# Solutions to problem set 1

**Notation.** $I := [0,1]$; we omit $\circ$ in compositions: $fg := f \circ g$.

1. By [Bredon, Prop. I.14.5], contractibility of $X$ means that $\mathrm{id}_X$ is homotopic to a map whose image is a singleton $\{x_0\} \in X$; that is, there exists a map $h : X \times I \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for all $x \in X$. Consider now the map $h' = r \circ h|_{A\times I} : A \times I \to A$. For $a \in A$, it satisfies $h'(a,0) = r(h(a,0)) = r(a) = a$ by the defining property of retraction, and $h'(a,1) = r(x_0)$ ; in other words, $\mathrm{id}_A$ is homotopic to a map with image $\{r(x_0)\} \subset A$, and hence $A$ is contractible (again by [Bredon, Prop. I.14.5]).

2. Recall that $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. We claim that the expression
$$(x,t) \mapsto \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|},$$
yields a well-defined map $\phi : X \times I \to S^n$. Pretending that this is true, note that $\phi(x,0) = f(x)$ and $\phi(x,1) = g(x)$ for all $x \in X$, and hence $f$ and $g$ are homotopic.

   To prove well-definedness, we must show that the denominator never vanishes. To do so, we assume the contrary, i.e., that there exists some $(x,t) \in X \times I$ such that $(1-t)f(x) + tg(x) = 0$. This is equivalent to $(1-t)f(x) = -tg(x)$, from which we obtain $(1-t)\cdot\|f(x)\| = t\cdot\|g(x)\|$; since $\|f(x)\| = \|g(x)\| = 1$, it follows that $1 - t = t$, and hence $t = \frac{1}{2}$. Inserting this into the first equation, we obtain $f(x) = -g(x)$, which contradicts our assumption.

3. Using the assumptions $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$, we obtain
$$fh \;=\; fh\,\mathrm{id}_Y \;\simeq\; fhfg \;\simeq\; f\,\mathrm{id}_X\,g \;=\; fg \;\simeq\; \mathrm{id}_Y.$$
Hence $h$ is in fact a homotopy inverse of $f$, and thus $f$ is a homotopy equivalence. (Similarly, one can show that $g$ is a homotopy inverse of $f$.)

4. (a) Let $q : Y \sqcup (X \times I) \to M_f$ be the quotient map. By definition of the quotient topology, a subset $V \subset M_f$ is open if and only if $q^{-1}(V) \subset Y \sqcup (X \times I)$ is open. Hence we get the following:
$$\begin{aligned}
&\phi \text{ is continuous}\\
&\iff \forall U \subset Z \text{ open}: \ \phi^{-1}(U) \subset M_f \text{ is open}\\
&\iff \forall U \subset Z \text{ open}: \ q^{-1}\circ\phi^{-1}(U) \subset Y \sqcup (X\times I) \text{ is open}\\
&\iff \forall U \subset Z \text{ open}: \ \begin{cases} \phi_{X\times I}^{-1}(U) = (\phi\circ q\circ i_{X\times I})^{-1}(U) \subset X \times I \\ \phi_Y^{-1}(U) = (\phi\circ q\circ i_Y)^{-1}(U) \subset Y \end{cases} \text{ are open}\\
&\iff \phi_{X\times I} \text{ and } \phi_Y \text{ are continuous.}
\end{aligned}$$

   (b) By definition of the maps that are involved we have $ri_X = f$ and hence the diagram is commutative.

   Let $F : M_f \times I \to M_f$ be defined by:
$$\begin{cases} F([x,t],t') = [x, t(1-t')], \text{ for } x \in X, t, t' \in I \\ F([y],t') = [y], \text{ for } y \in Y, t' \in I. \end{cases}$$

All we have to show is that $F$ is continous. To see this, we note that $M_f \times I$ is homeomorphic to $M_{f\times id_I}$, that is the mapping cylinder of the map $f \times id_I \colon X \times I \to Y \times I$, and then use exercise 4a.
We prove that $M_f \times I$ is homeomorphic to $M_{f\times id_I}$. This is a direct corollary of the following lemma:
**Lemma.** *Let $X$ be a topological space, $\sim$ be an equivalence relation on $X$, and $Y$ be a locally compact topological space. Then $X/\sim \times Y$ is homeomorphic to $(X \times Y)/\sim'$, where $\sim'$ is the equivalance relation on $X \times Y$ defined via*

$$(x, y) \sim' (x', y') \Leftrightarrow x \sim x' \text{ and } y = y'.$$

**Proof of the lemma.** We define the following maps:

$$\alpha\colon ([x], y) \in (X/\sim) \times Y \mapsto [x, y] \in (X \times Y)/\sim'$$

and

$$\beta\colon [x, y] \in (X \times Y)/\sim' \mapsto ([x], y) \in (X/\sim) \times Y$$

It is easy to see that both maps are well defined. For instance, let $[x, y] = [x', y'] \in (X\times Y)/\sim'$, i.e. $(x, y) \sim' (x', y')$, then $x \sim x'$ and $y = y'$, so that $\beta([x, y]) = \beta()[x', y'])$. Moreover, $\alpha$ and $\beta$ are inverses of each other.
Note that $\beta$ is continuos by the universal property of quotient maps: if $q'\colon X \times Y \to (X \times Y)/\sim'$ is the quotient map, then $\beta \circ q' = q \times id_Y$, where $q\colon X \to X/\sim$ is the quotient map for $\sim$.
We show that $\alpha$ is continous as well, by showing that $q \times id_Y : X \times Y \to (X/\sim) \times Y$ is a quotient map. This is enough by the universal property of quotient maps, as $\alpha \circ (q \times id_Y) = q'$. As the cartesian product of continous maps is continuous, it remains to prove that if $U \subset (X/\sim) \times Y$ is a subset such that $(q \times id_Y)^{-1}(U) \subset X \times Y$ is open, then $U$ itself is open.
Let $U \subset (X/\sim) \times Y$ as above and consider $([x], y) \in U$. As $Y$ is locally compact there is a compact neighbourhood $K$ of $y$ in $Y$. We can pick $K$ small enough such that $(q \times id_Y)(\{x\} \times K) \subset U$. We define the set

$$V := \left\{[z] \in X/\sim\colon\ \{z\} \times K \subset (q \times id_Y))^{-1}(U)\right\}$$

We have

$$([x], y) \in V \times \mathrm{int}(K) \subset U$$

Hence if we prove that $V \subset X/\sim$ is open we are done. Note that

$$q^{-1}(V) = \left\{z \in X :\ \{z\} \times K \subset (q \times id_Y)^{-1}(U)\right\} = X\backslash\Big(\mathrm{pr}_1\left((X \times K) \setminus (q \times id_Y)^{-1}(U)\right)\Big)$$

where $\mathrm{pr}_1$ is the projection onto the first factor. As $K$ is compact, hence closed, and since $(q\times id_Y)^{-1}(U)$ is open by assumption, it follows that $\mathrm{pr}_1\left((X \times K) \setminus (q \times id_Y)^{-1}(U)\right)$ is closed in $X$, hence $q^{-1}(V)$ is open in $X$. As $q$ is a quotient map, it follows that $V$ is open in $X/\sim$, finishing the proof.

(c) For a very nice solution see the proof of Proposition 7.46 (p. 206) in John Lee's book *Introduction to Topological Manifolds*.

By definition a subspace $A \subset B$ is a deformation retract if there is a retraction $r\colon B \to A$ which is a right homotopy inverse of the inclusion map $i : A \hookrightarrow B$. Explicitely, this means that $r \circ i = id_A$ and $i \circ r =\simeq id_B$. In particular, if $A \subset B$ is a deformation retract, then $A$ and $B$ have the same homotopy type. So if $X$ and $Y$ can be embedded as weak deformation retracts of the same space $Z$, then $X$ and $Y$ both have the same homotopy type as $Z$ and hence they are homotopy equivalent.

Conversely, suppose that $f : X \to Y$ is a homotopy equivalence. We will show that both $X$ and $Y$ are deformation retracts of $M_f$.

The retraction $r\colon M \to Y$ from 4(b) satisfies $r \circ i_y$ and $i_y \circ r \simeq id_{M_f}$. This shows that $Y$ is a (strong) deformation retract of $M_f$.

Again by 4(b) we have $f = r \circ i_X$ and thus $X \xrightarrow{i_X} M_f \xrightarrow{r} Y$ is a homotopy equivalence. Since $r$ is also a homotopy equivalence, it follows that $i_X$ is a homotopy equivalence as well. Now let $g : M_f \to X$ be a homotopy inverse of $i_X$, i.e. $g \circ i_X \simeq id_X$ and $i_X \circ g \simeq id_{M_f}$. The idea is to modify $g$ to construct a *retraction* $q\colon M_f \to X$ with $i_X \circ q \simeq id_{M_f}$. Denote by $G : X \times I \to X$ a homotopy $G : g \circ i_X \simeq id_X$. Define the homotopy $H : M_f \times I \to X$ by $H([y], t') = g([y])$ for $y \in Y$ and $t' \in I$ and

$$H([x,t],t') = \begin{cases} g\left(\left[x, \frac{2t}{2-t'}\right]\right), & 0 \le t' \le 2(1-t) \le 2,\ x \in X,\ t, t' \in I \\ G\left(x, \frac{2t-(2-t')}{t}\right), & 0 \le 2(1-t) \le t' \le 1,\ x \in X,\ t, t' \in I. \end{cases}$$

First note that $H$ is well-defined as a map on $(Y \sqcup (X \times I)) \times I$ because for $t' = 2(1-t)$ one has

$$g\left(\left[x, \frac{2t}{2-t'}\right]\right) = g([x,1]) = G(x,0) = G\left(x, \frac{2t-(2-t')}{t}\right).$$

Moreover, $H$ descends to a well-defined map on $M_f \times I$ because

$$H([x,0],t') = g([x,0]) = g([f(x)]) = H([f(x)],t').$$

Consider $q\colon M_f \to X$, $z \mapsto H(z,1)$. $q$ is a retraction: $q \circ i_X(x) = H([x,1],1) = G(x,1) = x$, so $q \circ i_X = id_X$. Moreover, $i_X \circ H$ is a homotopy from $i_X \circ g$ to $i_X \circ q$. Thus $id_{M_f} \simeq i_X \circ g \simeq i_X \circ q$. We conclude that $X$ is also embedded as a deformation retract of $M_f$.

5. We view $\mathbb{R}P^2$ as $D^2/\sim$, where $\sim$ is the equivalence relation that identifies antipodal points on the boundary. That is, $x, y \in D^2$ satisfy $x \sim y$ if and only if $x = y$, or $x$ and $y$ both lie on $S^1 \subset D^2$ and satisfy $x = -y$.

   Consider the map $F : S^1 \sqcup (S^1 \times I) \to \mathbb{R}P^2$ defined on $S^1$ by $e^{2\pi it} \mapsto [e^{\pi it}]$ for $t \in [0,1]$, and on $S^1 \times I$ by $(e^{2\pi it}, s) \mapsto [(1-s)e^{2\pi it}]$. (Note that the first part is well-defined and continuous, because $e^{\pi i0} = 1 \sim -1 = e^{\pi i1}$ and hence $[e^{\pi i0}] = [e^{\pi i1}] \in \mathbb{R}P^2$.)

   Observe that $F$ descends to a well-defined map $F' : M_f \to \mathbb{R}P^2$, because $F(f(e^{2\pi it})) = F(e^{4\pi it}) = [e^{2\pi it}] = F(e^{2\pi it}, 0)$, and moreover $F'$ is clearly surjective. Since $F'$ maps all of $S^1 \times \{1\}$ to a single point, namely $[0] \in \mathbb{R}P^2$, it descends further to a map $F'' : C_f \to \mathbb{R}P^2$. $F''$ is still surjective. Note that $F''$ is also injective, since $F'$ is injective on $M_f \setminus (S^1 \times \{1\})$ with image in $\mathbb{R}P^2 \setminus \{0\}$.

   Since $C_f$ is compact and $\mathbb{R}P^2$ is Hausdorff, we conclude using [Bredon, Theorem I.7.8] that $F''$ is a homeomorphism.