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Solutions to problem set 2

1. Recall that we may view singular 0-chains in X as finite formal sums $\sum_x n_x x$ with $x \in X$ and $n_x \in \mathbb{Z}$. In particular, a zero-simplex in X is a point $x \in X$.

   By definition, the image of $[x] \in H_0(X)$ under $f_* : H_0(X) \to H_0(X)$ is the class of $f(x) \in X$, viewed as a 0-simplex. Since X is path-connected, we can choose a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = f(x)$; regarding this path as a 1-simplex $\gamma : \Delta_1 \to X$, we obtain

   $$\partial_1 \gamma = \gamma(1) - \gamma(0) = f(x) - x.$$

   Hence the 0-chain $f(x) - x$ is a boundary, and thus $f_*[x] - [x] = [f(x) - x] = 0 \in H_0(X)$.

2. Let $\gamma : I \to X$ be a loop based at x_0, and recall that we can also consider γ as a singular 1-cycle; we denote the corresponding classes by $[\gamma] \in \pi_1(X, x_0)$ and $[\![\gamma]\!] \in H_1(X)$. It follows straight from the definitions of $f_\sharp, f_*$ and the Hurewicz homomorphisms ϕ_X and ϕ_Y that

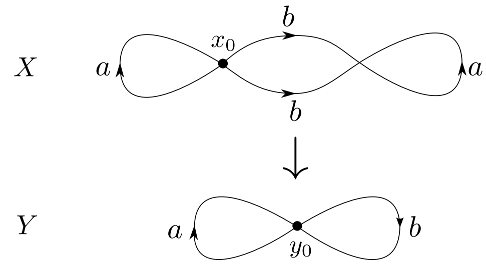
   $$f_*(\phi_X([\gamma])) = f_*[\![\gamma]\!] = [\![f \circ \gamma]\!] = \phi_Y([f \circ \gamma]) = \phi_Y(f_\sharp[\gamma]).$$

   Since this works for every γ, we conclude $f_* \circ \phi_X = \phi_Y \circ f_\sharp$.

3. Denote by $p_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ the map induced by p. Let $\gamma : I \to X$ be a loop based at x_0 and suppose that $p_\sharp([\gamma]) = 0 \in \pi_1(Y, y_0)$, which is equivalent to saying that the loop $p \circ \gamma : I \to Y$ is null-homotopic. This means that there exists a homotopy $F : I \times I \to Y$ such that $F(\cdot, 0) = p \circ \gamma$ and $F(\cdot, 1) \equiv y_0$ is constant. Since γ lifts $F(\cdot, 0)$, the Covering Homotopy Theorem tells us that there is a (unique) homotopy $G : I \times I \to X$ such that $G(\cdot, 0) = \gamma$ and such that $p \circ G = F$. In particular, this implies that $p(G(\cdot, 1)) = F(\cdot, 1)$ is constant, and thus that $G(\cdot, 1)$ is constant, because p is a covering map and hence a local homeomorphism. It follows that $[\gamma] = 0 \in \pi_1(X, x_0)$, and thus $p_\sharp$ is a monomorphism.

   It is not true that $p_* : H_1(X) \to H_1(Y)$ needs to be a monomorphism. For example, take any space Y with $H_1(Y) \neq 0$, set $X = Y \sqcup Y$, and consider the obvious double cover $p : X \to Y$; the induced map $p_* : H_1(X) \cong H_1(Y) \oplus H_1(Y) \to H_1(Y)$, $(\alpha, \beta) \mapsto \alpha + \beta$, is clearly not injective.

   For a slightly more involved example, consider $X = S^1 \vee S^1 \vee S^1$, $Y = S^1 \vee S^1$ and the covering map $p : X \to Y$ indicated by the following picture (convince yourself that this is a covering map!):

   Consider now the loop γ in X that starts at x_0 and then winds once around all of X in clockwise direction. This loop defines a non-zero element $[\![\gamma]\!] \in H_1(X)$; but note that

   $$p_*[\![\gamma]\!] = \phi_Y(p_\sharp[\gamma]) = \phi_Y[b^{-1}a^{-1}ba] = 0 \in H_1(Y),$$

because $[b^{-1}a^{-1}ba]$ lies in the commutator of $\pi_1(Y, y_0)$, which is the kernel of the Hurewicz homomorphism ϕ_Y. Thus $p_* : H_1(X) \to H_1(Y)$ is not a monomorphism.

4. We use the Seifert-Van Kampen and the Hurewicz Theorem.

   Let U be the open set consisting of the interior of the polygon and let V be a small open neighbourhood of the boundary of the polygon. More precisely, if P denotes the polygon and $q\colon P \to \Sigma_g$ the projection coming from gluing the edges, then $U = q(\mathrm{int}(P)$ and $V = q(N)$ where N is a small neighbourhood of ∂P in P.)

   U is homeomorphic to a disc, hence $\pi_1(U) \cong 0$.

   Consider the loop $\gamma = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = [a_1, b_1] \dots [a_g, b_g]$, which comes from following the boundary of the polygon in counter-clockwise direction. V deformation retracts to the image of γ, which is a bouquet of $2g$ many spheres. Therefore, $\pi_1(V)$ is the free group generated by $a_1, b_1, \dots, a_g, b_g$.

   $U \cap V$ is homotopy equivalent to a sphere, hence $\pi_1(U \cap V) \cong \mathbb{Z}$. Its generator gets mapped to γ via the map $(i_U)_\#\colon \pi_1(U \cap V) \to \pi_1(V)$ induced by the inclusion $i_U\colon U \cap V \to V$.

   Seifert-Van Kampen (Theorem 9.4 in Bredon) now gives

$$\begin{aligned} \pi_1(\Sigma_g) &\cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \\ &\cong \pi_1(U)/\langle\gamma\rangle \\ &\cong \big\langle a_1, b_1, \dots, a_g, b_g \big| [a_1, b_1] \dots [a_g, b_g] \big\rangle. \end{aligned}$$

   With the Hurewicz theorem we compute the first homology

$$\begin{aligned} H_1(\Sigma_g) &\cong \big(\big\langle a_1, b_1, \dots, a_g, b_g \big| [a_1, b_1] \dots [a_g, b_g] \big\rangle\big)^{ab} \\ &\cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \cong \mathbb{Z}^{2g}, \end{aligned}$$

   where the $2g$ factors are generated by $a_1, b_1, \dots a_g, b_g$.

5. We first relate the fundamental groups of $X \vee Y$, X and Y using Seifert-Van Kampen. Denote by $p \in X \vee Y$ the point where X and Y are glued, i.e. $p = q(x) = q(y)$ for the projection $q\colon X \sqcup Y \to X \vee Y$. Wlog assume that $A \subset X$ and $B \subset Y$ are open. We sometimes view A, X, B, Y as subsets of $X \vee Y$ via the obvious inclusions. The sets $U := X \cup B \subset X \vee Y$ and $V := A \cup Y \subset X \vee Y$ are open in $X \vee Y$. Moreover, U and V are both path-connected, because A and B both deformation retract to $\{p\}$. We show that $U \cap V = A \cup B \subset X \vee Y$ is simply connected: Let $F\colon A \times [0, 1] \to A$ be a strong deformation retraction to x and $G\colon B \times [0, 1] \to B$ be a strong deformation retraction to y. Then the map

$$\begin{aligned} H\colon (A \cup B) \times [0, 1] &\to A \cup B \\ H(a, t) &= F(a, t) \text{ for } a \in A \\ H(b, t) &= G(b, t) \text{ for } b \in B \end{aligned}$$

   is a (strong) deformation retraction from $A \cup B$ to p. (Here it is crucial that F and G are *strong* deformation retractions: H only is well-defined, because for all $t \in [0, 1]$, $F(p, t) = p = G(p, t)$.) We conclude that $A \cup B$ is homotopy equivalent to a point. In particular, it is simply-connected.

   So Seifert-Van Kampen (Corollary 9.5 in Bredon) applies to $X \vee Y = U \cup V$ and implies

$$\pi_1(X \vee Y, p) \cong \pi_1(U, p) * \pi_1(V, p), \tag{1}$$

where the isomorphism is induced by the inclusions.

As X resp. Y are deformation retracts of U resp. V, the inclusions $X \hookrightarrow U$ and $Y \hookrightarrow V$ are homotopy equivalences. Hence, $\pi_1(U,p) \cong \pi_1(X,x)$ and $\pi_1(V,p) \cong \pi_1(Y,y)$. Together with (1), we get

$$\pi_1(X \vee Y, p) \cong \pi_1(X,x) * \pi_1(Y,x),$$

where the isomorphism is still induced by the inclusions.

The Hurewicz theorem now implies the claim:

$$\begin{aligned} H_1(X \vee Y) &\cong \pi_1(X \vee Y, p)^{ab} \\ &\cong (\pi_1(X,x) * \pi_1(Y,y))^{ab} \\ &\cong \pi_1(X,x)^{ab} \oplus \pi_1(Y,y)^{ab} \\ &\cong H_1(X) \oplus H_1(Y) \end{aligned}$$

The isomorphism $H_1(X) \oplus H_1(Y) \cong H_1(X \vee Y)$ in homology is still induced by the inclusions, as can been shown similarly as in problem 2.