in D_r^*(z_0)$. Note that h has a zero of order m at z_0 because

$$h(z) = (z - z_0)^m \frac{1}{g(z)}$$

and g is a non-vanishing holomorphic function in $D_r(z_0)$.

$\boxed{3. \implies 1.}$ Suppose there exists $r > 0$ such that $D_r(z_0) \subset \Omega$ and $h \in \mathcal{H}(D_r(z_0))$ such that $h(z) \neq 0$ for all $z \in D_r^*(z_0)$ $h(z)$ has a zero of order m at z_0 and

$$f(z)) = \frac{1}{h(z)} \quad \forall\, z \in D_r^*(z_0).$$

Since h has a zero of order m at z_0, there exists $g \in \mathcal{H}(D_r(z_0))$ such that

$$h(z) = (z - z_0)^m g(z)$$

and there exists $s > 0$ such that $g(z) \neq 0$ for all $z \in D_s(z_0) \subset D_r(z_0)$. Since g is holomorph and non vanishing $1/g$ is holomorph in $D_s(z_0)$. But then

$$f(z) = \frac{1}{h(z)} = (z - z_0)^m \frac{1}{g(z)} \quad \forall\, z \in D_s^*(z_0)$$

would imply that $z - z_0^m(z) = \frac{1}{g(z)}$ is holomorphic on $D_s^*(z_0)$ and has a holomorphic extension $1/g(z)$ in $D_s(z_0)$ ($1/g$ is holomorphic onn$D_s(z_0)$ since $g \neq 0$ on $D_s(z_0)$). By Riemann's extendability theorem $(z - z_0)^m f(z)$ is bounded in a neighbourhood of z_0. Moreover

$$(z - z_0)^{m-1} f(z) = \left(\frac{1}{g(z)}\right)\left(\frac{1}{z - z_0}\right)$$

is not bounded since

$$\frac{1}{g(z_0)} \neq 0 \text{ and } \frac{1}{z - z_0} \to \infty \text{ as } z \to z_0.$$

Hence m is minimal and f has a pole of order m at z_0. □

Example 3.1.10

1. $f(z) = \frac{1}{e^z - 1}$ has a pole of order 1 at $z = 0$. We can see this using

   $$\frac{1}{f(z)} = e^z - 1 = z\left(1 + \frac{z}{2!} + \cdots\right).$$

   $1/f$ has a zero of order 1, hence f has a pole of order 1. f also has simple poles at $z = 2n\pi i$.
2. $f(z) = \frac{z}{z^2 - 1}$ has a zero at $z = 0$ and a pole of order 1 at $z = \pm 1$.

   $$f(z) = \frac{1}{z - 1}\left(\frac{z}{z + 1}\right) = (z - 1)^{-1} g(z)$$

   We can do the same for $z + 1$.

Theorem 3.1.11: Expansion of f at a pole

If f has a pole of order n at $z = z_0$, then there exists $r > 0$ such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

for all $z \in D_r^*(z_0)$, where $G \in \mathcal{H}(D_r(z_0))$.

Proof. f has a pole of order n at z_0, so $f(z) = (z - z_n)^{-n} g(z)$ for all $z \in D_r^*(z_0)$ and $g \in \mathcal{H}(D_r(z_0))$. We expand $g(z)$ in a power series

$$g(z) = \sum_{k=0}^{\infty} g^{(k)}(z_0) \frac{(z - z_0)^k}{k!}.$$

For $z \in D_r^*(z_0)$,

$$\begin{aligned} f(z) &= \frac{g(z)}{(z-z_0)^n} \\ &= \frac{1}{(z-z_0)^n}\left(g(z_0) + g'(z_0)(z-z_0) + \cdots + g^{(n)}(z_0)\frac{(z-z_0)^n}{n!} + \cdots\right) \\ &= \frac{g(z_0)}{(z-z_0)^n} + \frac{g'(z_0)}{(z-z_0)^{n-1}} + \cdots + \frac{g^{(n-1)}(z_0)}{(n-1)!(z-z_0)} \\ &\quad + \underbrace{\left(g^{(n)}(z_0)\frac{1}{n!} + g^{(n+1)}(z_0)\frac{z-z_0}{(n+1)!} + \cdots\right)}_{=G(z)}. \end{aligned}$$

□

Remark. In the expansion above, the part

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}$$

is called the **principle part** and $a_{-1} = \operatorname{res}_{z_0} f$ the **residue** of f at z_0. Lecture 14

Remark. If f has a pole of order 1 ($f(z) = \frac{a_{-1}}{z-z_0} + G(z)$), then

$$\lim_{z \to z_0} (z-z_0) f(z) = a_{-1} = \operatorname{res}_{z_0} f.$$

Conversely, if $\lim_{z \to z_0}(z-z_0)f(z)$ exists then $(z-z_0)f(z)$ is bounded in some neighbourhood of z_0. Hence z_0 is a pole of $f(z)$ of order at most 1. If the limit is actually 0, z_0 is a removable singularity.

More generally, we have

Theorem 3.1.12: Method to Calculate res_{z_0}

If f has a pole of order n at z_0, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\big((z-z_0)^n f(z)\big).$$

Proof. We have

$$f(z) = P_{z_0}(z) + G(z)$$

and

$$(z-z_0)^n f(z) = a_{-n} + a_{-(n-1)} + \cdots + a_{-1}(z-z_0)^{n-1} + (z-z_0)^n G(z).$$

□

Example 3.1.13

1.
$$\operatorname{res}_i \frac{1}{z^2+1}$$
$$\lim_{z\to i}(z-i)\frac{1}{z^2+1} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.$$

2.
$$f(z) = \frac{1}{(z^2+1)^2}$$
$$\operatorname{res}_i f = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\frac{(z-i)^2}{(z^2+i)^2} = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+1)^2} = \frac{1}{4i}.$$

Lemma 3.1.14: Residue of f/g

If f, g are holomorphic at z_0 and g has a simple zero at z_0. Then f/g has a simple pole at z_0 and $\operatorname{res}_{z_0} f/g = f(z_0)/g'(z_0)$.

Proof. If g has a zero of then

$$g(z) = (z-z_0)\tilde{g}(z)$$

for some neighbourhood $D_r(z_0)$ with $\tilde{g}(z) \neq 0$ in $D_r(z_0)$. Then

$$\frac{f}{g} = (z-z_0)^{-1}\frac{f(z)}{\tilde{g}(z)}$$

in $D_r^*(z_0)$. Hence f/g has a pole of order 1 at z_0. To calculate the residue we use Theorem 3.1.12

$$\begin{aligned} a_{-1} = \operatorname{res}_{z_0}\frac{f}{g} &= \lim_{z\to z_0}(z-z_0)\frac{f(z)}{g(z)} \\ &= \lim_{z\to z_0} f(z)\frac{(z-z_0)}{g(z)-g(z_0)} \\ &= \frac{f(z_0)}{g'(z_0)}. \end{aligned}$$

□

Example 3.1.15

$$\operatorname{res}_i \frac{z^3}{z^2+1} = -\frac{1}{2}$$

Remark. If $f(z) = P_{z_0}(z) + G(z)$, where P_{z_0} is the principle part and $G(z)$ holomorphic, for all $z \in D_r^*(z_0)$. Let C be a circle centred at z_0 and contained in $D_r(z_0)$. Then

$$\int_C P_{z_0}(z)\mathrm{d}z = \int_C \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}\mathrm{d}z = a_{-1}2\pi i.$$

This is because

$$\int_C \frac{\mathrm{d}z}{z-z_0} = 2\pi i$$

and

$$\int_C \frac{1}{(z-z_0)^n}\mathrm{d}z = 0$$

for $n > 1$ using Corollary 2.4.4 (Cauchy Integral Formula for derivatives). The integral of G vanishes due to Cauchy's Theorem as G is holomorphic.

3.2 The Residue Formula

Theorem 3.2.1: Residue Formula

Let $\Omega \subset \mathbb{C}$ open and $F = \{z_1, \ldots, z_n\}$. Suppose $f \in \mathcal{H}(\Omega \setminus F)$ except for poles in F. Let γ be any circle contained in Ω, counter-clockwise oriented such that $\gamma \cap F = \varnothing$. Let D be the open disc bounded by γ. Then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{z_i \in F \cap D} \operatorname{res}_{z_i} f.$$

Remark. Only the poles inside the curve do contribute for the residue theorem, not all poles in Ω.

Example 3.2.2

Let γ be a circle $|z| = 3$. We want to evaluate

$$\int_\gamma \frac{\mathrm{d}z}{(z^2+1)^2}.$$

The function has to double poles $i, -i$ which are both inside the circle. According to the residue theorem 3.2.1 we get

$$\int_\gamma \frac{\mathrm{d}z}{(z^2+1)^2} = 2\pi i(\mathrm{res}_i f + \mathrm{res}_{-i} f)$$

Where $\mathrm{res}_i = \frac{1}{4i}, \mathrm{res}_{-i} = \frac{-1}{4i}$ (Lemma 3.1.14), so we finally obtain

$$\int_\gamma \frac{\mathrm{d}z}{(z^2+1)^2} = 0.$$

Lecture 15

Mock exam. Lecture 16

Mock exam discussion.

In the following we give the proof of residue formula. Lecture 17

Proof of Theorem 3.2.1. Let's first assume f is holomorphic in an open set Ω containing a circle and its interior except for a single pole at z_0, inside γ. By theorem 3.1.11

$$f(z) = P_{z_0}(z) + G(z) \quad \forall\, z \in D_r^*(z_0),$$

where $G(z)$ is holomorphic in a neighbourhood $D_r(z_0)$. The principle part is given by

$$P_{z_0}(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}.$$

The function $f(z) - P_{z_0}(z)$ extends holomorphically to Ω in the following way. Note that $P_{z_0}(z)$ is holomorphic in all $\mathbb{C} \setminus \{z_0\}$ and let

$$g(z) = \begin{cases} f(z) - P_{z_0}(z) & z \in \Omega \setminus \{z_0\} \\ G(z) & z \in D_r(z_0). \end{cases}$$

Then $g(z)$ is the holomorphic extension of $f(z) - P_{z_0}(z)$. Hence

$$\int_\gamma f(z) - P_{z_0}(z)\mathrm{d}z = 0,$$

which means

$$\begin{aligned} \int_\gamma f(z)\mathrm{d}z &= \int_\gamma P_{z_0}(z)\mathrm{d}z \\ &= \int_\gamma \left(\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} \right) \mathrm{d}z. \end{aligned}$$

Recall Theorem 2.4.1 (Cauchy Integral Formula), for $C = \partial D$, $D \subset \Omega$, and $F \in$

$\mathcal{H}(\Omega)$, for any $z \in D$ we have

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{F(w)}{(w-z)^{n+1}} \mathrm{d}w.$$

Applying the formula with $F = 1$, we see

$$\int_C \frac{\mathrm{d}z}{(z-z_0)^n} = 2\pi i \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}(1) = \begin{cases} 0 & \text{if } n-1 \geq 1 \\ 2\pi i & \text{if } n = 1. \end{cases}$$

For the general case we assume f is holomorphic in Ω except for poles $F = \{z_0, \dots, z_n\}$. For each z_i, let $P_{z_i}(z)$ be the principle part of f at $z = z_i$. Define

$$g(z) = f(z) - \sum_{z_i \in F} P_{z_i}(z).$$

If $z \notin F$, then $g \in \mathcal{H}(\Omega \setminus F)$ and in fact g can be extended holomorphically to all of Ω. To see this, let $z_0 \in F$ and $r > 0$ such that $D_r(z_0) \subset D, D_r^*(z_0) \cap F = \varnothing$. Then, $f(z) - P_{z_0}(z)$ is holomorphic in $D_r^*(z_0)$ and for $z \in D_r^*(z_0)$,

$$g(z) = \sum_{\substack{z_i \in F \\ z_i \neq z_0}} P_{z_i}(z) + (f(z) - P_{z_0}(z)).$$

Now the first sum is holomorphic in $D_r(z_0)$ and $f(z) - P_{z_0}(z) = G(z)$ extends holomorphically to $D_r(z_0)$. Hence $g(z)$ has an extension to $(\Omega \setminus F) \cup \{z_0\} = \Omega \setminus \{z_1, \dots, z_n\}$. We can do this for each z_i to get an extension to all of Ω. By Cauchy's Theorem 2.2.5

$$\int_\gamma g(z) = 0,$$

hence

$$\int_\gamma f(z)\mathrm{d}z = \sum_{z_i} \int_\gamma P_{z_0}(z)\mathrm{d}z = 2\pi i \sum_{z_i \in D \cap F} \operatorname{res}_{z_i} f.$$

□

Remark.

1. Another way to prove this is to use a "keyhole" contour . Inside $\Gamma_{\varepsilon,\delta}$ f is holomorphic, so by Cauchy's Theorem 2.2.5

$$\int_{\Gamma_{\varepsilon,\delta}} f\mathrm{d}z = 0.$$

We then make the width of the corridor $\delta \to 0$ and use continuity of f to show that the two sides cancel. The remaining part becomes two circles, the large circle γ and the small circle C_ε.

$$\int_\gamma f(z)\mathrm{d}z + \underbrace{\int_{C_\varepsilon} f(z)\mathrm{d}z}_{-2\pi i\,\mathrm{res}_{z_0}(f)} = 0$$

This gives the result.

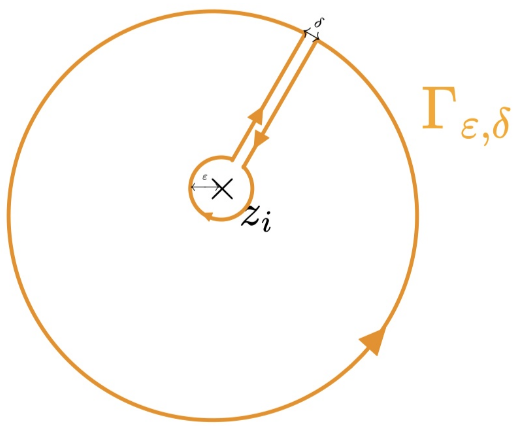


Figure 3.1: Keyhole contour [1]

2. The best way to understand Cauchy Integral Formula 2.4.1, Cauchy's Theorem 2.2.5 or residue formula is using homotopy. If one path can be continuously deformed into another path while staying in the region where f is holomorphic, the integrals over these two paths are equal.

Remark. If γ is not a circle but a rectangle, polygon or any curve which has a parametrisation

$$\begin{aligned}\gamma : [a,b] &\longrightarrow \mathbb{C} \setminus \{z_0\} \\ t &\longmapsto z_0 + r(t)e^{i\theta(t)}\end{aligned}$$

for some C^1 functions $r, \theta : [a,b] \to \mathbb{R}$ satisfying $r(t) > 0$, $r(a) = r(b)$, and $\theta(a) = 0$, $\theta(b) = 2\pi$. It holds that

$$r(t) = |\gamma(t) - z_0|$$

[1] This figure was created by Joshua Dreier. His personal lecture notes and more brilliant diagrams you will find under his personal webpage [1].

and

$$e^{i\theta(t)} = \frac{\gamma(t) - z_0}{|\gamma(t) - z_0|}.$$

Then $\gamma'(t) = r'(t)e^{i\theta(t)} + r(t)i\theta'(t)e^{i\theta(t)}$, hence

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - z_0} &= \int_a^b \frac{\gamma'(t)}{r(t)e^{i\theta(t)}}\mathrm{d}t \\ &= \underbrace{\int_a^b \frac{r'(t)}{r(t)}\,\mathrm{d}t}_{=0} + i\underbrace{\int_a^b \theta'(t)\mathrm{d}t}_{2\pi i} = 2\pi i.\end{aligned}$$

Example 3.2.3

Show that

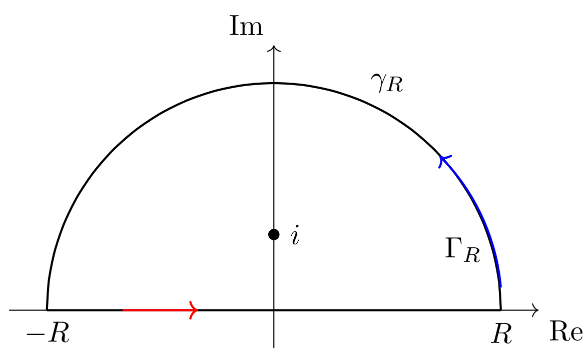
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1 + x^2} = \pi.$$

Idea: To choose a function f and a closed contour γ so that part of the contour leads to the real integral.

In this particular case we guess

$$f(z) = \frac{1}{1 + z^2}$$

and we choose the path along the boundary of a semicircle with radius R.



For fixed R we get

$$\int_{\gamma_R} f(z)\mathrm{d}z = 2\pi i \operatorname{res}_i f = \pi.$$

Hence,

$$\pi = \int_{\gamma_R} f(z)\mathrm{d}z = \int_{-R}^{R} \frac{1}{1 + x^2}\mathrm{d}x + \int_{\Gamma_R} \frac{\mathrm{d}z}{1 + z^2},$$

where Γ_R is the curved part of γ_R. We take the limit $R \to \infty$ to get

$$\pi = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} + \lim_{R\to\infty} \int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^2}.$$

On Γ_R, $|z^2+1| \geq R^2 - 1$ implies that $\left|\frac{1}{z^2+1} \leq \frac{1}{R^2-1}\right|$, hence

$$\left|\int_{\Gamma_R} \frac{1}{1+z^2}\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \to 0 \quad \text{as } R \to \infty.$$

Example 3.2.4

The same technique works well to evaluate integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$$

where $P, Q \in \mathbb{C}[x]$ and Q has no zeros on the real line. We also require $\deg Q \geq \deg P + 2$, because this is needed to show that

$$\int_{\Gamma_R} \frac{P(z)}{Q(z)}\mathrm{d}z \to 0 \quad \text{as } R \to 0,$$

which appears in

$$\int_{\gamma_R} \frac{P(z)}{Q(z)}\mathrm{d}z = \int_{-R}^{R} \frac{P(x)}{Q(x)}\mathrm{d}x + \int_{\Gamma_R} \frac{P(z)}{Q(z)}\mathrm{d}z.$$

To show exactly why we need this, let $\deg Q = n$ and $\deg P = m$. On the semicircle Γ_R of radius R, for large R we have that $|Q(z)| > B|z|^n$ for some B, so we can give bound

$$\left|\frac{P(z)}{Q(z)}\right| < C\frac{R^m}{R^n} = CR^{m-n}.$$

For the integral we have

$$\left|\int_{\Gamma_R} \frac{P(z)}{Q(z)}\mathrm{d}z\right| < CR^{m-n}\pi R < C\frac{1}{R^{m-n-1}}.$$

To obtain

$$\int_{\Gamma_R} f(z)\mathrm{d}z \to 0 \quad \text{as } R \to \infty,$$

we need $n - m - 1 > 0$, which is equivalent to $n \geq m + 2$.

Example 3.2.5

Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$

Example 3.2.6

Some contour can by used to evaluate the integrals of rational functions multiplied with $\sin(ax)$ or $\cos(ax)$, for example

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\mathrm{d}x.$$

As $f(z)$ we take

$$f(z) = \frac{P(z)}{Q(z)} e^{iaz}$$

and **not**

$$\frac{P(z)}{Q(z)} \cos(az)$$

because $\cos(az)$ behaves "badly" on the upper half-plane.[2] On the imaginary axis for example

$$\cos(it) = \frac{e^t + e^{-t}}{2} = \cosh(t)$$

grows exponentially as $t \to \infty$, whereas

$$\left|e^{iz}\right| = \left|e^{i(x+iy)}\right| = e^{-y} \leq 1$$

is bounded by 1 for $z \in \mathbb{H}^+$.

Example 3.2.7

Show that

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} \mathrm{d}x = \pi e^{-a}.$$

Let

$$f(z) = \frac{e^{iaz}}{z^2+1}$$

then it holds that

$$\operatorname{res}_i f = \frac{e^{-a}}{2i}.$$

[2]Note that $\cos z + i \sin z = e^{iz}$ is not the real-imaginary decomposition of e^{iz}. Nonetheless e^{iz} is still bounded for $\mathrm{Im}(z) > 0$ because $\cos z$ and $\sin z$ cancel each other out.

So (choosing γ_R as in Example 3.2.3) we get

$$\int_{\gamma_R} f(z)\mathrm{d}z = \pi e^{-a}.$$

We bound the integrand

$$\left|\frac{e^{iaz}}{z^2+1}\right| \leq \frac{1}{R^2-1}$$

on Γ_R and hence

$$\left|\int_{\Gamma_R} f(z)\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \to 0 \quad \text{as } R \to \infty.$$

Taking real parts on both sides gives us

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{x^2+1}\mathrm{d}x = \pi e^{-a}.$$

Example 3.2.8

An other class of integrals that we can solve using the residue theorem is

$$\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\mathrm{d}t,$$

where P, Q are polynomials and $Q(x,y) \neq 0$ for all $x, y \in \mathbb{R}$ with x^2+y^2+1. A specific example of this is

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}, \quad a > 1.$$

On the unit circle $z = e^{i\theta}$ we have

$$\frac{\mathrm{d}z}{iz} = \mathrm{d}\theta.$$

Also note that cos can be written as

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}.$$

Hence we can write $a+\cos\theta = a + \frac{1}{2}(z+1/z)$, which gives

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} = \int_{|z|=1} \frac{1}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{\mathrm{d}z}{iz}$$
$$= \frac{2}{i}\int_{|z|=1} \frac{\mathrm{d}z}{z^2+2az+1}.$$

A minor computation shows that the poles are at $-a \pm \sqrt{a^2-1}$, and both points are inside the unit circle. This gives us the final result

$$\int_{|z=1|} \frac{\mathrm{d}\theta}{a+\cos\theta} = 2\pi i \operatorname{res}_{z_0} f = \frac{2\pi}{\sqrt{a^2-1}}.$$

Lecture 18

Proposition 3.2.9:

Suppose f has an isolated singularity at z_0. Then z_0 is a pole of f if an only if the limit $\lim_{z\to z_0}|f(z)| = \infty$.

Proof. If $f(z)$ has a pole of order $k \geq 1$ at z_0, then there exists $r > 0$ such that

$$f(z) = g(z)(z-z_0)^{-k}$$

on $D_r^*(z_0)$, where $g \in \mathcal{H}(D_r(z_0))$ and $g(z_0) \neq 0$. This gives us

$$\lim_{z\to z_0}|f(z)| = \lim_{z\to z_0}|g(z)||z-z_0|^{-k} = \infty,$$

since $\lim_{z\to z_0} g(z) = g(z_0)$ is finite. Conversely, if $\lim_{z\to z_0}|f(z)| = \infty$. Find $r > 0$ such that $|f(z)| \geq 1 \quad \forall\, z \in D_r^*(z_0)$. In particular $f(z) \neq 0$ in $D_r^*(z_0)$. Then

$$h(z) = \frac{1}{f(z)}$$

for $z \in D_r^*(z_0)$ is holomorphic in $D_r^*(z_0)$ and $|h(z)| \leq 1$. Hence by Riemann's Theorem 3.1.4 $h(z)$ extends to a holomorphic function in $D_r(z)$, by defining $h(z_0) = \lim_{z\to z_0} 1/f(z) = 0$. If N is the order of the zero of h at $z = z_0$, then $f(z)$ has a pole of order N at z_0. □

We have seen that if z_0 is removable then $\lim_{z\to z_0} f(z)$ exists and finite. If z_0 is a pole the $\lim_{z\to z_0}|f(z)| = \infty$. Recall that if z_0 is an isolated singularity, then z_0 is called an essential singularity if it is neither a pole nor removable. For example $e^{1/z}$ has an essential singularity at $z = 0$. Note that $e^{1/x} \to 0$ as $x \to 0$ along the negative reals and $e^{1/x} \to \infty$ as $x \to 0$ along the positive reals.

Theorem 3.2.10: Casorati-Weierstrass

Suppose f is holomorphic in $D_r^*(z_0)$ and has an essential singularity at z_0. Then the image of $D_r^*(z_0)$ under f is dense in $\mathbb{C}$.

Proof. We want to show that for every $w \in \mathbb{C}$ and every $\varepsilon > 0$ there exists $z \in D_r^*(z_0)$ such that $|f(z) - w| \leq \varepsilon$. We argue by contradiction. Assume that this does not hold, then we show that z_0 is either removable or a pole. Assume on the contrary that there exists $w_0 \in \mathbb{C}$ and $\delta > 0$ such that for all $z \in D_r^*(z_0)$,

$$|f(z) - w_0| \geq \delta > 0.$$

Let

$$g(z) := \frac{1}{f(z) - w_0},$$

with $z \in D_r^*(z_0)$. $g(z)$ is bounded by $1/\delta$ in $D_r^*(z_0)$, hence by Riemann's Theorem there is a holomorphic extension of g to $D_r(z_0)$, and in particular, the limit $\lim_{z\to z_0} g(z)$ exists. Since $|f(z) - w_0| \geq \delta$ and by definition of g, it is zero free in $D_r^*(z_0)$. Hence its reciprocal has an isolated singularity in $D_r(z_0)$. This singularity of $1/g$ is either a pole or removable depending on whether $\lim_{z\to z_0} g(z) = 0$ or not. In particular

$$f(z) = w_0 + \frac{1}{g}$$

has at most a pole at z_0, which contradicts the assumption that z_0 is an essential singularity. □

Remark. There is another theorem of Picard (1879) that states that if $f \in \mathcal{H}(D_r^*(z_0))$ and has an essential singularity at z_0. Then $\mathbb{C} \setminus f(D_r^*(z_0))$ contains at most one point. This exceptional point can exist! For instance $\exp(1/z)$ never takes the value 0.

3.3 Meromorphic functions

We now look at functions whose singularities are poles. Since around a pole z_0, $\lim_{z\to z_0} |f(z)| = \infty$, we might want to add the "infinity" to $\mathbb{C}$.

Definition 3.3.1: Extended complex plane

Define $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where ∞ is an ideal point at infinity and is unsigned. $\hat{\mathbb{C}}$ is called the extended complex plane. We can supplement the rules in $\mathbb{C}$ by

1. $\infty \pm z = z \pm \infty = \infty$.
2. $\infty \cdot z = z \cdot \infty = \infty$
3. $z/\infty = 0$
4. $z/0 = \infty$

The expressions $\infty \pm \infty$, ∞/∞, $0/0$, $0 \cdot \infty$ are not assigned a meaning in $\hat{\mathbb{C}}$. A sequence $(z_n)_{n=0}^{\infty} \subset \mathbb{C}$ converges to ∞ if $\lim_{n\to\infty} |z_n| = \infty$. Similarly we say $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0} |f(z)| = \infty$.

Lecture 19

Definition 3.3.2: Meromorphic functions

Let Ω be an open subset of $\mathbb{C}$. $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called meromorphic on Ω if the following the conditions are satisfied.

1. The set $S_f = \{z \in \Omega : f(z) = \infty\}$ has no limit points in Ω.
2. The points in S_f are poles of f.
3. The restriction $f|_{\Omega \setminus S_f}$ is holomorphic.

Let $\mathcal{M}(\Omega)$ be the set of all meromorphic functions on Ω.

Example 3.3.3

1. Let $P(z), Q(z) \in \mathbb{C}[z]$ with no common zeros. Then $f(z) = P(z)/Q(z) \in \mathcal{M}(\mathbb{C})$, where we let

$$f(z) = \begin{cases} P(z)/Q(z) & Q(z) \neq 0 \\ \infty & Q(z) = 0. \end{cases}$$

   In this case S_f is the set of zeros of Q.
2. $f(z) = \cot \pi z = \frac{\cos \pi z}{\sin \pi z}$ then f is meromorphic in $\mathbb{C}$ and $S_f = \mathbb{Z}$.
3. $f(z) = e^{1/z}/(z^2 - 1)$ is meromorphic for $\mathbb{C} \setminus \{0\}$, gut is not meromorphic for $\mathbb{C}$, because it has an essential singularity at 0.

Now we want to investigate more properties of meromorphic functions. To begin with, we consider the following situation. If we have two functions $f, g \in \mathcal{M}(\Omega)$ with pole sets S_f, S_g, then $(f+g)(z) = f(z)+g(z)$ is $z \in \Omega \setminus (S_f \cup S_g)$. If $z_0 \in S_f \cup S_g$ then we can write $f(z) = P_f(z)+\tilde{f}(z), \forall\, z \in D_r^*(z_0), \tilde{f} \in \mathcal{H}(D_r(z_0))$, $g(z) = P_g(z)+\tilde{g}(z)$.[3] Then

$$(f+g)(z) = P_f(z) + P_g(z) + \tilde{f}(z) + \tilde{g}(z).$$

This shows that $f+g \in \mathcal{M}(\Omega)$, where $S_{f+g} \subset S_f \cup S_g$.

Proposition 3.3.4:

Let $\Omega \subset \mathbb{C}$ open, then

1. $\mathcal{M}(\Omega) \supseteq \mathcal{H}(\Omega)$.
2. *If $f, g \in \mathcal{M}(\Omega)$, then so is $af+bg \in \mathcal{M}(\Omega)$ with $a, b \in \mathbb{C}$. Hence $\mathcal{M}(\Omega)$ is a vector space.*
3. *$f, g \in \mathcal{M}(\Omega)$, $z_0 \in S_f \cup S_g$. Let $f = P_f + \tilde{f}$ and $g = P_g + \tilde{g}$, with $\tilde{f}, \tilde{g} \in \mathcal{H}(\Omega)$. Then*

$$(fg) = (P_f + \tilde{f})(P_g + \tilde{g}) = P_{fg} + G,$$

   where $G \in \mathcal{H}(\Omega)$.
4. *If $0 \not\equiv f \in \mathcal{M}(\Omega), \Omega$ connected and the zeros of f do not have a limit point in Ω, then $1/f \in \mathcal{M}(\Omega)$.*

Proof.

1. Obvious but note we identified a holomorphic function $f : \Omega \to \mathbb{C}$ with the corresponding function $\tilde{f} : \Omega \to \hat{\mathbb{C}}$ where $\tilde{f} = i \circ f, i : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$.
2. The same argument for $f+g$ works with $af+bg$.
3. Let $f = P_f + \tilde{f}, g = P_g + \tilde{g}$. Let $z_0 \in S_f \cup S_g$ then

$$\begin{aligned} fg &= (P_f + \tilde{f})(P_G + \tilde{g}) \\ &= P_{fg} + G \end{aligned}$$

   where P_{fg} is a linear combination and G is holomorphic in $D_r(z_0)$. Now (consider example 3.3.5)

$$\begin{aligned} fg &= \left(\sum_{k=-n}^{\infty} a_k (z-z_0)^k\right)\left(\sum_{\ell=-m}^{\infty} b_\ell (z-z_0)^\ell\right) \\ &= \sum_{N=-(n+m)}^{\infty}\left(\sum_{\substack{k,\ell \\ k+\ell=N}} a_k b_{N-\ell}\right)(z-z_0)^N. \end{aligned}$$

[3] If $z_0 \in S_f$ but not S_g, g if holomorphic in a neighbourhood if z_0 and the argument still applies.

Similar to $f+g$ we can define

$$fg = \begin{cases} f(z)g(z) & \text{if } z \in \Omega \setminus (S_f \cup S_g) \\ \infty & \text{if } z \in (S_f \cup S_g). \end{cases}$$

then fg is meromorphic in $\mathcal{M}(\Omega)$ with $S_{fg} \subseteq S_f \cup S_g$.

4. If $f \in \mathcal{M}(\Omega)$. If $z_0 \in \Omega \setminus S_f$ and $f(z_0) \neq 0$ then $1/f$ is holomorphic at z_0. If $z_0 \in \Omega \setminus S_f$ and $f(z_0) = 0$ then $1/f$ has a pole of order $k =$ order of zero of f at z_0. If $z_0 \in S_f$ then

$$\left| \frac{1}{f(z)} \right| \underset{z \to z_0}{\longrightarrow} 0$$

hence $1/f$ has a removable singularity at z_0. So if zeroes of f has no limit point in Ω then the poles of $1/f$ have no limit point in Ω and hence $1/f \in \mathcal{M}(\Omega)$.

□

Example 3.3.5

$$f = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$g = \frac{b_{-2}}{(z - z_0)^2} + \frac{b_{-1}}{z - z_0} + \sum b_n (z - z_0)$$

Then

$$fg = \frac{b_{-2}a_{-1}}{(z - z_0)^3} + \frac{b_2 a_0 + b_{-1}a_{-1}}{(z - z_0)^2} + \frac{a_{-1}b_0 + a_0 b_{-1} + b_{-2}a_1}{z - z_0} + G.$$

Recall from Theorem 2.4.15 that if $f : \Omega \to \mathbb{C}$, Ω connected, and $f \in \mathcal{H}(\Omega)$, then the zeros of f have no limit point in Ω.

Proposition 3.3.6:

If Ω open and connected, $0 \not\equiv f \in \mathcal{M}(\Omega)$, let $\mathscr{Z} := \{z \in \Omega : f(z) = 0\}$, then $\mathscr{Z}$ has no limit point in Ω.

Proof. Assume on the contrary there exists $(z_n)_n \subset \mathscr{Z}$, such that $\lim_{n\to\infty} z_n = b \in \Omega$ and $f(z_n) = 0$ for all z_n. Let $S_f =$ poles of f. Then $\Omega \setminus S_f$ is also connected. By theorem 2.4.15, the limit point $b \notin \Omega \setminus S_f$. But now $b \notin S_f$ either, because if b is a pole of f,

$$\lim_{z \to b} f(z) = \infty,$$

i.e., $|f(z)| > 0$ with $|z - b| < \varepsilon$ for some $\varepsilon > 0$. But this contradicts that $z_n \to b$, i.e., $|z_n - b| < \varepsilon$ for $n > n_0$ and $f(z_n) = 0$. $\square$

Remark. Let $f \in \mathcal{M}(\Omega)$ z_0 a pole of f. Then there exists $r > 0$ such that $D_r^*(z_0) \cap S_f = 0$. If the order of pole at z_0 is k then we can write f as

$$f(z) = (z - z_0)^{-k} g(z), \quad g(z_0) \neq 0,\ g(z) \in \mathcal{H}(D_r(z_0)).$$

We see that locally we can write a meromorphic function as the quotient of holomorphic functions. It is non-trivial but true that if Ω is connected then we can do this globally,

$$\mathcal{M}(\Omega) = \left\{ \frac{f(z)}{g(z)} : f, g \in \mathcal{H}(\Omega), g \not\equiv 0 \right\}.$$

Remark. This is the analogue of constructions of $\mathbb{Q}$ as field of fractions of the integral domain $\mathbb{Z}$. Recall $\mathcal{H}(\Omega)$ has no zero devisors if Ω is connected.

Definition 3.3.7: Order for meromorphic functions

Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$, $f \in \mathcal{M}(\Omega)$, $f \not\equiv 0$. Define the valuation of f at z_0 or the order of f at z_0, denoted $\operatorname{ord}_{z_0} f$, $\nu_{z_0} f$ to be the integer $k \in \mathbb{Z}$ such that

1. If z_0 is not a pole of f, i.e. $f(z_0) \neq \infty$, then $k \geq 0$ is the order of zero of f at z_0.
2. If $f(z_0) = \infty$, i.e., z_0 is a pole of f, then $k \leq -1$ is minus the order of pole of f.

I.e. if $\operatorname{ord}_{z_0} f > 0$ then z_0 is a zero, if $\operatorname{ord}_{z_0} f < 0$ then z_0 is a pole and if $\operatorname{ord}_{z_0} f = 0$ then $f(z_0) \neq 0, f(z_0) \neq \infty$.

Proposition 3.3.8:

If $f \in \mathcal{M}(\Omega)$, $f \not\equiv 0$, $z_0 \in \Omega$.

1. $k = \operatorname{ord}_{z_0} f \Leftrightarrow$ *there exists $r > 0$ and $h \in \mathcal{H}(D_r(z_0))$ such that $h(z_0) \neq 0$ and $f(z) = (z - z_0)^k h(z)$.*
2. $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g$.
3. *If $f + g \neq 0$, then $\operatorname{ord}_{z_0}(f + g) \geq \min\{\operatorname{ord}_{z_0} f, \operatorname{ord}_{z_0} g\}$.*

Example 3.3.9

$$f(z) = \frac{z}{(e^z - 1)^2}$$

z has a zero of order 1 at $z = 0$, $e^z - 1$ has zero of order 2 at $z = 0$, $e^z - 1$ has zero of order 2 at $z = 2\pi i n$, $n \neq 0$.

$$\operatorname{ord}_0 f = \operatorname{ord}_0 z - \operatorname{ord}_0 (e^z - 1)^2 = 1 - 2 = -1.$$

So f has a pole of order 1 at $z = 0$.

$$\operatorname{ord}_{2\pi i n} f = \operatorname{ord}_{2\pi i n} z - \operatorname{ord}_{2\pi i n} (e^z - 1)^2$$

f has poles or order 2 at $z = 2\pi i n$.

Remark. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and Riemann sphere. Let $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$ be the 2-sphere. We identify $\mathbb{C}$ with the plane (x_1, x_2) sitting in $\mathbb{R}^3$. Let $N = (0, 0, 1)$ be the north pole. Define $\Pi : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ as follows. For $p \in \mathbb{S}^2$, $p \neq N$, let $\Pi(p)$ be the intersection of $\mathbb{C}$ with the ray that starts at N and passes through p. Explicitly

$$\begin{aligned} \Pi(p) &= \Pi(x_1, x_2, x_3) \\ &= \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}, 0 \right) \\ &=: \frac{x_1}{1 - x_3} + \frac{x_2}{1 - x_3} i \in \mathbb{C} \end{aligned}$$

Note the equation of the ray that starts at N and goes through p is

$$N + t(p - N)$$

for $t \geq 0$ and

$$\Pi(p) = N + t_0(p - N)$$

where t_0 is the unique real number so that

$$(0, 0, 1) + t_0(x_1, x_2, x_3 - 1) = (y_1, y_2, 0)$$

for some suitable $y_1, y_2 \in \mathbb{R}$. Defining $\Pi(n) = \infty$ gives a bijection

$$\Pi : \mathbb{S}^2 \to \hat{\mathbb{C}}.$$

Conversely given $z \in \mathbb{C}$ one checks

$$\Pi^{-1}(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \in \mathbb{S}^2 \setminus \{\mathbb{N}\}.$$

Π is called a stereographic projection.

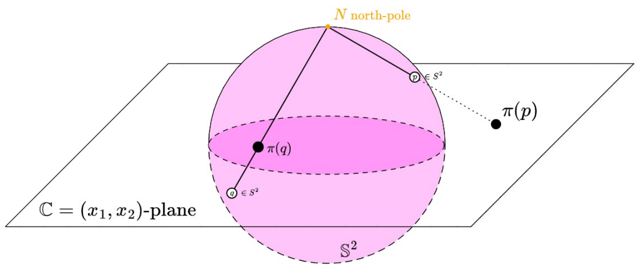


Figure 3.2: Riemann sphere. [4]

Definition 3.3.10: Isolated singularity at ∞

For a function f which is analytic for $|z| > 1/R$ for some $R > 0$, we say f has an isolated singularity at infinity (which will be called removable, pole, or essential) if $g(z) := f(1/z)$ has an isolated singularity at $z = 0$ (removable, pole, essential respectively).
A meromorphic function in the complex plane that is either holomorphic at infinity (i.e. f is holomorphic at 0) or has a pole at infinity is called meromorphic in $\hat{\mathbb{C}}$.

We will use the following notation $D_R^*(\infty) := \left\{ z \in \mathbb{C} \middle| |z| > \frac{1}{R} \right\}, D_r^*(\infty) \subset D_s^*(\infty)$ when $R < S$.

Example 3.3.11

1. An entire function is analytic in $D_R^*(\infty)$ for every $R > 0$ for any $R > 0$. For example $f(z) = e^z$ is holomorphic in $\mathbb{C}$ but has an essential singularity at ∞ since $f(1/z) = e^{1/z}$ has an essential singularity at 0.
2. $p(z) \in \mathbb{C}[z]$ has a pole at ∞ since

$$p(1/z) = \frac{a_n}{z^n} + \cdots + a_0.$$

3. $f(z) = \tan(z)$ does not have an isolated singularity at infinity because $f(z)$

[4] This figure was created by Joshua Dreier. His personal lecture notes and more brilliant diagrams you will find under his personal webpage [1].

has a pole at $z = \pi/2 + k\pi$, hence

$$g(z) = f(1/z) = \tan(1/z)$$

has singularities at $S = \{(\pi/2 + k\pi)^{-1} : k \in \mathbb{Z}\}$ accumulate at 0. The singularity of $\tan(1/z)$ at $z = 0$ is not isolated.

Theorem 3.3.12:

If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ meromorphic, then f is a rational function:

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P, Q \in \mathbb{C}[z]$.
Hence

$$\mathcal{M}(\hat{\mathbb{C}}) = \left\{ \frac{P(z)}{Q(z)} \middle| P(z), Q(z) \in \mathbb{C}[z] \right\}.$$

3.4 Applications of the Residue Theorem

Our goal is to show the argument principle 3.4.2, which allows us to count the number of zeros and poles of a given function $f \in \mathcal{M}(\Omega)$ inside a closed curve.

Lemma 3.4.1: f'/f

Let $\Omega \subset \mathbb{C}$ an open and connected set, $f \in \mathcal{M}(\Omega)$, $f \not\equiv 0$. Then f'/f, called the ***logarithmic derivative*** *of f, is also meromorphic in Ω. Moreover, f'/f has poles of order 1 at $z_0 \in \Omega$ for which $\operatorname{ord} f_{z_0} \neq 0$,i.e. z_0 is either a zero or a pole of f. Then $\operatorname{res}_{z_0} f'/f = \operatorname{ord}_{z_0} f$.*

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Proof. By assumption the zeros of f do not have limit points, hence $1/f \in \mathcal{M}(\Omega)$. Clearly $f' \in \mathcal{M}(\Omega \setminus S_f)$, where S_f is the set of poles. If $z_0 \in S_f$ is a pole of order n, then we can write f as

$$f(z) = (z - z_0)^{-n} g(z)$$

for all $z \in D_r^*(z_0)$, where $g \in \mathcal{H}(D_r(z_0))$ and $g(z_0) \neq 0$. Taking the derivative we get

$$\begin{aligned} f'(z) &= \frac{-n}{(z-z_0)^{n+1}} g(z) + \frac{1}{(z-z_0)^n} g'(z) \\ &= \frac{1}{(z-z_0)^{n+1}} \underbrace{\big((z-z_0)g'(z) - ng(z)\big)}_{:=h(z)} \end{aligned}$$

where $h \in \mathcal{H}(D_r(z_0))$ and $h(z_0) = -ng(z_0) \neq 0$. Hence $f'(z)/f(z)$ has a pole of order 1 at z_0. Similarly, if z_0 is a zero of f of order n, then f' has a zero of order $n-1$ at z_0, this gives

$$\operatorname{ord}_{z_0} f'/f = \operatorname{ord}_{z_0} f' - \operatorname{ord}_{z_0} f = \begin{cases} -(n+1) - (-n) = -1 & z_0 \text{ is a pole} \\ (n-1) - n = -1 & z_0 \text{ is a zero} \\ \geq 0 & \text{otherwise.} \end{cases}$$

Next, we want to compute the residue at zeros/poles. If z_0 is a zero, so that $f(z) = (z-z_0)^n g(z)$, then

$$\begin{aligned} \frac{f'}{f} &= \frac{n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} \\ &= \frac{n}{z-z_0} + \frac{g'(z)}{g(z)} \end{aligned}$$

That shows that $\operatorname{res}_{z_0} f'/f = n = \operatorname{ord}_{z_0} f$. The calculation is similar in the case when z_0 is a pole of f. □

Theorem 3.4.2: Argument Principle

Let $\Omega \subset \mathbb{C}$ be open and connected, $f \in \mathcal{M}(\Omega)$ with $f \not\equiv 0$, and let γ be a circle (or any closed path so that the Theorem 3.2.1 (Residue Formula) applies). If f has no zeros or poles on γ, then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'}{f} \mathrm{d}z = \sum_{z_0 \in \mathscr{Z}_f \cap \, int(\gamma)} \operatorname{ord}_{z_0} f + \sum_{z_0 \in S_f \cap \, int(\gamma)} \operatorname{ord}_{z_0} f = \sum_{\substack{z \in \, int(\gamma) \\ \operatorname{ord}_z f \neq 0}} \operatorname{ord}_z f = Z - P.$$

Where $\mathscr{Z}$ = zero set of f, S_f = poles of f, and $int(\gamma)$ denotes the interior enclosed by γ. Also Z = number of zeroes and P = number of poles (counted with multiplicity and order) of f inside γ.

Proof. This follows from Lemma 3.4.1 and $\text{res}_{z_0}(f/f') = \text{ord}_{z_0} f$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\mathrm{d}z = \sum_{z_i\in\text{int}\gamma} \text{res}_{z_i}\left(\frac{f'}{f}\right) = Z - P$$

where $Z =$ number of zeroes of f inside γ counted with multiplicity and $P =$ number of poles of f inside γ counted with multiplicity. □

Remark. The contour integral

$$\oint_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z$$

can be interpreted as $2\pi i$ times the winding number of the path $f(\gamma)$ around the origin.

Theorem 3.4.3: Rouché's Theorem

Suppose f, g holomorphic in an open set Ω which contains a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f + g$ have the same number of zeros inside C.

Proof. Let $t \in [0, 1]$ and define $f_t(z) := f(z)+tg(z)$, so that $f_0(z) = f$ and $f_1 = f+g$. We have that

$$|f_t(z)| = |f(z) + tg(z)| \geq ||f(z)| - t|g(z)|| \,.$$

For $z \in C$ we have $|f| > |g|$, hence $|f_t(z)| > (1 - t)|g(z)| \geq 0$, hence $|f_t(z)| > 0$ for $z \in C$. Applying the argument principle 3.4.2 to f_t gives

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}\mathrm{d}z,$$

where n_t is the number of zeros of f_t in C. Since f_t is continuous in both t and z, n_t is a continuous function of t.[5] Since n_t is an integer-valued function and continuous, it must be constant. In particular n_0 is the number of zeroes of f and n_1 is the number of zeroes of $f + g$ and they are equal. □

Example 3.4.4

Let $p(z) = z^6+8z^4+z^3+2z+3$ and let's prove that the numbers of zeros of $p(z)$ inside $|z| = 1$ is 4. In this case choose $f(z) = 8z^4$ (big on C) and $g(z) = z^6 + z^5 + 2z + 3$ (small on C). For $|z| = 1$, we have $7 = |z^6 + z^3 + 2z + 3| \leq |8z^4| = 8$ by Rouché,

[5] Recall from real analysis; $f : [a, b] \times [c, d] \to \mathbb{R}$ continuous on $[a, b] \times [c, d]$ then $h(t) := \int_c^d f(t, x)\mathrm{d}x$ is continuos on $[a, b]$.

$f, f+g = p(z)$ has the same number of zeroes inside $|z| = 1$, since f has four zeroes, so does $P(z)$.

Example 3.4.5

One can give a "nice" proof of the Fundamental Theorem of Algebra using Rouché's Theorem 3.4.3. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. For $|z|$ large $|z^n|$ dominates the rest. If $|z| = R$, with R large enough $f(z) = z^n, f(z) = a_n z^{n-1} + \cdots + a_0$ satisfies $|f(z)| > |g(z)|$ on $|z| = R$. Hence $f, f+g = p$ has the same number of zeroes inside $|z| = R$. Since f has n zeros, so does $f + g = p$. Quod erat demonstrandum.

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Theorem 3.4.6: Open Mapping Theorem

Let Ω be an open and connected set in $\mathbb{C}$ and $f \in \mathcal{H}(\Omega)$ a non-constant function, then f is open.

Proof. Let $z_0 \subset U \subset \Omega$, where U is an open set, and let $w_0 := f(z_0)$. We want to show that a neighbourhood of w_0 is also contained in $f(U)$, i.e., if w is near w_0, then there exists $z \in U$ such that $w = f(z)$. Let $r > 0$ such that $\overline{D_r(z_0)} \subset U$ and such that $f(z) - w_0 \neq 0$ in $\overline{D_r^*(z_0)}$ (we can do this since zeros of holomorphic functions are isolated). In particular $f(z) - w_0 \neq 0$ on $C_r(z_0)$. $C_r(z_0)$ is compact, hence we can find $\delta > 0$ such that $|f(z) - w_0| \geq \delta$ for all $z \in C_r(z_0)$. Let $w \in \mathbb{C}$ such that $|w - w_0| < \delta$, i.e. $w \in D_\delta(w_0)$. Define

$$F(z) := f(z) - w = (f(z) - w_0) + (w_0 - w).$$

We want to show that $F(z)$ has a zero inside the circle $C_r(z_0)$, because this would mean that there exists $z \in D_r(z_0)$ such that $f(z) = w$. We can now apply Rouché's Theorem 3.4.3 with $\tilde{f} := f(z) - w_0$ and $\tilde{g} = w_0 - w$. On the circle $C_r(z_0)$ we have $|\tilde{f}| \geq \delta$ and $|\tilde{g}| < \delta$. Hence by Rouché's theorem 3.4.3, $\tilde{f}$ and $\tilde{f} + \tilde{g} = F$ has the same number of zeroes inside $C_r(z_0)$. In particular since $\tilde{f}$ has a zero, namely z_0, inside $C_r(z_0)$, so does F. □

Remark. This theorem says for example that if f is a holomorphic function, then the image of a disc D under f cannot be completely contained in $\mathbb{R}$, since any subset of $\mathbb{R}$ is not open in $\mathbb{C}$.

Theorem 3.4.7: Maximum Modulus Principle

Let $\Omega \subset \mathbb{C}$ be an open and connected set, and let $f \in \mathcal{H}(\Omega)$ be a non-constant function. Then there is no $z_0 \in \Omega$ such that

$$|f(z)| \leq |f(z_0)|$$

for all $z \in \Omega$, i.e. $|f|$ cannot attain a maximum in Ω. In particular if $\overline{\Omega}$ is bounded and f is continuous on $\overline{\Omega}$, then

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \overline{\Omega} \setminus \Omega} |f(z)|.$$

Proof. Suppose $f \in \mathcal{H}(\Omega)$, non-constant and suppose f attains a maximum at $z_0 \in \Omega$. By Theorem 3.4.6 (Open Mapping Theorem) f is an open map, hence if $D = D_r(z_0) \subset \Omega$ then $f(D)$ is open. Hence $f(D)$ contains a disc around $f(z_0)$. But this means there are points $z \in D$ such that $|f(z_0)| \leq |f(z)|$ which contradicts that f attains its maximum at z_0. □

Remark. The assumption about $\overline{\Omega}$ being bounded is important. For example consider

$$\Omega := \left\{ z \in \mathbb{C} \,\middle|\, -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2} \right\}$$

is open and connected, but its closure is not bounded. Let $f(z) = \exp(e^z)$, then $f|_{\partial\Omega} = \exp\big(e^{x \pm i\pi/2}\big) = \exp(\pm i e^x)$. So on the boundary

$$\Big| f|_{\partial\Omega} \Big| = 1,$$

but on the interior of Ω, $f(x) = \exp(e^x) \to \infty$ as $x \to \infty$.

3.5 Homotopy and simply connected domains

The key idea to understand the homotopy version of Cauchy's Theorem 3.5.3 is the following: If $f : \Omega \to \mathbb{C}$ holomorphic and if we can deform $\gamma_1 \subset \Omega$ into another curve γ_2 (both γ_1, γ_2 are either closed or have the same endpoints) continuously while staying in Ω, then

$$\int_{\gamma_1} f \mathrm{d}z = \int_{\gamma_2} f \mathrm{d}z.$$

Definition 3.5.1: Homotopy

Let $\Omega \subset \mathbb{C}$ be open and $\gamma_0 : [a,b] \to \mathbb{C}$ and $\gamma_1 : [a,b] \to \Omega$ be two curves such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. We say that γ_0 is **homotopic** to γ_1 in Ω **with fixed endpoints** if there exists $H : [a,b] \times [0,1] \to \Omega, (t,s) \mapsto H(t,s)$ continuous, such that

1. $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$ for all $t \in [a,b]$.
2. $H(t,s) =: \gamma_s(t)$ is continuous for all $s \in [0,1]$ and $t \in [a,b]$, and $H(a,s) = \gamma_0(a) = \gamma_1(a)$, $H(b,s) = \gamma_1(b) = \gamma_1(b)$, i.e. γ_s has the same endpoints as γ_0 and γ_1.

Similarly, if γ_0 and γ_1 are closed curves, we say γ_0 is **homotopic** to γ_1 in Ω is there exists $H : [a,b] \times [0,1] \to \Omega$ such that

1. $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$ for all $t \in [a,b]$.
2. $\gamma_s = H(t,s)$ is a continuous curve and $H(a,s) = H(b,s)$ for all $s \in [0,1]$.

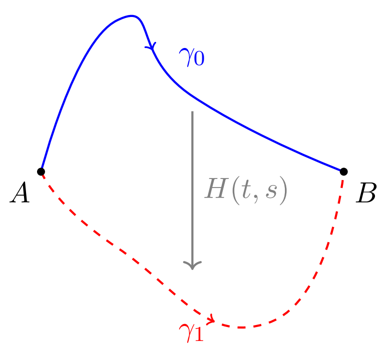

Figure 3.3: Homotopic curves γ_0 and γ_1 with fixed endpoints

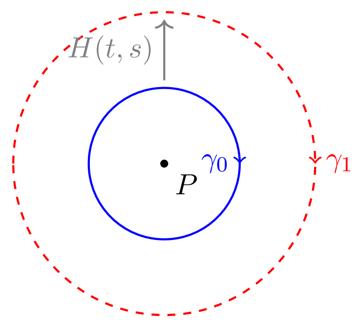

Figure 3.4: Closed homotopic paths γ_0 and γ_1

Example 3.5.2

1. Let $\Omega = \mathbb{C}$ and γ_1, γ_2 be two closed curves. Then they are homotopic in $\mathbb{C}$. In fact there are homotopic to the constant curve $\sigma : [a,b] \to \mathbb{C}, t \mapsto c \in \mathbb{C}$.

We can define a homotopy between the two paths as

$$\begin{aligned} H : [a,b] \times [0,1] &\longrightarrow \mathbb{C} \\ (t,s) &\longmapsto (1-s)\gamma_0(t) + s\gamma_1(t). \end{aligned}$$

H is continuous, being a combination of continuous functions, satisfying

$$\begin{aligned} H(t,0) &= \gamma_0(t) \\ H(t,1) &= \gamma_1(t) \\ H(a,s) &= (1-s)\gamma_0(a) + s\gamma_1(a) \\ H(b,s) &= (1-s)\gamma_0(b) + s\gamma_1(b). \end{aligned}$$

Since $\gamma_0(a) = \gamma_0(b)$ and $\gamma_1(a) = \gamma_1(b)$, we have $H(a,s) = H(b,s)$ for all $s \in [0,1]$. Note here we defined the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ as our homotopy. The same works for *any* domain in $\mathbb{C}$ which is *convex*.

2. Take $\Omega = \mathbb{C} \setminus \{0\}$. Let

$$\begin{aligned} \gamma_0(t) : [0,\pi] &\longrightarrow \Omega \\ t &\longmapsto e^{it} \end{aligned}$$

and

$$\begin{aligned} \gamma_1(t) : [0,\pi] &\longrightarrow \Omega \\ t &\longmapsto e^{-it}. \end{aligned}$$

We will see that they are not homotopic once we see the homotopy version of Cauchy's theorem.

3. Take $\Omega = \mathbb{C} \setminus (-\infty, 0]$, then Ω is not convex, but we can still deform a closed curve γ_0 to γ_1 as follows: choose a point on $\mathbb{R}^+$, say 1, and the constant curve $\sigma : [a,b] \to \Omega, t \mapsto 1$. We can deform γ_0 to 1, and 1 to γ_1. Explictly,

$$H(s,t) = \begin{cases} 1 + (1-2s)(\gamma_0(t) - 1) & 0 \leq s \leq \frac{1}{2} \\ 1 + (2s-1)(\gamma_1(t) - 1) & \frac{1}{2} < s \leq 1. \end{cases}$$

Remark. If γ_0 is homotopic to γ_1 in Ω we write $\gamma_0 \sim_\Omega \gamma_1$. If Ω is fixed we just write $\gamma_0 \sim \gamma_1$. Note that $\sim$ is an equivalence relation.

Theorem 3.5.3: Homotopy Theorem

Let $\Omega \subset \mathbb{C}$ be an open set and γ_0, γ_1 two curves in Ω such that either (i) γ_0 and γ_1 are both closed and homotopic, or (ii) γ_0 is homotopic to γ_1 with fixed endpoints. Then for $f \in \mathcal{H}(\Omega)$,

$$\int_{\gamma_0} f \mathrm{d}z = \int_{\gamma_1} f \mathrm{d}z.$$

Proof. We will give proofs for different versions of the theorem. We will begin with a simpler version under more assumptions.

Simpler version

If we also assume that $H(t,s)$ has continuous second partial derivatives, then by Schwarz's Theorem

$$\frac{\partial^2 H}{\partial s \partial t} = \frac{\partial^2 H}{\partial t \partial s}$$

for all $(t,s) \in [a,b] \times [0,1]$. Recall from analysis that for

$$\begin{aligned} h : [a,b] \times [0,1] &\longrightarrow \mathbb{R} \\ (t,s) &\longmapsto h(t,s), \end{aligned}$$

if $\partial h / \partial s$ exists and is continuous, and we define

$$\begin{aligned} G : [0,1] &\longrightarrow \mathbb{R} \\ s &\longmapsto G(s) = \int_a^b h(t,s) \mathrm{d}t, \end{aligned}$$

then G is differentiable and

$$G'(s) = \int_a^b \frac{\partial h}{\partial s}(t,s) \mathrm{d}t.$$

We will apply this to the real and imaginary parts of the following integral. Define

$$\begin{aligned} I(s) &:= \int_a^b \underbrace{f(H(s,t)) \frac{\partial H}{\partial t}(t,s)}_{h(t,s)} \mathrm{d}t \\ &= \int_a^b f(\gamma_s(t)) \gamma_s'(t) \mathrm{d}t. \end{aligned}$$

We have that

$$I(0) = \int_{\gamma_0} f \mathrm{d}z$$

and

$$I(1) = \int_{\gamma_1} f \mathrm{d}z.$$

We want to prove $I(0) = I(1)$ and we will do this by showing that I is constant. Taking the derivative we get

$$\begin{aligned} I'(s) &= \int_a^b \frac{\partial}{\partial s} \left(f(H(t,s)) \frac{\partial H}{\partial t}(t,s) \right) \mathrm{d}t \\ &= \int_a^b \frac{\partial}{\partial s} \left((f \circ H) \frac{\partial H}{\partial t} \right) \mathrm{d}t. \end{aligned}$$

Using chain rule we have

$$I'(s) = \int_a^b \left[f'(H(t,s)) \frac{\partial H}{\partial s}(t,s) \frac{\partial H}{\partial t}(t,s) + f(H(t,s)) \frac{\partial^2 H}{\partial s \partial t}(t,s) \right] \mathrm{d}t.$$

Notice that the part inside [...] is equal to

$$\frac{\partial}{\partial t} \left(f(H(t,s)) \frac{\partial}{\partial s} H(t,s) \right).$$

Hence we have

$$\begin{aligned} I'(s) &= \int_a^b \frac{\partial}{\partial t} \left(f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \right) \mathrm{d}t \\ &= f(H(t,s)) \left. \frac{\partial H}{\partial s}(t,s) \right|_{t=a}^{t=b} \\ &= f(H(t,s)) \frac{\partial H}{\partial s}(b,s) - f(H(t,s)) \frac{\partial H}{\partial s}(a,s) \\ &= 0. \end{aligned}$$

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Full version

We now prove the theorem with no extra assumptions. We will use the following two facts:

1. If $K = H([a,b] \times [0,1])$, then K is compact. This allows us to use the following lemma.

   Lemma. If K is compact, then there exists $\varepsilon > 0$ such that for all $z \in K$, the disc $D_\varepsilon(z)$ is contained in Ω.

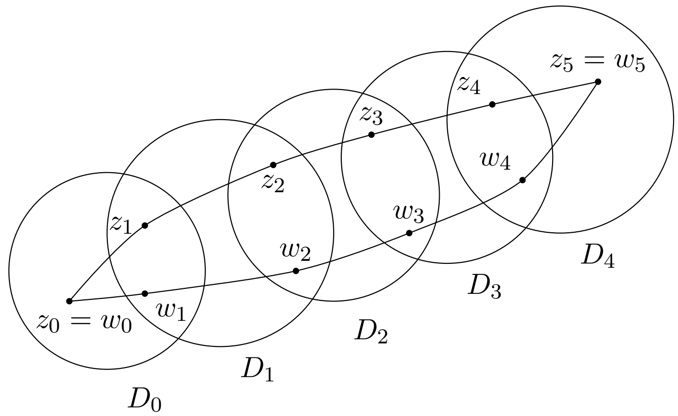

Figure 3.5: Two curves, $\gamma_{n/N}$ and $\gamma_{(n-1)/N}$, covered by discs

Proof of lemma. Assume that such an ε does not exist. Then for all $n \geq 1$ there exists $z_n \in K$ such that $D_{1/n}(z_n)$ is not contained in Ω, i.e. there exists $w_n \in \mathbb{C} - \Omega$ such that $|z_n - w_n| < 1/n$. $(z_n)_{n=0}^{\infty}$ is a sequence in K and since K is compact, there exists a convergent subsequence $(z_{n_k})_{k=0}^{\infty}$ with $\lim_{k\to\infty} z_{n_k} = z \in K$ due to the closedness of K. Since $|w_n - z_n| < 1/n$, we also have $|w_{n_k} - z_{n_k}| < 1/n_k$, hence $w_{n_k} \to z$ as well. But $w_{n_k} \in \mathbb{C} \setminus \Omega$, which is also a closed set, this means that $z \in \mathbb{C} \setminus \Omega$, which is a contradiction. □

2. H is continuous on the compact set $[a, b] \times [0, 1]$ implies that H is uniformly continuous. Now we divide $[a, b] \times [0, 1]$ into $N \times N$ small rectangles, each with size $1/N \times (b - a)/N$, we consider the image of the rectangle with vertices at $x_{m,n}$, $x_{m-1,n}$, $x_{m,n-1}$, and $x_{m+1,n+1}$ (we write $x_{m,n}$ for (t_m, s_n)). Due to uniform continuity, there exists $N > 0$ such that

$$|H(t, s) - H(t_m, s_n)| < \varepsilon$$

whenever $|(t, s) - (t_m, s_n)| < 2/N$. Since the diameter of $Q_{mn} = [t_m, t_{m+1}] \times [s_n, s_{n+1}]$ is $\sqrt{2}/N$, we can use the above remark to get $H(Q_{mn}) \subset D_\varepsilon(z_{mn})$, where $z_{mn} = H(x_{m,n})$.

We use induction on n, $0 \leq n \leq N$ to show that

$$\int_{\gamma_{n/N}} f(z)\mathrm{d}z = \int_{\gamma_0} f(z)\mathrm{d}z.$$

If $n = 0$ it is clear. Assume for $n \geq 1$ that

$$\int_{\gamma_{(n-1)/N}} f \mathrm{d}z = \int_{\gamma_0} f \mathrm{d}z,$$

then it is enough to show that

$$\int_{\gamma_{(n-1)/N}} f \mathrm{d}z = \int_{\gamma_{n/N}} f \mathrm{d}z.$$

For each $0 \leq m \leq N$, let $\gamma^{(m)}_{(n-1)/N} = \gamma_{n-1/N}\big|_{[t_m, t_{m+1}]}$. Let σ_m be the line segment between $z_{m,n-1}$ and $z_{m,n}$ and analogously for σ_{m+1}. Now we apply Cauchy's Theorem in the disc $D_\varepsilon(z_{m,n-1})$, which we have chosen to be contained in Ω and to contain the segments $\gamma^{(m)}_{(n-1)/N}$ and $\gamma^{(m)}_{(n-1)/N}$.

$$\int_{\gamma^{(m)}_{n/N}} f(z)\mathrm{d}z - \int_{\sigma_{m+1}} f(z)\mathrm{d}z - \int_{\gamma^{(m)}_{(n-1)/N}} f(z)\mathrm{d}z + \int_{\sigma_m} f(z)\mathrm{d}z = 0.$$

Summing over m gives

$$\begin{aligned}
\int_{\gamma_{(n-1)/N}} f(z)\mathrm{d}z &= \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{(n-1)/N}} f(z)\mathrm{d}z \\
&= \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{n/N}} f(z)\mathrm{d}z + \sum_{m=0}^{N-1} \left(\int_{\sigma_m} f(z)\mathrm{d}z - \int_{\sigma_{m+1}} f(z)\mathrm{d}z \right) \\
&= \int_{\gamma_{n/N}} f(z)\mathrm{d}z + \int_{\sigma_0} f(z)\mathrm{d}z - \int_{\sigma_N} f(z)\mathrm{d}z.
\end{aligned}$$

Now if we have two homotopic curves with fixed endpoints, then σ_0 and σ_N are trivial and both integrals vanish. If we have two closed curves, then $\sigma_0 = \sigma_N$ because $\gamma_{(n-1)/N}(a) = \gamma_{(n-1)/N}(b)$ and $\gamma_{n/N}(a) = \gamma_{n/N}(b)$. This concludes the proof. □

Example 3.5.4

In Example 3.5.2 (2), $\gamma_0 \not\sim \gamma_1$ in Ω, because if they were homotopic, then that would say for $f(z) = 1/z \in \mathcal{H}(\Omega)$,

$$\int_{\gamma_0} f \mathrm{d}z = \int_{\gamma_1} f \mathrm{d}z,$$

which implies

$$\int_{\gamma_0 - \gamma_1} \frac{1}{z} \mathrm{d}z = 0,$$

but we know the integral is $2\pi i$.

Lecture 23

3.6 The complex logarithm

Definition 3.6.1: Simply Connected

An open set $\Omega \subset \mathbb{C}$ is called simply connected if it is connected and if every 2 curves with the same endpoints are homotopic.

Theorem 3.6.2: Simply connected and primitive

Any holomorphic function on a simply connected domain has a primitive. In particular,

$$\int_\gamma f \mathrm{d}z = 0$$

for every closed curve. Any 2 primitives differ by a constant.

Proof. Fix $z_0 \in \Omega$. For $z \in \Omega$, since Ω is connected, there exists a curve γ connecting z_0 to z. Define

$$F(z) = \int_\gamma f(w)\mathrm{d}w.$$

The function F is well defined since Ω is simply connected: if $\tilde{\gamma}$ is another curve from z_0 to z, then $\gamma \sim \tilde{\gamma}$ and by Theorem 3.5.3 (Homotopy Theorem),

$$\int_\gamma f(w)\mathrm{d}w = \int_{\tilde{\gamma}} f(w)\mathrm{d}w.$$

Now we show that F is indeed the primitive of f. Choose h small so that the line segment connecting z and $z+h$ is contained in Ω. Then, by definition

$$F(z+h) - F(z) = \int_z^{z+h} f(w)\mathrm{d}w.$$

Arguing as in the proof of Theorem 2.1 (this is the numbering in the book, not in our notes), or using continuity of f as below we get

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

This concludes the proof. In the following we elaborate on the argument using continuity. By uniform continuity in a compact set (here $[z, z+h]$ denotes the line segment from z to $z+h$),

$$\left|\int_{[z,z+h]} f(w)-f(z)\mathrm{d}w\right| \leq \left(\sup_{w\in[z,z+h]} |f(w)-f(z)|\right) h.$$

Hence

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \leq \sup_{w\in[z,z+h]} |f(w)-f(z)|.$$

But f is continuous. Hence $\sup_{w\in[z,z+h]} |f(w)-f(z)| \to 0$ as $h\to 0$. □

For a given $z\in\mathbb{C}\setminus\{0\}$, we want to define a logarithm as a complex number w such that $e^w = z$. If $z = re^{i\theta}$ we can set $\log z = \log r + i\theta$. The problem with this is that this is not single valued. For example if we choose $z = 1$, we have

$$e^0 = 1$$

but also

$$e^{2\pi ik} = 1, \quad k\in\mathbb{Z}.$$

Definition 3.6.3: Branch of the logarithm

Let $\Omega\subset\mathbb{C}$ be an open set. A branch of the logarithm, $\log_\Omega$, on Ω is a holomorphic function such that $\exp(\log_\Omega(z)) = z$.

Remark.

1. Since $\exp z \neq 0$ for all $z\in\mathbb{C}, \log_\Omega$ function can exist only if $0\notin\Omega$.
2. Let $\Omega = \mathbb{C}\setminus\{0\}$. Even though

$$\exp : \mathbb{C}\to\mathbb{C}\setminus\{0\}$$

is surjective, there is no holomorphic choice of logarithm in $\mathbb{C}\setminus\{0\}$. If it existed $f\in\mathcal{H}(\mathbb{C}\setminus\{0\})$

$$\exp(f(z)) = z$$

differentiating we get

$$f'(z)\underbrace{\exp(f(z))}_{z} = 1 \quad \forall z\in\Omega$$

where

$$f'(z) = \frac{1}{z} \quad \forall z\in\mathbb{C}\setminus\{0\}.$$

But then

$$\int_\gamma \frac{1}{z} \mathrm{d}z$$

should be 0 for any closed curve in Ω because f is a primitive of $1/z$, which is not true.

3. If Ω is open and connected, and $\ell = \log_\Omega : \Omega \to \mathbb{C}$ is a logarithm on Ω. If $\tilde{\ell}$ is also a logarithm on Ω then $\tilde{\ell} - \ell = 2\pi i n$ for $n \in \mathbb{Z}$. That is because

$$\exp(\ell(z)) = z, \exp\left(\tilde{\ell}(z)\right) = z$$

which implies

$$\exp\left(\ell(z) - \tilde{\ell}(z)\right) = 1 \implies \tilde{\ell}(z) - \ell(z) \in 2\pi i\mathbb{Z} \text{ for all } z \in \Omega$$

i.e. $\frac{\tilde{\ell}(z) - \ell(z)}{2\pi i}$ is an integer valued continuous function on a connected set Ω. Hence it is a single point n. Conversely if $\tilde{\ell} = \ell + 2\pi i n$ then $\exp\left(\tilde{\ell}(z)\right) = \exp(\ell(z))\exp(2\pi i n)) = \exp(\ell(z)) = z$.

Theorem 3.6.4: Existence of branch of logarithm

Let $\Omega \subset \mathbb{C} \setminus \{0\}$ be simply connected. Then there exists a branch of logarithm on Ω, i.e.

$$F : \Omega \to \mathbb{C}$$

such that F is holomorphic and $\exp(F(z)) = z$ for all $z \in \Omega$.

Proof. Since $0 \notin \Omega$ we have $1/z \in \mathcal{H}(\Omega)$. Since Ω is simply connected $1/z$ has a primitive, call it $f(z)$. Let

$$G(z) := z\exp(-f(z)).$$

The derivative is

$$G'(z) = -f'(z)z\exp(-f(z)) + \exp(-f(z)) = 0.$$

Since Ω is connected $G(z) = a = ze^{-f(z)}$ is a constant which is non zero since $z \neq 0$ and $\exp \neq 0$. So there exists b such that $a = \exp(b)$. Let $F(z) = f(z) + b$. Then

$$\exp(F(z)) = \underbrace{\exp(f(z))}_{z/a}\underbrace{\exp(b)}_{a} = z,$$

i.e. $F(z)$ is a branch of logarithm on Ω. ☐

Definition 3.6.5: Principal branch of logarithm

Let $\Omega := \mathbb{C}^- := \mathbb{C} \setminus \{-\infty, 0]$. The principal branch of logarithm is the unique $\log_\Omega \in \mathcal{H}(\Omega)$ such that $\log(1) = 0$.

Remark. Sometimes the principal branch of logarithm is denoted by Log.

Proposition 3.6.6: Calculation of the logarithm

If $z = re^{i\theta} \in \mathbb{C}^-$ with $r > 0, -\pi < \theta < \pi$ then the principal branch is given by

$$\log z = \log r + i\theta.$$

Proof. Let

$$\log z := \int_{\gamma_z} \frac{\mathrm{d}w}{w}$$

be a primitive of $1/z$ where we take the path γ_z which starts at 1 and ends at z. Note that by definition $\log 1 = 0$. If $z = re^{i\theta}$, $r < 1$, take the path γ_z which goes on the real line from 1 to r, then on the circular arc to z. We have

$$\log z = \underbrace{-\int_r^1 \frac{\mathrm{d}x}{x}}_{\text{on the } x_- \text{ axis}} + \underbrace{\int_0^{-\theta} \frac{-ire^{-i\theta}}{re^{-it}} \mathrm{d}t}_{\text{on the arc } z=re^{-it},\ 0<t<\theta} = \log r + i\theta.$$

☐

Remark. $r > 1$ is similar and was an exercise.

Remark.

1. The identity
$$\log z + \log w = \log(zw)$$
does *not* hold for all $z, w, zw \in \mathbb{C}^-$. If $w = re^{i\alpha}$, $z = se^{i\beta}$, $zw = rse^{i\theta}$ and $-\pi < \alpha, \beta, \theta < \pi$. Then there exists $\gamma \in \{-2\pi, 0, 2\pi\}$ such that $\theta = \alpha + \beta + \gamma$. Then we have
$$\begin{aligned}\log wz = \log rs + i\theta &= \log r + \log s + i(\alpha + \beta + \gamma) \\ &= (\log r + i\alpha) + (\log s + i\beta) + i\gamma.\end{aligned}$$
This implies $\log wz = \log w + \log z \Leftrightarrow \gamma = 0 \Leftrightarrow \alpha + \beta \in (-\pi, \pi)$.

2. For the principal branch of log one has the Taylor expansion

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-1)^n}{n}, \quad |z-1| < 1.$$

Take the derivative of both sides then the left hand side has derivative $(\mathrm{LHS})' = 1/z$ and the right hand side

$$(\mathrm{RHS})' = \sum_{n=0}^{\infty}(1-z)^n = \frac{1}{1-(1-z)} = \frac{1}{z}, \quad |z-1| < 1.$$

The right and left hand side differ by a constant. Choose $z = 1$ to see that the constant is 0.

3. The image of the punctured circle $\{z \in \mathbb{C} \mid |z| = r,\ -\pi < \arg z < \pi\}$ is the vertical interval $\{w \in \mathbb{C} \mid \mathrm{Re}(w) = \log|z|,\ -\pi < \mathrm{Im}(w) < \pi\}$, as shown in Figure 3.6

$$\text{if } r < 1 \text{ then } \operatorname{Re} w < 0,$$
$$\text{if } r > 1 \text{ then } \operatorname{Re} w > 0.$$

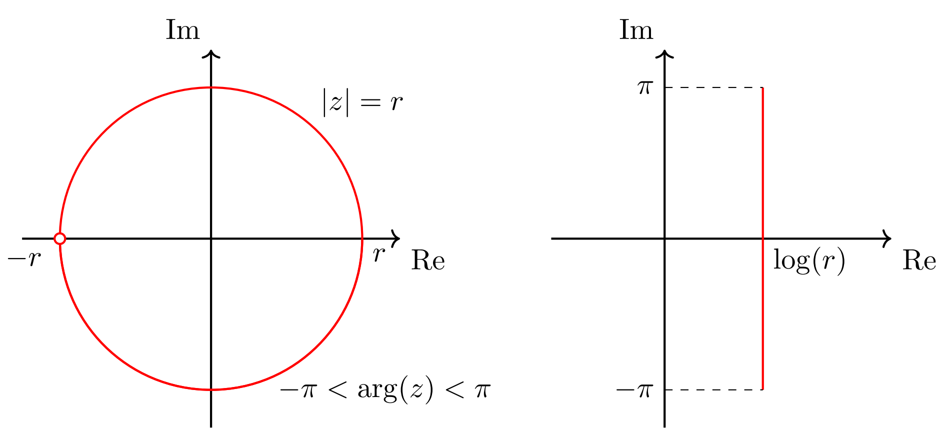

Figure 3.6: The punctured circle $\{z \in \mathbb{C} : |z| = r,\ -\pi < \arg(z) < \pi\}$(left) and its image under the principle branch of log (right)

4. The image of $\{z \in \mathbb{C} \mid \operatorname{Arg} z = \theta\}$, a ray from 0 to ∞, is the horizontal line $\{w \in \mathbb{C} \mid \operatorname{Im} w = \theta\}$, as shown in Figure 3.7.

5. We can define a holomorphic branch of logarithm for any

$$\Omega := \mathbb{C} \setminus (\{z \mid \operatorname{Arg} z = \alpha\} \cup \{0\}).$$

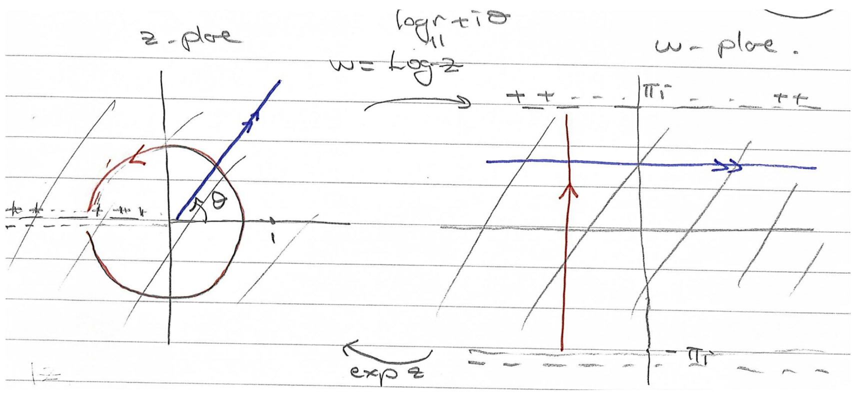

Figure 3.7: Image of $\{z \in \mathbb{C} \mid \operatorname{Arg} z = \theta\}$

Definition 3.6.7: Power function

Let $\Omega \subset \mathbb{C} \setminus \{0\}$ which is simply connected and $\log_\Omega : \Omega \to \mathbb{C}$ a branch of logarithm. Let $\alpha \in \mathbb{C}, z \in \Omega$, we define

$$z^\alpha := \exp(\alpha \log_\Omega(z)).$$

Note that this definition depends on the choice of $\log_\Omega$. If we choose $\log_\Omega + 2\pi i k$ instead, then

$$\exp(\alpha(\log_\Omega z + 2\pi i k)) = z^\alpha e^{2\pi i k \alpha}.$$

In particular if we choose the principal branch with $\log 1 = 0$ and $\alpha = 1/m$ then $z^{1/m} = e^{1/m \log z}$. We get

$$\begin{aligned}\left(z^{1/m}\right)^m &= \exp\left(\frac{\log z}{m}\right) \cdots \exp\left(\frac{\log z}{m}\right)\\ &= \exp\left(\frac{m}{m} \log z\right) = z.\end{aligned}$$

Theorem 3.6.8:

If $f \in \mathcal{H}(\Omega)$, Ω simply connected, $f(z) \neq 0$ for all $z \in \Omega$. Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$, called logarithm of f such that $e^{g(z)} = f(z)$.

Proof. Exercise. Define g as a primitive of f'/f. □

Corollary 3.6.9:

If $f \in \mathcal{H}(\Omega)$, non vanishing and Ω simply connected. Then f has a square root in Ω, i.e. there exists $h : \Omega \to \mathbb{C}$ holomorphic such that $h^2(z) = f(z)$.

Proof. Take $h = \exp\left(\frac{1}{2}\log f\right) = \exp\left(\frac{1}{2}g(z)\right)$. Then $h^2 = \exp g(z) = f(z)$. □

Lecture 24

Definition 3.6.10: Winding number

Let $z_0 \in \mathbb{C}$ and γ a closed curve in $\mathbb{C}$, such that $z_0 \notin \gamma$. The **winding number** of γ around z_0 is

$$w_\gamma(z_0) = \mathrm{ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - z_0}.$$

Remark. Let $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi n$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - z_0} = n.$$

If $\gamma = z_0 + re^{it}$, $0 \le t \le 2\pi$ and $z_1 \neq z_0$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - z_1} = \begin{cases} 1 & z_1 \in \mathrm{int}(\gamma) \\ 0 & z_1 \notin \mathrm{int}(\gamma). \end{cases}$$

To get an intuition, let γ be a smooth curve $\gamma : [0,1] \to \mathbb{C}$, $\gamma(0) = \gamma(1)$,

$$\int_\gamma \frac{\mathrm{d}z}{z - z_0} = \int_0^1 \frac{\gamma'(t)\mathrm{d}t}{\gamma(t) - z_0}.$$

This looks like $\log(\gamma(t) - z_0)\big|_{t=0}^{t=1} = 0$, which is wrong. We need to use the complex logarithm: with the principle branch we get

$$\begin{aligned} &\log(\gamma(1) - z_0) - \log(\gamma(0) - z_0) \\ &= \log|\gamma(1) - z_0| - i\arg(\gamma(1) - z_0) - (\log|\gamma(0) - z_0| + i\arg(\gamma(0) - z_0)) \\ &= i(\arg(\gamma(1) - z_0) - \arg(\gamma(0) - z_0)). \end{aligned}$$

There is an ambiguity in defining the argument, but geometrically we can understand it as the total change in argument as $\gamma(t)$ moves around z_0. In particular, since the start and endpoints are the same, the difference in argument is a multiple of 2π.[6]

[6] The argument here is not completely rigorous, it is mainly to show the intuition behind the definition and later we will prove the key properties of the winding number.

Proposition 3.6.11:

Let γ be a closed curve in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \mathrm{im}(\gamma)$, which is open. Then the map $w_\gamma : \Omega \to \mathbb{C}$ takes values in $\mathbb{Z}$ and is continuous. Hence it is constant on any connected subset of Ω. Moreover, $w_r(z) = 0$ if $|z|$ is large enough.

Proof. Let $\gamma : [a, b] \to \mathbb{C}$ be a parametrisation of γ. Let

$$G(t) = \int_a^t \frac{\gamma'(s)\mathrm{d}s}{\gamma(s) - z}.$$

Note $G(a) = 0$ and $G(b) = 2\pi i w_\gamma(z)$. $G(t)$ is continuous and except possibly for finitely many points, it is differentiable, with

$$G'(t) = \frac{\gamma'(t)}{\gamma(t) - z}.$$

Let $H(t) := (\gamma(t) - z)e^{-G(t)}$, then

$$\begin{aligned} H'(t) &= \gamma'(t)e^{-G(t)} - (\gamma(t) - z)G'(t)e^{-G(t)} \\ &= 0. \end{aligned}$$

Hence H is constant: $H(t) = (\gamma(t) - z)e^{-G(t)} = c \in \mathbb{C}$, so $\gamma(t) - z = ce^{G(t)}$ for all $t \in [a, b]$. Moreover, we have $c = ce^{G(a)} = \gamma(a) - z$, which implies $\gamma(b) - z = ce^{G(b)} = c$ as well. This gives $e^{G(b)} = 1$, so $G(b) = 2\pi i w_\gamma(z) \in 2\pi i\mathbb{Z}$.

If $M := \sup_{t\in[a,b]} |\gamma(t)|$. Let $|z| > M$, then

$$|w_\gamma(z)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}w}{w - z} \right| \leq \frac{1}{2\pi} \frac{\mathrm{length}(\gamma)}{|z| - M},$$

since $|w - z| \geq ||w| - |z|| \geq ||z| - M|$. When $|z| \to \infty$, LHS goes to 0, and in particular, there exists R such that for $|z| > R$, $|w_\gamma(z)| < 1/2$. Since $w_\gamma(z)$ is an integer, this implies $w_\gamma(z) = 0$ for z large enough. □

Theorem 3.6.12: Residue Formula version 2

Let $\Omega \in \mathbb{C}$ be simply connected and $f \in \mathcal{M}(\Omega)$, $V = \Omega \setminus S_f$, where S_f is the set of poles. For a closed curve γ in V, we have

$$\int_\gamma f\mathrm{d}z = 2\pi i \sum_{z_0 \in S_f} w_\gamma(z_0) \operatorname{res}_{z_0} f.$$

Proof. For any $z_0 \in S_f$, let $P_{z_0}(z)$ be the principle part of f at z_0,

$$P_{z_0}(z) = \sum_{j=1}^{N(z_0)} \frac{a_{-j}(z_0)}{(z-z_0)^j},$$

where $N(z_0)$ is the order of the pole.

Case 1. S_f is finite. Then

$$\tilde{f} = f - \sum_{z_0 \in S_f} P_{z_0}$$

has removable singularities at $z_0 \in S_f$ and hence has a holomorphic extension. This means the integral of $\tilde{f}$ along γ is 0 and hence

$$\int_\gamma f \mathrm{d}z = \sum_{z_0 \in S_f} \int_\gamma \frac{a_{-1}(z_0)}{z - z_0} \mathrm{d}z = \sum_{z \in S_f} w_\gamma(z_0) a_{-1}(z_0).$$

Case 2. S_f is infinite. Pick $R > 0$ such that $D_R(0)$ contains γ and $w_\gamma(z) = 0$ if $|z| \geq R$. Then $S_f \cap D_R(0)$ is finite because S_f is a discrete set. Similar to the previous case, let

$$\tilde{f} = f - \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} P_{z_0},$$

which is holomorphic in $\Omega \cap D_R(0)$. Hence we have

$$\int_\gamma \tilde{f} \mathrm{d}z = 0,$$

which implies

$$\begin{aligned} \int_\gamma f \mathrm{d}z &= \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} \int_\gamma P_{z_0}(z) \mathrm{d}z \\ &= 2\pi i \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} (\operatorname{res}_{z_0} f) w_\gamma(z_0) \\ &= 2\pi i \sum_{z_0 \in S_f} (\operatorname{res}_{z_0} f) w_\gamma(z_0), \end{aligned}$$

since $w_\gamma(z_0) = 0$ for any $z_0 \in S_f$ outside $D_R(0)$. $\square$

Chapter 4

Conformal Mappings

Lecture 25

Motivation: We want to answer the following questions:

1. Given $U, V \subset \mathbb{C}$, when does there exist a holomorphic function

$$f : U \to V$$

that is a bijection? We will see that $f^{-1} : V \to U$ is automatically holomorphic.

2. When does there exist a holomorphic bijection between $\Omega \subset \mathbb{C}$ and the unit disk?

4.1 Conformal equivalence

Definition 4.1.1: Conformal map

Let $U, V \subset \mathbb{C}$ be open sets. An injective holomorphic map $f : U \to V$ is called a **conformal map**. If f is bijective, then it is called a **conformal equivalence**, and we say U and V are **conformally equivalent**. Other terms used to describe conformal equivalences include biholomorphism and holomorphic isomorphism.

Proposition 4.1.2: Holomorphicity of the inverse

If $f : U \to V$ is conformal (i.e. holomorphic and injective), then for all $z \in U, f'(z) \neq 0$. The inverse of f defined on its image

$$f^{-1} : f(U) \to U$$

is also holomorphic.

Proof. Assume on the contrary that there exists $z_0 \in U$ such that $f'(z_0) = 0$. We want to show that in this case f cannot be injective. Let $h(z) = f(z) - f(z_0)$, then $h(z_0) = h'(z_0) = 0$. This implies that $k := \operatorname{ord}_{z_0} h \geq 2$. If $k = \infty$, then $f(z) \equiv f(z_0) = \text{const}$, which is not possible due to injectivity. Hence $k < \infty$ and there exists $r > 0$ such that for all $z \in D_r(z_0)$,

$$f(z) - f(z_0) = \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + G(z)(z - z_0)^{k+1}$$

and

$$\frac{f^k(z_0)}{k!} =: a \neq 0.$$

Since the zeros of f' (which is holomorphic) are isolated, we can choose $r > 0$ such that $f'(z) \neq 0$ in $D_r^*(z_0)$. Now we want to use Rouché's Theorem 3.4.3 to show that for some $w \in \mathbb{C}$,

$$g(z) := f(z) - f(z_0) - w$$

has the same number of zeroes as $a(z - z_0)^k - w$ in some disc around z_0. To do this, we write for $z \in D_r^*(z_0)$

$$\begin{aligned} f(z) - f(z_0) - w &= a(z - z_0)^k + G(z)(z - z_0)^{k+1} - w \\ &= \Big(a(z - z_0)^k - w\Big) + G(z)(z - z_0)^{k+1}. \end{aligned}$$

We apply Rouché 3.4.3 as follows: Let $C = \sup_{|z-z_0|=r/2} |G(z)|$. Pick $0 < s < r/2$ with $s < 1$, and let

$$|w| < |a| \left(\frac{s}{2}\right)^k.$$

Then on the circle $|z - z_0| = s$,

$$\Big|a(z - z_0)^k - w\Big| \geq \left(|a|s^k - |a|\left|\frac{s}{2}\right|^k\right) \geq |a| \left(\frac{s}{2}\right)^k.$$

We also have that

$$\left|G(z)(z-z_0)^{k+1}\right| \le Cs^{k+1}$$

on $|z-z_0| = s$. So if we choose $a(s/2)^k > Cs^{k+1}$, i.e. $s < a/(c \cdot 2^k)$, then we can apply Rouché 3.4.3 to conclude that $g(z) = f(z) - f(z_0) - w$ has the same number of zeros in $|z - z_0| < s$ as $a(z-z_0)^k - w$ for $|w| < |a|(s/2)^k$. Note that if $w = re^{i\theta}$, then the zeros of $a(z-z_0)^w$ are at z_n for $n = 0, 1, \ldots, k-1$, satisfying

$$z_n - z_0 = \left|\frac{w}{a}\right|^{1-k} e^{i\left(\frac{\theta+2n\pi}{k}\right)}.$$

But then $|z_n - z_0| = |w/a|^{1/k} < s/2 < s$. Hence, all roots are inside $D_s(z_0)$, with $s < \min\{|a|/(C \cdot 2^k), r/2, 1\}$ and $|w| < |a|(s/2)^k$. Hence for a suitable choice of w, g has the same number of zeros as $a(z-z_0)^k$, namely k zeros. Let $z_1, z_2, \ldots, z_k$ be the zeros of g. If we choose $w \neq 0$, which we can do, then these zeros are not equal to z_0, because if $z_k = z_0$, it implies $0 = g(z_k) = f(z_0) - f(z_0) - w \neq 0$. Since $f'(z) \neq 0$ in $D_r^*(z_0)$, we have that $g'(z) = f'(z) \neq 0$ for $z \in D_r^*(z_0)$. Hence each zero has order 1 and they are distinct. That means $f(z_\ell) - f(z_0) - w = 0$ for k distinct z_ℓ, which contradicts injectivity. This proves that $f'(z) \neq 0$ for all $z \in U$.

Note that $f : U \to f(U)$ is bijective, so without loss of generality assume $f(U) = V$, which means $f^{-1} : V \to U$ is bijective and continuous, as f is an open map. Let $w_0 \in V$ and $w \in V$ be close to w_0. Write $w = f(z), w_0 = f(z_0)$. For $w \neq w_0$ we have

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}.$$

Now we take the limit $w \to w_0$ to show that $(f^{-1})'$ exists. □

Remark.

1. Proposition 4.1.2 says that if $f : U \to V$ is a conformal equivalence then f^{-1} is also a conformal equivalence.

2. Conformal equivalence is indeed an equivalence relation because

   (a) $U \sim_C U$ as $\mathrm{id} : U \to U$ is a conformal equivalence;

   (b) if $U \sim_C V$ by $f : U \to V$, then $V \sim_C U$ by $f^{-1} : V \to U$;

   (c) if $U \sim_C V$ by $f : U \to V$, $V \sim_C W$ by $g : V \to W$, then $U \sim_C W$ by $g \circ f : U \to W$.

COROLLARY 4.1.3:

If $f : U \to V$ is a conformal equivalence, then the map

$$\begin{aligned} T : \mathcal{H}(V) &\longrightarrow \mathcal{H}(U) \\ \phi &\longmapsto \phi \circ f \end{aligned}$$

is a linear isomorphism between two vector spaces.

Example 4.1.4

Consider half plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$ and the unit disc $\mathbb{D} := D_1(0)$. The map

$$\begin{aligned} f : \mathbb{H} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{z - i}{z + i} \end{aligned}$$

is a conformal equivalence (a Möbius transformation), as shown in Figure 4.1, with inverse

$$f^{-1}(w) = i\frac{1 + w}{1 - w}.$$

For any $z \in \mathbb{H}$ with $z + i \neq 0$, so f is clearly holomorphic on $\mathbb{H}$. The function $g = f^{-1}$ is also holomorphic. To see that $g(w) \in \mathbb{H}$ we calculate

$$\begin{aligned} g(w) &= \frac{i\left(\frac{1+w}{1-w}\right) - \overline{i\left(\frac{1+w}{1-w}\right)}}{2i} \\ &= \frac{1 - |w|^2}{|1 + w|^2} > 0. \end{aligned}$$

One can check directly that $f(g(w)) = w$ and $g(f(w)) = z$.

Example 4.1.5

Consider the map

$$\begin{aligned} f : U &\longrightarrow \mathbb{H} \\ z &\longmapsto z^2, \end{aligned}$$

where $U = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/2\}$, then $g(z) = f^{-1}(w) = w^{1/2} = \exp(\log(z)/2)$ (with log the principle branch). Note f is injective: if $z_1^2 = z_2^2$, then $z_1 = \pm z_2$ and only one of them can be in U. To show surjectivity, let $w := re^{i\theta}$,

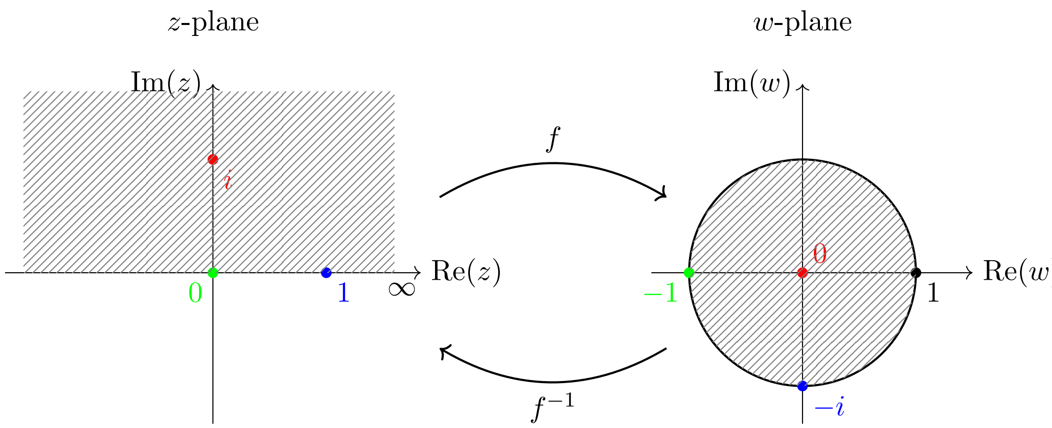

Figure 4.1: Conformal equivalence between $\mathbb{H}$ and $\mathbb{D}$

$0 < \theta < \pi$. Then $z = r^{1/2}e^{i\theta/2} \in U$ satisfies $z^2 = w$. In general the map $z \to z^n$ maps a sector $S = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/n\}$ to $\mathbb{H}$ conformally.

Example 4.1.6

Any horizontal strip of length 2π is conformally equivalent to a cut plane (split plane). Someone make this diagram.

Example 4.1.7

An important **non example**: Let $U = \mathbb{C}, V = \mathbb{D}$, then there is no conformal equivalence between $\mathbb{C}$ and $\mathbb{D}$ because otherwise $f : \mathbb{C} \to \mathbb{D}$ would be a holomorphic and bounded function. Hence by Liouville's Theorem 2.4.6, it would be constant.

4.2 Riemann Mapping Theorem

Theorem 4.2.1: Riemann Mapping Theorem

Suppose $\Omega \subsetneq \mathbb{C}$ non-empty, simply connected. If $z_0 \in \Omega$, then there exists a unique conformal equivalence $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

Idea of proof:

1. *Uniqueness*: this boils down to finding all conformal automorphisms of the unique disc. $f_1 : \Omega \to \mathbb{D}$, $f_2 :\to \Omega \to \mathbb{D}$ two conformal equivalences, then $f_2 \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}$.

2. If $\Omega \neq \mathbb{C}$ we will show that there is a conformal map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$. Hence Ω is conformally equivalent to a subset of the unit disk $\overline{D_1(0)}$.

3. Step 2 shows that the set $\mathcal{F} := \{f : \Omega \to \mathbb{D}, f \text{ conformal}, f(z_0) = 0\}$. We will see that $s : \sup_{f\in\mathcal{F}} |f'(z)|$ exists and there exists $f \in \mathcal{F}$ such that $|f'(z_0)|$ is maximal. This f has maximal expansion speed.

4. The function f found in step 3 is actually surjective. Lecture 26

Proof. **Step 1. Automorphisms of $\mathbb{D}$ and uniqueness.**

Theorem 4.2.2: Automorphisms of $\mathbb{D}$

If $f : \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$. Then there exists $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that

$$f(z) = e^{i\theta}\left(\frac{\alpha - z}{1 - \overline{\alpha}z}\right).$$

Then $f(0) = e^{i\theta}\alpha$ and $f'(0) = e^{i\theta}(|\alpha|^2 - 1)$. Conversely every map of this form is an automorphism of $\mathbb{D}$.

The next corollary will give the uniqueness in Riemann's Theorem.

Corollary 4.2.3:

The map in Riemann's Theorem is unique, i.e. $f_i : \Omega \to \mathbb{C}, f_i(z_0) = 0, f_i'(z_0) > 0, i \in \{1, 2\}$, then $f_1 = f_2$.

Proof of Corollary 4.2.3. Let $g = f_2 \circ f_1^{-1}$, then $g : \mathbb{D} \to \mathbb{D}$ an automorphism of $\mathbb{D}$. Hence by Theorem 4.2.2,

$$g(z) = e^{i\theta}\left(\frac{\alpha - z}{1 - \overline{\alpha}z}\right)$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Since $f_i(z_0) = 0$, $g(0) = 0$, which implies $\alpha = 0$ and $g(z) = -e^{i\theta}z$ for some $z \in \mathbb{D}$.

$$\begin{aligned} g'(0) &= -e^{i\theta} \\ &= (f_2 \circ f_1^{-1})'(0) \\ &= f_2'(f_1^{-1}(0))(f_1^{-1})'(0) \\ &= \frac{f_2'(z_0)}{f_1'(z_0)} > 0 \end{aligned}$$

$-e^{i\theta} > 0$ implies that $\theta = \pi + 2\pi k$, hence $g(z) = z$ so we have $f_1 = f_2$. □

Remark. If $f(0) = 0$, then $f(0) = e^{i\theta}\alpha = 0 \implies \alpha = 0$. If $\alpha = 0$ then $f(z) = -e^{i\theta}z = e^{i\tilde{\theta}}z$, i.e. f is a rotation.

To prove Theorem 4.2.2 we need another important lemma.

Lemma 4.2.4: Schwarz Lemma

Let $f : \mathbb{D} \to \mathbb{D}$ holomorphic with $f(0) = 0$. Then the following holds:

1. $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.
2. *If for some* $z_0 \neq 0$ *with* $|f(z_0)| = |z_0|$, *then* f *is a rotation (i.e. there exists* $\theta \in \mathbb{R}$ *such that* $f(z) = e^{i\theta} z$*).*
3. $|f'(0)| \leq 1$. *We have equality* $|f'(0)| = 1$ *if and only if* f *is a rotation.*

Proof of Lemma 4.2.4. We will use Maximum Modulus Principle 3.4.7.

1. Since $f(0) = 0$, $\operatorname{ord}_0 f \geq 1$. Let

$$g(z) = \frac{f(z)}{z} \in \mathcal{H}(\mathbb{D}).$$

Fix $z \in \mathbb{D}$, choose $0 \leq |z| < r < 1$. For $|w| = r$, we have by Maximum Modulus Principle 3.4.7,

$$|g(z)| \leq \max_{|w|=r} |g(w)|.$$

We write

$$\begin{aligned} |g(z)| \leq \max_{|w|=r} |g(z)| &= \frac{1}{r} \max_{|w|=r} |f(w)| \\ &\leq \frac{1}{r}, \end{aligned}$$

since $|f(w)| < 1$ for all $z \in \mathbb{D}$ (f is an automorphism). The above is true for any r, letting $r \to 1$ we have $|g(z)| \leq 1$, which means $|f(z)| \leq |z|$.

2. Let g be as in the first part, so $\sup_{z \in D_1(0)} |g(z)| \leq 1$. But if we have $|f(z_0)| = |z_0|$ for some $z_0 \neq 0, z_0 \in D_1(0)$, then the maximum of g is attained inside the disc. This will contradict the Maximum Modulus Principle 3.4.7 for g, unless it is constant. This means that there exists $c \in \mathbb{C}$ so that

$$\frac{f(z)}{z} = g(z) \equiv c,$$

hence $f(z) = cz$ for all $z \in D_1(0)$. We also have $|f(z_0)| = |z_0|$ for some z_0, this implies $|c| = 1$, so $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and $f(z) = e^{i\theta} z$, a rotation.

3. We have

$$
\begin{aligned}
g(0) &= \lim_{z\to 0} \frac{f(z)}{z} \\
&= \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} \\
&= f'(0).
\end{aligned}
$$

Hence $|f'(0)| = |g(0)| \le 1$. If $|f'(0)| = 1$ then again 0 is a local maximum of g, which can not happen unless (by the Maximum Modulus principle 3.4.7) g is a constant. As above this gives f is a rotation.

□

Proof of Theorem 4.2.2. Let

$$\varphi_\alpha(z) := \frac{\alpha - z}{1-\overline{\alpha}z}$$

for $\alpha \in \mathbb{C}$ with $|\alpha| < 1$.

Claim. $\varphi_\alpha(z)$ is an automorphism of $\mathbb{D}$.

Proof of claim.

1. For $|\alpha| < 1$ we have $|1-\overline{\alpha}z| \neq 0$ for $|z| < 1$. Hence $\varphi_\alpha \in \mathcal{H}(\mathbb{D})$.
2. If $\varphi_\alpha(z) = \varphi_\alpha(w)$, then

$$\frac{\alpha - z}{1-\overline{\alpha}z} = \frac{\alpha - w}{1-\overline{\alpha}w} \iff (1-|\alpha|^2)z = (1-|\alpha|^2)w \iff z = w.$$

   This shows φ_α is injective.
3. Finally we show $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$. If $|z| = 1$ then $z = e^{i\theta}$

$$\varphi_\alpha(e^{i\theta}) = \left| \frac{\overbrace{\alpha - e^{i\theta}}^{:=w}}{e^{i\theta}(e^{-i\theta}-\overline{\alpha})} \right| = \left|e^{-i\theta}\right|\left|\frac{w}{\overline{w}}\right| = 1.$$

   By Maximum Modulus Principle 3.4.7, for $z \in \mathbb{D}$, $|\varphi_\alpha(z)| < 1$.
4. One sees that $(\varphi_\alpha \circ \varphi_\alpha)(z) = z$, hence $\varphi_\alpha^{-1} = \varphi_\alpha$.

This shows the claim. ■ Lecture 27

Now suppose f is an automorphism of $\mathbb{D}$, there exists a unique $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Now consider $g = f \circ \varphi_\alpha$, then by definition $g(0) = 0$. By Schwarz Lemma

4.2.4 applied to g, we have $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Since $g^{-1}(0) = 0$ and g^{-1} is also an automorphism of $\mathbb{D}$, we get that $\left|g^{-1}(w)\right| \leq |w|$ for all $w \in \mathbb{D}$. Applying this to $w = g(z)$ we get that $|z| \leq |g(z)|$ which implies that $|z| = |g(z)|$, and from this it follows that g is a rotation by Schwarz Lemma 4.2.4. Hence

$$g(z) = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$ and by construction,

$$e^{i\theta} z = (f \circ \varphi_\alpha)(z) = g(z).$$

Replace z by $\varphi_\alpha(z)$ we obtain

$$\begin{aligned} e^{i\theta}\varphi_\alpha(z) = (f \circ \varphi_\alpha)(\varphi_\alpha(z)) &= (f \circ \overbrace{\varphi_\alpha \circ \varphi_\alpha}^{\mathrm{id}})(z) \\ &= f(z) \end{aligned}$$

This proves Theorem 4.2.2. □

Remark. Combining automorphisms of $\mathbb{D}$ with the Cayley map

$$\begin{aligned} F : \mathbb{H} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{z-i}{z+i} \end{aligned}$$

one can show the following.

Theorem. *Every automorphism of $\mathbb{H}$ is of the form*

$$g(z) = \frac{az+b}{cz+d},$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}),\ ad - bc > 0.$

Proof. Read in the book. □

Step 2. If Ω is a proper and simply connected subset of $\mathbb{C}$, then Ω is conformally equivalent to a subset of $\mathbb{D}$.

Proposition 4.2.5:

Let $\Omega \subsetneq \mathbb{C}, \Omega \neq \varnothing$, simply connected and open. Then there exists $f : \Omega \to \mathbb{D}$ such that f is conformal and $0 \in f(\Omega)$, i.e. Ω is conformally equivalent to a subset of $\mathbb{D}$.

Proof. By assumption Ω is a proper subset, i.e. there exists $\alpha \in \mathbb{C}$ such that $\alpha \notin \Omega$. By replacing Ω with $\Omega - \alpha = \{z - \alpha : z \in \Omega\}$, we may assume $\alpha = 0$. Hence $\Omega \subset \mathbb{C}\setminus\{0\}$. Then there exists $\log_\Omega : \Omega \to \mathbb{C}$ with $\log_\Omega \subset \mathcal{H}(\Omega)$, a holomorphic branch of logarithm. Note $\log_\Omega$ is also injective since if $\log_\Omega z = \log_\Omega w$, exponentiating gives

$$\begin{aligned} z &= \exp(\log_\Omega z) \\ &= \exp(\log_\Omega w) \\ &= w, \end{aligned}$$

i.e. $\log_\Omega$ is conformal. Now let $w \in \Omega$. Then note that for any $z \in \Omega$, $\log_\Omega z \neq \log_\Omega w + 2\pi i$, since otherwise exponentiating gives $z = w$, but then $\log_\Omega z = \log_\Omega w$. In fact, $\log_\Omega(z)$ stays away from $\log_\Omega w + 2\pi i$ in the sense that there exists $\delta > 0$ such that $D_{2\delta}(\log_\Omega w + 2\pi i) \cap \log_\Omega(\Omega) = \varnothing$. Assume not, then for every $n > 0$ we get a sequence $(z_n)_{n=0}^\infty \subset \Omega$ such that $|\log_\Omega z_n - (\log_\Omega w + 2\pi i)| < 1/n$. Hence $\log_\Omega z_n \to \log_\Omega w + 2\pi i$. Exponentiating gives $z_n \to w$ and hence, using continuity of the logarithm, $\log_\Omega z_n \to \log_\Omega w$, which is a contradiction to $\log_\Omega z_n \to \log w + 2\pi i$. In the sequel we omit Ω in $\log_\Omega$, but we always refer to this branch of logarithm, $\log_\Omega$, defined on Ω. Now we consider the following map

$$\begin{aligned} F : \Omega &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{1}{\log z - (\log w + 2\pi i)}. \end{aligned}$$

Since $\log z \neq \log w + 2\pi i$ for any $z \in \Omega$, F is holomorphic. Now F is also injective since $\log z$ is injective, i.e. F is a conformal map. Since $D_{2\delta}(\log w + 2\pi i) \cap \log \Omega = \varnothing$ we have $\log z - (\log w + 2\pi i) \geq 2\delta$. Hence

$$|F(z) - 0| = \left| \frac{1}{\log z + (\log w + 2\pi i)} \right| \leq \frac{1}{2\delta} < \frac{1}{\delta}.$$

Hence $F(\Omega) \subset D_{1/\delta}(0)$. We can now translate and rescale to find the function we are looking for, namely

$$f(z) := \frac{\delta}{4}(F(z) - F(w)).$$

Then $f : \Omega \to \mathbb{C}$ is conformal, because F is conformal. We also have $f(w) = 0$ and

$$|f(z)| \leq \frac{\delta}{4}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) \leq \frac{1}{2},$$

which implies $f(\Omega) \subset D_1(0)$. □

Step 3. An extremal problem.

Define $\mathscr{F} := \{F : \Omega \to \mathbb{D} : F \text{ conformal}, f(z_0) = 0\}$. By Step 2, $\mathscr{F} \neq \varnothing$. We start with the following lemma.

Lemma 4.2.6:

The set of values $\{|f'(z_0)| : f \in \mathscr{F}\}$ is bounded in $[0, \infty)$, that is, $\sup_{f\in\mathscr{F}} |f'(z_0)| = s < \infty$.

Proof. Let $\delta > 0, D_{2\delta}(z_0) \subset \Omega,\ f \in \mathscr{F}$. Cauchy Integral Formula for derivatives 2.4.4 gives

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_\delta(z_0)} \frac{f(z)}{(z - z_0)^2}\mathrm{d}z,$$

which implies

$$\begin{aligned} \left|f'(z_0)\right| &\leq \frac{1}{2\pi i} \cdot 2\pi\delta \cdot \frac{\max_{z\in C_\delta(z_0)} |f(z)|}{\delta^2} \\ &\leq \frac{1}{\delta}, \end{aligned}$$

since $|f(z)| < 1$. □

Proposition 4.2.7:

There exists $f \in \mathscr{F}$ such that $|f'(z_0)| = s$.

Step 4. Surjectivity of f from Proposition 4.2.7.

Proposition 4.2.8:

Let $f \in \mathscr{F}$ be such that $|f'(z_0)| = s = \sup_{f\in\mathscr{F}}|f'(z_0)|$, then f is a conformal equivalence, i.e. f is also onto $\mathbb{D}$.

Proof. To show that f is surjective, we assume this is not the case and find $g \in \mathcal{F}$, such that $g'(z_0) > |f'(z_0)|$. To do this, we will use φ_α for some appropriate α, and the squaring map. Since f is assumed to be not surjective, there exists $\alpha \in \mathbb{D}$

such that $f(z) \neq \alpha$ for all $z \in \Omega$. Recall that $\varphi_\alpha(0) = \alpha$ and $\varphi_\alpha(\alpha) = 0$. Then $\varphi_\alpha \circ f : \Omega \to \mathbb{D}$ is conformal and $0 \notin (\varphi_\alpha \circ f)(\Omega)$. If 0 were in the image, then $(\varphi_\alpha \circ f)(z) = 0$, which implies $f(z) = \alpha$ for some z. Since Ω is simply connected, $0 \notin (\varphi_\alpha \circ f)(\Omega)$, a logarithm and thus the square root of $\varphi_\alpha \circ f$ exists. There exists a holomorphic map $\tilde{f} : \Omega \to \mathbb{C}$ such that $\tilde{f}^2(z) = (\varphi_\alpha \circ f)(z)$, for all $z \in \Omega$. Note $\tilde{f}$ is injective, because if $\tilde{f}(z) = \tilde{f}(w)$, then $(\varphi_\alpha \circ f)(z) = (\varphi_\alpha \circ f)(w)$, this means $z = w$ as φ_α and f are both injective. We also notice that $\tilde{f}(z_0) \neq 0$ as $\varphi_\alpha(f(z_0)) \neq 0$. Let $\tilde{f}(z_0) = \beta$ and consider

$$\begin{aligned} \varphi_\beta : \mathbb{D} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{\beta - z}{1 - \overline{\beta} z}. \end{aligned}$$

Let $g : \Omega \to \mathbb{D}$ defined by $g(z) = \varphi_\beta \circ \tilde{f}$, then $g(z_0) = 0$.

Claim. $|g'(z_0)| > |f'(z_0)|$.

Proof of claim. We first formally construct $g : \Omega \to \mathbb{D}$ as described above. We already have the function $\varphi_\alpha \circ f : \Omega \to \mathbb{D} \to \mathbb{D}$. Let h be the square root function,

$$\begin{aligned} h : (\varphi_\alpha \circ f)(\Omega) &\longrightarrow \mathbb{D} \\ w &\longmapsto \exp\left(\frac{1}{2} \log w\right), \end{aligned}$$

so $h \circ \varphi_\alpha \circ f : \Omega \to \mathbb{D}$. We set $\tilde{f} = h \circ \varphi_\alpha \circ f$. Then we compose this with φ_β to get $g : \Omega \to \mathbb{D}$, where

$$\begin{aligned} g &= \varphi_\beta \circ \tilde{f} \\ &= \varphi_\beta \circ \underbrace{h \circ \varphi_\alpha \circ f}_{\tilde{f}}, \end{aligned}$$

and $\varphi_\beta^{-1} \circ g = h \circ \varphi_\alpha \circ f = \tilde{f}$. By construction

$$(\varphi_\beta^{-1} \circ g)^2 = \varphi_\alpha \circ f.$$

This implies

$$\varphi_\alpha^{-1} \circ (\varphi_\beta^{-1} \circ g)^2 = f.$$

Let $s(z)$ be the squaring map, so

$$f = \underbrace{\varphi_\alpha^{-1} \circ s \circ \varphi_\beta^{-1}}_{\Phi} \circ g = \Phi \circ g.$$

$\Phi : \mathbb{D} \to \mathbb{D}$ is holomorphic and *not* injective, because the squaring function is not. We have that

$$\begin{aligned} \Phi(0) &= (\varphi_\alpha^{-1} \circ s \circ \varphi_\beta^{-1})(0) \\ &= \varphi_\alpha^{-1}(\beta^2), \end{aligned}$$

since $\varphi_\beta^{-1}(0) = \beta$. It holds that

$$\beta^2 = (\tilde{f}(z_0))^2 = (\varphi_\alpha \circ f)(z_0)$$

since $\beta = \tilde{f}(z_0)$. Hence $\Phi(0) = (\varphi_\alpha^{-1} \circ \varphi_\alpha \circ f)(z_0) = f(z_0) = 0$.

Applying part 3 of Schwarz Lemma 4.2.4 we get $|\Phi'(0)| < 1$. We note that $|\Phi'(0)| \neq 1$, otherwise Φ would be a rotation and hence injective, but as said before Φ is not injective. By construction $f = \Phi \circ g$. Applying the chain rule we get

$$|f'(z_0)| = \left|\Phi'(g(z_0))g'(z_0)\right| = \left|\Phi'(0)g'(z_0)\right| < |g'(z_0)|.$$

■

Hence we can conclude. □ Lecture 28

Proof of Proposition 4.2.7. Recall:

1. For a bounded set $U \subset \mathbb{R}$ there exists a non-decreasing sequence $(a_n)_{n=0}^\infty \subset U$ such that $\lim_{n\to\infty} a_n = s = \sup U$. Hence if $f_n \in \mathscr{F}$ with $|f_n'(z_0)| \to s$, we want to show that $f_n \to f \in \mathscr{F}$.
2. We have seen that a sequence of holomorphic functions that converge uniformly on compact set has a limit which is also holomorphic. But we can not expect that an arbitrary sequence of holomorphic functions f_n is uniformly convergent on compact set.
3. In a finite-dimensional vector space $\mathbb{R}^n$, every bounded sequence has a convergent subsequence.

We are looking for an analogue of this, which is provided by

Theorem 4.2.9: Montel's Theorem

Let $\Omega \subset \mathbb{C}$ be open, $(f_n)_n \subset \mathcal{H}(\Omega)$. Suppose for all compact sets $K \subset \Omega$ there exists $M_k > 0$ such that $|f_n(z)| < M_k$ for all $n \geq 1$ and for all $z \in K$. Then there exists a subsequence (f_{n_k}) which converges uniformly on compacta.

Proof. No proof. □

We apply this theorem in our case, since $(f_n) \subset \mathscr{F}$, $f_n : \Omega \to \mathbb{D}$, so $|f_n(z)| < 1$ for all n and for all $z \in \Omega$. Hence by Montel's Theorem 4.2.9 there exists $(f_{n_k}) \subset \mathscr{F}$ which converges uniformly on compact sets with $\lim_{k\to\infty} f_{n_k} = f$, f holomorphic. We still need $f \in \mathscr{F}$, i.e. we want to show that f is injective and $f(\Omega) \subset \mathbb{D}$ (we already have $f(z_0) = 0$ since $f_{n_k}(z_0) = 0$).

Proposition 4.2.10:

Let (f_n) be a sequence in $\mathscr{F}$ and suppose $f_n \to f$ for all $z \in \Omega$ uniformly on compact sets. Then either f is constant or $f \in \mathscr{F}$, $\lim_{n\to\infty} f_n'(z_0) = f'(z_0)$.

Proof. First note $f_n \to f$ uniformly on compact sets, hence $f \in \mathcal{H}(\Omega)$ and $f_n'(z_0) \to f'(z_0)$. We still need

1. $f(\Omega) \subset \mathbb{D}$,
2. f is either injective for constant.

For 1. note $|f_n(z)| \leq 1$, since $f_n : \Omega \to \mathbb{D}$. Hence $|f(z)| = \lim |f_n(z)| \leq 1$. But if for some $z \in \Omega$, $f(z) = 1$, then f attains its maximum in a point inside $\mathbb{D}$. By maximum modulus principle, it is constant. This means that if $f(z)$ is not constant, then indeed $f(\Omega) \subset \mathbb{D}$.

To prove injectivity we use the following lemma.

Lemma 4.2.11:

$\Omega \subset \mathbb{C}$ open, connected. $f_n : \Omega \to \mathbb{D}$ conformal. If $f_n \to f$ uniformly on compacta, then f is either constant or injective.

Proof. Suppose f is not injective, we will show that f it constant. Let $z_1 \neq z_2 \in \Omega$ such that $f(z_1) = f(z_2)$ and suppose f is not constant. Since zeros of non-constant holomorphic functions are isolated, we can find a disc $D_\delta(z_2) \subset \Omega$ such that

$$f(z) - f(z_2) \neq 0$$

in $z \in D^*_\delta(z_2)$. In particular for all $z \in C_{\delta/2}(z_2)$, $f(z) - f(z_2) \neq 0$ and $z_1 \notin C_{\delta/2}(z_2)$. We apply argument principle 3.4.2.

$$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z) - f(z_1)} \mathrm{d}z \geq 1,$$

since z_2 is a zero of $f(z) - f(z_1)$ in $D_{\delta/2}(z_2)$. Moreover $f_n \to f$ uniformly on compact sets, in particular on $C_{\delta/2}(z_2)$. Also, $f_n(z) \neq f_n(z_1)$ for all $z \in C_{\delta/2}(z_2)$ (f_n's are injective and $z_1 \notin C_{\delta/2}(z_2)$. Now consider

$$\frac{f_n'(z)}{f_n(z) - f_n(z_1)} \to \frac{f'(z)}{f(z) - f(z_1)},$$

which converges uniformly on $C_{\delta/2}(z_2)$. Hence

$$\int_{C_{\delta/2}} \frac{f_n'(z)}{f_n(z) - f_n(z_1)} \to \int_{C_{\delta/2}} \frac{f'(z)}{f(z) - f(z_1)}.$$

However, the function on LHS is holomorphic, so the integral is always zero and cannot converge to a non-zero value, thus we have a contradiction. This proves the injectivity. □

This concludes the proof of proposition 4.2.10 □

This shows Proposition 4.2.7 and completes Step 3. □

This finally proves the Riemann Mapping Theorem. □

Corollary 4.2.12:

Any two proper open subsets of $\mathbb{C}$ which are simply connected are conformally equivalent.

4.3 Exam preparation

General suggestions:

1. For every definition, know an example and a counterexample.
2. For all theorems and formulas, have some examples in mind and know where and how they are used.
3. Know the ideas of proofs of main theorems.

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