Prof. Paul Biran | Algebraic Topology I | HS 2024

ETH Zürich

Problem set 2

1. Suppose that X is a path-connected space and let $f : X \to X$ be a map. Prove that the induced map $f_* : H_0(X) \to H_0(X)$ is the identity. What happens if X is not path-connected?

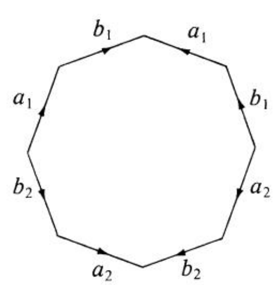
2. Let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_* : H_1(X) \to H_1(Y)$. Prove commutativity of the diagram

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_\sharp"] \arrow[d, "\phi_X"] & \pi_1(Y,y_0) \arrow[d, "\phi_Y"] \\ H_1(X) \arrow[r, "f_*"] & H_1(Y) \end{tikzcd}$$

   where ϕ_X and ϕ_Y are the Hurewicz homomorphisms.

3. Let $p : X \to Y$ be a covering map, and let $x_0 \in X$ and $y_0 = p(x_0)$. Prove that the induced map $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism. Is it true in general that $p_* : H_1(X) \to H_1(Y)$ is a monomorphism?

4. Consider a polygon with $4g$ edges which are grouped into g tuples, each consisting of 4 consecutive edges labelled in clockwise order by $a_k, b_k, a_k^{-1}, b_k^{-1}$ for $1 \leq k \leq g$ (the figure shows the case $g = 2$). By identifying the edges according to the labelling, one obtains a closed orientable surface Σ_g of genus g.



   Compute $H_1(\Sigma_g)$.
   Hint: Use the Seifert-Van Kampen Theorem (see e.g. Theorem 9.4 in Bredon).

5. Let X and Y be topological spaces and fix two points $x \in X$ and $y \in Y$. The *wedge* of (X, x) and (Y, y) is the topological space obtained by the disjoint union of X and Y and then identifying x with y:

$$X \vee Y = (X \sqcup Y)/\sim, \;\; x \sim y.$$

Assume that X and Y are path-connected. Assume there are neighbourhoods $A \subset X$ of x and $B \subset Y$ of y such that $\{x\}$ is a strong deformation retract of A and $\{y\}$ is a strong deformation retract of B. Show that

$$H_1(X \vee Y) \cong H_1(X) \oplus H_1(Y),$$

where the isomorphism is induced by the inclusions $i_X \colon X \to X \vee Y$ and $i_Y \colon Y \to X \vee Y$.

Hint: Use the Seifert-Van Kampen Theorem (see e.g. Corollary 9.5 in Bredon).