## Problem set 3

Let H be a homology theory.

- 1. Prove from the axioms that  $H_i(\emptyset) = 0$  for all i and that  $H_i(X, X) = 0$  for all i and all spaces X.
- 2. Check that the long exact sequence for reduced homology continues to hold for pairs of the type  $(X, \emptyset)$ , including the case  $X = \emptyset$ .
- 3. Let  $A \subset X$  be a non-empty subset and assume that  $\widetilde{H}_*(A) = 0$  (that is, A is acyclic). Prove that  $H_*(X, A) \cong \widetilde{H}_*(X)$ .
- 4. Define the unreduced suspension  $\Sigma X$  of a space X to be the quotient space of  $[0, 1] \times X$  obtained by identifying  $\{0\} \times X$  and  $\{1\} \times X$  to points. Show that there is a natural isomorphism

$$\tilde{H}_i(X) \xrightarrow{\approx} \tilde{H}_{i+1}(\Sigma X).$$

Here *natural* means that for a map  $f: X \to Y$ , and its suspension  $\Sigma f: \Sigma X \to \Sigma Y$  the following diagram commutes:

$$\begin{split} \tilde{H}_{i}(X) & \xrightarrow{\approx} \tilde{H}_{i+1}(\Sigma X) \\ & \downarrow_{f_{*}} & \downarrow_{(\Sigma f)_{*}} \\ \tilde{H}_{i}(Y) & \xrightarrow{\approx} \tilde{H}_{i+1}(\Sigma Y). \end{split}$$

*Hint:* Consider the two cones  $C_+X := \{[t,x] \in \Sigma X | t \ge \frac{1}{2}\}$  and  $C_-X := \{[t,x] \in \Sigma X | t \le \frac{1}{2}\}.$ 

5. Let X and Y be topological spaces. Let  $x_0 \in X$  and  $y_0 \in Y$  be points. Assume that x has a closed neighborhood N in X, of which  $\{x_0\}$  is a strong deformation retract. Recall the wedge  $X \vee Y$  of X and Y we defined in Problem Set 2, Exercise 5. Show that the inclusion maps induce isomorphisms

$$\tilde{H}_i(X) \oplus \tilde{H}_i(Y) \xrightarrow{\approx} \tilde{H}_i(X \lor Y)$$

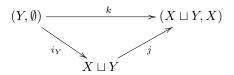
whose inverse is induced by the projections of  $X \vee Y$  to X and Y.

6. Let H be a "theory" satisfying axioms 1-4 of a homology theory, but not necessarily axiom 5. Show that

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \to H_p(X \sqcup Y)$$

is an isomorphism for all spaces X, Y and for all  $p \in \mathbb{Z}$ , where  $i_X : X \hookrightarrow X \sqcup Y$ ,  $i_Y : Y \hookrightarrow X \sqcup Y$  denote the inclusions into the disjoint union. *Hints.* 

- (a) Consider the long exact sequence of the pair  $(X \sqcup Y, X)$ .
- (b) Consider the excision  $(Y, \emptyset) = ((X \sqcup Y) \setminus X, X \setminus X) \stackrel{k}{\hookrightarrow} (X \sqcup Y, X)$ and the resulting isomorphism  $k_* : H_*(Y) \xrightarrow{\cong} H_*(X \sqcup Y, X)$ .
- (c) Note that the following diagram commutes:



(d) Deduce that in the long exact sequence

$$\cdots \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to \dots$$

all maps  $j_*$  are surjective, and that thus the sequence gives rise to short exact sequences

$$0 \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to 0.$$

(e) Find a right inverse for  $j_*$  to show that these short exact sequences split.