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Solutions to problem set 3

1. (See notes from the exercise class from 15.11.2024 for more details.) By the exactness axiom, there is a long exact sequence for the pair (X, X):

$$\cdots \to H_i(X) \xrightarrow{id_*=id} H_i(X) \to H_i(X,X) \to H_{i-1}(X) \xrightarrow{id_*=id} H_{i-1}(X) \cdots$$

Since id is an isomorphism, it follows from exactness that $H_i(X,X) = \{0\}$. The exactness axiom for the pair $(X, \emptyset)$ for some space X reads

$$\cdots \to H_i(X) \xrightarrow{id_*=id} H_i(X) \cong H_i(X,\emptyset) \to H_i(\emptyset) \to H_i(X) \xrightarrow{id_*=id} H_i(X) \cong H_i(X,\emptyset) \to \ldots$$

Again it follows from exactness that $H_i(\emptyset) = 0$ for all i.

2. Let $A = \emptyset$ but $X \neq \emptyset$. It's enough to consider the reduced sequence in degrees ≤ 1. Consider the diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & \mathbb{Z} & & & & & & \\
 & & & & \epsilon_* \uparrow & & & & & & \\
H_1(X,\emptyset) & \longrightarrow & H_0(\emptyset) = 0 & \longrightarrow & H_0(X) & \xrightarrow{\text{id}} & H_0(X,\emptyset) = H_0(X) & \longrightarrow & 0 & & \\
\| & & \| & & \uparrow & & \text{id} \uparrow & & & & \\
H_1(X,\emptyset) & \longrightarrow & \tilde{H}_0(\emptyset) = 0 & \longrightarrow & \tilde{H}_0(X) & \longrightarrow & H_0(X,\emptyset) = H_0(X) & \xrightarrow{\epsilon_*} & \tilde{H}_{-1}(\emptyset) = \mathbb{Z} & \longrightarrow & 0 \\
 & & & & \uparrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

The first row comes from the (non-reduced) LES for $(X, \emptyset)$. The lower row is the sequence of reduced homologies, for which we need to show exactness. The SES occuring as a column comes from the definition of reduced homology. By comparing this SES with the lower row it follows that the reduced sequence is also exact.

Assume now that $X = \emptyset$. Then the sequence for reduced homology is 0 except in degree -1:

$$H_0(X,A) = 0 \to H_{-1}(A) = \mathbb{Z} \xrightarrow{id_*=id} \mathbb{Z} = H_{-1}(X) \to 0 = H_{-1}(X,A).$$

This is exact.

3. Consider the reduced long exact sequence for the pair (X, A):

$$\cdots \to \tilde{H}_p(A) \to \tilde{H}_p(X) \to H_p(X,A) \to \tilde{H}_{p-1}(A) \to \ldots$$

Since $\widetilde{H}_*(A) = 0$ by assumption, this long exact sequence yields exact sequences

$$0 \to \tilde{H}_p(X) \to H_p(X,A) \to 0$$

which tell us that

$$\widetilde{H}_p(X) \cong H_p(X,A)$$

for all p.

4. Let U be a small open neighbourhood of the "bottom" point $[0, x] \in \Sigma X$. We apply excision to the pair $(\Sigma X, C_- X)$ and the open subset U. We get an isomorphism

$$H_i(\Sigma X \backslash U, C_- X \backslash U) \xrightarrow{incl_*} H_i(\Sigma X, C_- X).$$

Note that $\Sigma X \backslash U$ deformation retracts to $C_+ X$ relative to X. By the homotopy axiom, the inclusion of $(C_+ X, X)$ into $(\Sigma X \backslash U, C_- X \backslash U)$ induces an isomorphism in homology:

$$H_i(C_+ X, X) \xrightarrow{incl_*} H_i(\Sigma X \backslash U, C_- X \backslash U).$$

Put togehter, we get isomorphisms:

$$\widetilde{H}_{i-1}(X) \xleftarrow{\partial_*} H_i(C_+ X, X) \xrightarrow{incl_*} H_i(\Sigma X \backslash U, C_- X \backslash U) \xrightarrow{incl_*} H_i(\Sigma X, C_- X) \xleftarrow{incl_*} \widetilde{H}_i(\Sigma X),$$

where the first isomorphism comes from the LES of the pair $(C_+ X, X)$ and the last isomorphims comes from the LES of the pair $(\Sigma X, C_- X)$. (One uses that the cones $C_\pm X$ are contractible and hence acyclic.) Each of the isomorphism is natural, and hence the composition

$$\widetilde{H}_{i-1}(X) \xrightarrow{\approx} \widetilde{H}_i(\Sigma X)$$

is a natural isomorphism.

5. Consider the LES for the pair $(X \vee Y, X)$:

$$\cdots \to H_{n+1}(X \vee Y, X) \xrightarrow{\partial_*} \widetilde{H}_n(X) \xrightarrow{(i_X)_*} \widetilde{H}_n(X \vee Y) \xrightarrow{k_*} H_n(X \vee Y, X) \to \dots \qquad (1)$$

We claim that this long exact sequence splits into many short exact sequences. Denote by $i_Y : Y \to X \vee Y$ and $j : (Y, \emptyset) \to (X \vee Y, X)$ the inclusions. Then j induces an isomorphism in homology:

$$j_* = \left(\widetilde{H}_n(Y) \xrightarrow{incl_*, \approx} H_n(Y, \{y_0\}) \xrightarrow{incl_*, \approx} H_n(Y \cup N, N) \xrightarrow{incl_*, \approx} H_n(X \vee Y, X) \right). \qquad (2)$$

(where we identified x_0 with y_0 and see both X and Y as subspaces of their wedge product.) Indeed, the first isomorphism in (2) comes from the LES for the pair $(X, \{y_0\})$. The second isomorphism in (2) follows from the homotopy axiom because $Y \cup N$ deformation retracts to Y relative to $\{y_0\}$. Finally, the third isomorphism in (2) comes from excision for the pair $(X \vee Y, X)$ with respect to the open set $(X \vee Y) \backslash (Y \cup N)$.

Note that $k_* \circ (i_Y)_* \circ (j_*)^{-1} = id$. This implies that k_* is surjective and the LES (1) splits into short exact sequences:

$$0 \to \widetilde{H}_n(X) \xrightarrow{incl_*} \widetilde{H}_n(X \vee Y) \xrightarrow{k_*} H_n(X \vee Y, X) \to 0.$$

Replacing $H_n(X \cup Y, X)$ with $\widetilde{H}_n(Y)$ via j_*, we get SES's

$$0 \to \widetilde{H}_n(X) \xrightarrow{incl_*} \widetilde{H}_n(X \vee Y) \xrightarrow{(p_Y)_*} \widetilde{H}_n(Y) \to 0,$$

where $p_Y : X \vee Y \to Y$ denotes the projection. This SES's split with a right splitting given by $(i_Y)_*$ and a left splitting given by $(pr_X)_*$. Therefore, $(i_X)_* \oplus (i_Y)_* : H_n(X) \oplus H_n(Y) \to H_n(X \vee Y)$ is an isomorphism with inverse $p_X \oplus p_Y$.

6. The commutativity of the diagram

$$\begin{array}{ccc} (Y,\emptyset) & \xrightarrow{\quad k \quad} & (X \sqcup Y, X) \\ & {}_{i_Y}\searrow \qquad \nearrow_{j} & \\ & X \sqcup Y & \end{array}$$

implies that

$$k_* = j_* \circ (i_Y)_* : H_*(Y) \to H_*(X \sqcup Y, X) \tag{3}$$

by functoriality of H. Since k_* is an isomorphism by the Excision axiom, it follows that j_* is surjective. This implies that the connecting homomorphisms ∂_* in the long exact sequence for the pair $(X \sqcup Y, X)$ are all zero, and hence this long exact sequence gives rise to short exact sequences

$$0 \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to 0. \tag{4}$$

Now equation (3) is equivalent to $j_* \circ (i_Y)_* \circ k_*^{-1} = \mathrm{id}_{H_p(X \sqcup Y, X)}$, and thus j_* has a right inverse. Hence the short exact sequence (4) splits, and therefore

$$(i_X)_* \oplus ((i_Y)_* \circ k_*^{-1}) : H_p(X) \oplus H_p(X \sqcup Y, X) \to H_p(X \sqcup Y)$$

is an isomorphism; precomposing it with the isomorphism $\mathrm{id}_{H_p(X)} \oplus k_*$ yields the desired isomorphism

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \to H_p(X \sqcup Y).$$