Solutions Sheet 1

Radical ideals, decomposition, Zariski topology

Exercises 2 and 6 are taken from the book *Introduction to Commutative Algebra* by Atiyah and MacDonald.

1. (a) Show that if $\mathfrak{a}$ is an ideal in a ring R and $\mathrm{Rad}(\mathfrak{a})$ its radical ideal, then $V(\mathfrak{a}) = V(\mathrm{Rad}(\mathfrak{a}))$.

   (b) Show that a proper ideal $\mathfrak{p} \subsetneqq R$ is a prime ideal if and only if, for any ideals $\mathfrak{a}, \mathfrak{b} \subset R$ with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, we have $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

   (c) Show that every prime ideal is radical. Find an example which shows that the converse is *not* true.

2. Let X be a topological space. Show that

   (a) For any irreducible subspace Y of X, the closure $\overline{Y}$ of Y in X is irreducible.

   (b) Every irreducible subspace of X is contained in a maximal irreducible subspace.

   (c) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X.

   (d) What are the irreducible components of a Hausdorff space?

   Solution:

   (a) By definition of $\overline{Y}$, any non-empty open subset of $\overline{Y}$ has non-empty intersection with Y. From this, the statement follows immediately.

   (b) This is implied through Zorn's Lemma as follows: Let Y be an irreducible subset of X. Consider the set Σ of irreducible subsets of X which contain Y. It is partially ordered through the inclusion relation. We have $\Sigma \neq \varnothing$, as $Y \in \Sigma$. We claim that for any chain $(Y_i)_{i\in I}$ in Σ, the union $\tilde{Y} := \bigcup_{i\in I} Y_i$ is in Σ, and therefore is an upper bound for $(Y_i)_{i\in I}$. Indeed, assume U, U' are non-empty open subsets of $\tilde{Y}$. There must thus exist $i_1, i_2 \in I$ with $U \cap Y_{i_1} \neq \varnothing$ and $U' \cap Y_{i_2} \neq \varnothing$. Without loss of generality $Y_{i_2} \subset Y_{i_1}$. As Y_{i_1} is irreducible, we find $(U \cap Y_{i_1}) \cap (U' \cap Y_{i_1}) \neq \varnothing$ and in particular $U \cap U' \neq \varnothing$. Hence $\tilde{Y} \in M$. This guarantees, by Zorn's Lemma, a maximal element in Σ, which is precisely a maximal irreducible subset of X containing Y.

   (c) They are closed by (a). They cover X because each element of X forms an irreducible subset and is thus contained in an irreducible component by (b).

   (d) The irreducible components of a Hausdorff space are the one point subsets.

3. Determine the ideal in $\mathbb{R}[\underline{X}]$ of

   (a) the union of the three coordinate axes in $\mathbb{R}^3$,

   (b) the union of the lines containing the twelve edges of the cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$,

   (c) the set $\{(n, e^n) \mid n \in \mathbb{Z}^{\geqslant 0}\}$ in $\mathbb{R}^2$.

   Solution (Sketch):

   (a) The corresponding ideal is

   $$(Y, Z) \cap (X, Z) \cap (X, Y) = (XY, Z) \cap (X, Y) = (XY, XZ, YZ).$$

   (b) The ideal is the intersection of the twelve ideals $(X \pm 1, Y \pm 1), (X \pm 1, Z \pm 1)$ and $(Y \pm 1, Z \pm 1)$, which is

   $$\big((X^2 - 1)(Y^2 - 1), (X^2 - 1)(Z^2 - 1), (Y^2 - 1)(Z^2 - 1)\big).$$

   (c) The ideal of the set $X := \{(n, e^n) \mid n \in \mathbb{Z}^{\geqslant 0}\}$ is the zero ideal. Equivalently, for any non-zero polynomial $f \in \mathbb{R}[X, Y]$, there exists an integer $n \geqslant 0$ with $f(n, e^n) \neq 0$. To see this write $f = \sum_{k=0}^{d} f_k Y^k$ with $f_i \in \mathbb{R}[X]$ and $f_d \neq 0$. Then by looking at growth rates find that $|f_d(n) e^{dn}| > |\sum_{k=0}^{d-1} f_k(n) e^{kn}|$ and hence $f(n, e^n) \neq 0$ for all $n \gg 0$.

4. Compute the irreducible components of $V(XZ - Y^2, X^3 - YZ)$ in $\mathbb{C}^3$.

   Solution (Sketch): The irreducible components are the Z-axis, with ideal (X, Y), and the curve

   $$V(Y^2 - XZ, X^2 Y - Z^2, X^3 - YZ) \;=\; \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\}.$$

*5 Show that every ring homomorphism $\varphi \colon R \to R'$ induces a continuous map $\operatorname{Spec} R' \to \operatorname{Spec} R$, $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

6. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of a ring R. Show that

   (a) the set $\{\mathfrak{p}\}$ is closed in $\operatorname{Spec} R$ if and only if $\mathfrak{p}$ is maximal. In that case we call $\mathfrak{p}$ a *closed point*.

   (b) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

   (c) $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \iff \mathfrak{p} \subset \mathfrak{q}$.

   (d) $\operatorname{Spec} R$ is a T_0-space, i.e., for any distinct points $\mathfrak{p}, \mathfrak{q}$ of $\operatorname{Spec} R$, there exists a neighborhood of $\mathfrak{p}$ which does not contain $\mathfrak{q}$, or a neighborhood of $\mathfrak{q}$ which does not contain $\mathfrak{p}$.

Solution:

(a) By definition of the Zariski topology, the set $\{\mathfrak{p}\}$ is closed if and only if $\{\mathfrak{p}\} = V(S)$ for some subset $S \subset R$, hence if and only if $\mathfrak{p}$ is the only prime ideal containing S. Since every prime ideal is contained in a maximal ideal, which is also prime, we conclude this is the case if and only if $\mathfrak{p}$ is maximal.

(b) $\overline{\{p\}} = \bigcap_{\substack{S \subset R \\ \mathfrak{p} \in V(S)}} V(S) = \bigcap_{\substack{S \subset R \\ \mathfrak{p} \supset S}} V(S) = V(\bigcup_{\substack{S \subset R \\ \mathfrak{p} \supset S}} S) = V(\mathfrak{p})$

(c) This is immediate from (b) and the definition of $V(\mathfrak{p})$.

(d) Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime ideals of R. Then we cannot have both $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$. In the case $\mathfrak{p} \not\subset \mathfrak{q}$ we have $\mathfrak{q} \notin V(\mathfrak{p}) \ni \mathfrak{p}$; hence $X \setminus V(\mathfrak{p})$ is an open neighborhood of $\mathfrak{q}$ which does not contain $\mathfrak{p}$. Similarly, if $\mathfrak{q} \not\subset \mathfrak{p}$, then $X \setminus V(\mathfrak{q})$ is an open neighborhood of $\mathfrak{p}$ which does not contain $\mathfrak{q}$.