Prof. Paul Biran
ETH Zürich

Algebraic Topology I

HS 2024

# Problem set 3

Let $H$ be a homology theory.

1. Prove from the axioms that $H_i(\emptyset) = 0$ for all $i$ and that $H_i(X, X) = 0$ for all $i$ and all spaces $X$.

2. Check that the long exact sequence for reduced homology continues to hold for pairs of the type $(X, \emptyset)$, including the case $X = \emptyset$.

3. Let $A \subset X$ be a non-empty subset and assume that $\widetilde{H}_*(A) = 0$ (that is, $A$ is acyclic). Prove that $H_*(X, A) \cong \widetilde{H}_*(X)$.

4. Define the *unreduced suspension* $\Sigma X$ of a space $X$ to be the quotient space of $[0, 1] \times X$ obtained by identifying $\{0\} \times X$ and $\{1\} \times X$ to points. Show that there is a natural isomorphism

$$\tilde{H}_i(X) \xrightarrow{\approx} \tilde{H}_{i+1}(\Sigma X).$$

   Here *natural* means that for a map $f \colon X \to Y$, and its suspension $\Sigma f \colon \Sigma X \to \Sigma Y$ the following diagram commutes:

$$\begin{tikzcd} \tilde{H}_i(X) \arrow[r, "\approx"] \arrow[d, "f_*"] & \tilde{H}_{i+1}(\Sigma X) \arrow[d, "(\Sigma f)_*"] \\ \tilde{H}_i(Y) \arrow[r, "\approx"] & \tilde{H}_{i+1}(\Sigma Y). \end{tikzcd}$$

   *Hint:* Consider the two cones $C_+X := \{[t, x] \in \Sigma X | t \geq \frac{1}{2}\}$ and $C_-X := \{[t, x] \in \Sigma X | t \leq \frac{1}{2}\}$.

5. Let $X$ and $Y$ be topological spaces. Let $x_0 \in X$ and $y_0 \in Y$ be points. Assume that $x$ has a closed neighborhood $N$ in $X$, of which $\{x_0\}$ is a strong deformation retract. Recall the wedge $X \vee Y$ of $X$ and $Y$ we defined in Problem Set 2, Exercise 5. Show that the inclusion maps induce isomorphisms

$$\tilde{H}_i(X) \oplus \tilde{H}_i(Y) \xrightarrow{\approx} \tilde{H}_i(X \vee Y),$$

   whose inverse is induced by the projections of $X \vee Y$ to $X$ and $Y$.

6. Let $H$ be a "theory" satisfying axioms 1-4 of a homology theory, but not necessarily axiom 5. Show that

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \to H_p(X \sqcup Y)$$

is an isomorphism for all spaces $X, Y$ and for all $p \in \mathbb{Z}$, where $i_X : X \hookrightarrow X \sqcup Y$, $i_Y : Y \hookrightarrow X \sqcup Y$ denote the inclusions into the disjoint union.

*Hints.*

(a) Consider the long exact sequence of the pair $(X \sqcup Y, X)$.

(b) Consider the excision $(Y, \emptyset) = \big((X \sqcup Y) \setminus X, X \setminus X\big) \xhookrightarrow{k} (X \sqcup Y, X)$ and the resulting isomorphism $k_* : H_*(Y) \xrightarrow{\cong} H_*(X \sqcup Y, X)$.

(c) Note that the following diagram commutes:

$$\begin{tikzcd} (Y, \emptyset) \arrow[rr, "k"] \arrow[dr, "i_Y"'] & & (X \sqcup Y, X) \\ & X \sqcup Y \arrow[ur, "j"'] & \end{tikzcd}$$

(d) Deduce that in the long exact sequence

$$\cdots \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to \ldots$$

all maps $j_*$ are surjective, and that thus the sequence gives rise to short exact sequences

$$0 \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to 0.$$

(e) Find a right inverse for $j_*$ to show that these short exact sequences split.