Solutions to problem set 5

1. Define $f\colon X \vee Y \to W$ by $f(x) = (x, y_0)$ for $x \in X$ and $f(y) = (x_0, y)$ for $y \in Y$. This is clearly bijective and maps the base point $* \in X \vee Y$ to the base point (x_0, y_0). f is obviously a homeomorphism away from $*$. Open neighboorhoods of $(x_0, y_0) \in W$ are of the form $N = (U \times \{y_0\}) \cup (\{x_0\} \times V)$ for $U \subset X$ an open neighbourhood of x_0 and $V \subset Y$ an open neighbourhood of y_0. Its inverse image is $f^{-1}(N) = \pi(U) \cup \pi(V)$, where $\pi\colon X \sqcup Y \to X \vee Y$ denotes the projection. These are precisely the open neighbourhoods of $*$ in $X \wedge Y$. We conclude that f and f^{-1} are both continuos in $*$ and so f is a homeomorphism.

2. We denote the base-points by $x_0 \in X$, $y_0 \in Y$, $x_0' \in X'$ and $y_0' \in Y'$. By assumption we have $f(x_0) = x_0'$ and $g(y_0) = y_0'$. To show the claim, we can either work with the original definition of $X \vee Y$ or with the equivalent definition using the space W from Exercise 1.

   (a) *via the original definition.*
   We define
   $$f \vee g\colon X \vee Y \to X' \vee Y'$$
   via
   $$f \vee g(z) := \begin{cases} f(z), & \text{if } z \in X \\ g(z), & \text{if } z \in Y \end{cases}$$
   This map is well defined as $x_0' = f(x_0) = g(y_0) = y_0'$ in $X' \vee Y'$ by definition of wedge, and it still preserves base points. The map $f \vee g$ is obviously continuous away from the gluing point $x_0 \sim y_0$ of $X \vee Y$. Let U' be an open neighborhood of x_0' in $X' \vee Y'$. By definition of disjoint union topology and quotient topology, U has the form
   $$U' = \frac{U_x' \sqcup U_y'}{x_0' \sim y_0'} = \pi'(U_x') \cup \pi(U_y')$$
   where $x_0' \in U_x' \subset X'$ and $y_0' \subset U_y' \subset Y'$ are open and π' is defined as in exercise 1. It directly follows that
   $$(f \vee g)^{-1}(U') = \pi(f^{-1}(U_x')) \cup \pi(g^{-1}(U_y'))$$
   which is open in $X \vee Y$.
   The fact that this construction respects identity maps and composition is obvious from its definition.

   (b) *via the definition using W.*
   We identify $X \vee Y$ with
   $$W := (X \times \{y_0\}) \cup (\{x_0\} \times Y)$$
   and similarly for $X' \vee Y'$, but we keep using the notation with $\vee$.
   Define $f \vee g\colon X \vee Y \to X' \vee Y'$ by $(f \vee g)(x, y_0) := (f(x), y_0')$ and $(f \vee g)(x_0, y) := (x_0', g(y))$. $f \vee g$ is well-defined as the maps f and g preserve base points, and preserves the base point itself. Moreover, is clearly continous away from (x_0, y_0). An open neighbourhood $N' = (U' \times \{y_0'\}) \cup (\{x_0'\} \times V')$ of (x_0', y_0') has inverse image $(f \vee g)^{-1}(N') = (f^{-1}(U') \times \{y_0\}) \cup (\{x_0\} \times g^{-1}(V'))$. This is an open neighbourhood of (x_0, y_0). Therefore $f \vee g$ is also continuos in (x_0, y_0). Again, the fact that this construction respects identity maps and composition is obvious from its definition.

3. We denote by $[x, y] \in X \wedge Y$ the equivalence class of $(x, y) \in X \times Y$.

   (a) Define $f \wedge g\colon X \wedge Y \to X' \wedge Y'$ by setting $(f \wedge g)[x, y] := [f(x), g(y)]$. This is well-defined: if $(x, y) \in X \vee Y$ then $(f(x), g(y)) \in X' \vee Y'$ because f and g preserve base points. $f \wedge g$ is continuos because $f \times g$ is continuous. Moreover, $id_X \wedge id_Y = id_{X \wedge Y}$ and $(f' \circ f) \wedge (g' \circ g) = (f' \wedge g') \circ (f \wedge g)$ for maps $f'\colon X' \to X''$ and $g'\colon Y' \to Y''$. So $\wedge$ is functorial.

   (b) Define $\varphi_{X,Y}\colon X \wedge Y \to Y \wedge X$ by $\varphi_{X,Y}[x, y] = [y, x]$. It is easy to see that this is a homeomorphism and that the diagram commutes.

   (c) Define $\psi_{X,Y,Z}\colon (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$ by $\psi_{X,Y,Z}([[x, y], z]) = [x, [y, z]]$. Consider the compositions

$$(X \times Y) \times Z \xrightarrow{\pi_{X,Y} \times id} (X \wedge Y) \times Z \xrightarrow{\pi_{X \wedge Y, Z}} (X \wedge Y) \wedge Z$$

   and

$$X \times (Y \times Z) \xrightarrow{id \times \pi_{Y,Z}} X \times (Y \wedge Z) \xrightarrow{\pi_{X, Y \wedge Z}} X \wedge (Y \wedge Z).$$

   Since all the spaces are locally compact, $\pi_{X,Y} \times id$ and $id \times \pi_{Y,Z}$ are quotient maps (see e.g. J. H. C. Whitehead, *A note on a theorem of Borsuk*, Bull. Amer. Math. Soc, 54 (1958), 1125-1132, Lemma 4). Therefore, the two compositions are both quotient maps. It now follows from the universal property of quotient maps that $\psi_{X,Y,Z}$ is a homeomorphism.

   Naturality means that the following diagram commutes:

$$\begin{tikzcd}
(X \wedge Y) \wedge Z \arrow[r, "\psi_{X,Y,Z}"] \arrow[d, "(f \wedge g) \wedge h"] & X \wedge (Y \wedge Z) \arrow[d, "f \wedge (g \wedge h)"] \\
(X' \wedge Y') \wedge Z' \arrow[r, "\psi_{X',Y',Z'}"] & X' \wedge (Y' \wedge Z').
\end{tikzcd}$$

   This is easy to check.

   The assumption that X, Y, Z are locally compact Hausdorff spaces is necessary. A counterexample can be found in J. Peter May, Johann Sigurdsson, *Parametrized Homotopy Theory*, Amer. Math. Soc, 10 (2006), section 1.7.

4. (a) First of all, note that Q is a locally compact Hausdorff space and $X \wedge Y$ is a compact Hausdorff space. Denote by $\pi\colon X \times Y \to X \wedge Y$ the quotient map. Note that π sends $Q \subset X \times Y$ bijectively to $(X \wedge Y)\backslash\{*\}$. By Theorem 11.3 in Bredon, it is enough to show that the injection $\pi|_Q\colon Q \to X \wedge Y$ is a homeomorphism onto its image. Indeed, $\pi|_Q$ is open: An open set $U \subset Q$ is also open in $X \times Y$ because Q is open in $X \times Y$. Moreover, $\pi^{-1}(\pi(U)) = U$ and hence $\pi(U) \subset X \wedge Y$ is open. We conclude that $\pi|_Q$ is a homeomorphism onto its image and $X \wedge Y$ is the 1-point compactification of Q.

   (b) The compactification of $(S^m\backslash\{x_0\}) \times (S^n\backslash\{y_0\}) \approx \mathbb{R}^m \times \mathbb{R}^n \approx \mathbb{R}^{m+n}$ is S^{m+n}. It now follows from (a) that $S^m \wedge S^n \approx S^{m+n}$.

5. (a) The map

$$g_n\colon \mathbb{R}^{n+1}\backslash\{0\} \longrightarrow S^n$$
$$\underline{x} \longmapsto \frac{\underline{x}}{||\underline{x}||}.$$

descends to a homeomorphism $\mathbb{R}P^n \to S^n/(\underline{x} \sim -\underline{x})$. The map

$$f_n \colon B^n \longrightarrow \mathbb{R}P^n$$
$$x = (x_1, \ldots, x_n) \longmapsto [x_1, \ldots, x_n, \sqrt{1 - |x|^2}]$$

descends to a homeomorphism $(B^n/\sim) \to \mathbb{R}P^n$, where $x \sim y$ in B^n if and only if $x = -y \in \partial B^n$.

(b) $\mathbb{R}P^0$ is a point and so it's a CW-complex with one 0-cell. View $\mathbb{R}P^n$ as $B^n/\sim$. As such, $\mathbb{R}P^n$ can be obtained as a 2-cell B^n glued to $\partial B^n/(x \sim -x)$ along the boundary via the projection $\partial B^n \to \partial B^n/(x \sim -x)$. Note that

$$\partial B^n/(x \sim -x) \approx S^{n-1}/(\underline{x} \sim -\underline{x}) \approx \mathbb{R}P^{n-1}.$$

Hence $\mathbb{R}P^n$ is obtained by gluing precisely one n-cell to $\mathbb{R}P^{n-1}$. This provides CW-structures as claimed by proceeding inductivly over

$$\mathbb{R}P^0 \subset \mathbb{R}P^0 \cup B^1 \approx \mathbb{R}P^1 \subset \mathbb{R}P^1 \cup B^2 \approx \mathbb{R}P^2 \subset \ldots.$$

The characteristic map for the k-cell a_k is $f_{a_k} := f_k \colon B^k \to \mathbb{R}P^k \subset \mathbb{R}P^n$. Note that f_{a_k} is an embedding on $Int(B^k)$. Moreover, $f_{a_k}(\partial B^k) = \{[x_1, \ldots, x_k, 0] \in \mathbb{R}P^k\} \approx \mathbb{R}P^{k-1} \subset \mathbb{R}P^n$. The attaching map is its restriction to ∂B^k:

$$f_{\partial a_k} \colon \partial B^k \approx S^{k-1} \longrightarrow \mathbb{R}P^{k-1} \subset \mathbb{R}P^n.$$

(c) The cellular chain complex of $\mathbb{R}P^n$ has one copy of $\mathbb{Z}$ in each degree $0 \le k \le n$ and is 0 in all the other degrees. For the k-cell a_k consider the projection

$$p_{a_k} \colon \mathbb{R}P^k \approx \left(B^k/\sim\right) \to \left(B^k/\partial B^k\right) \approx S^k.$$

The differential $d_k \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ in degree $1 \le k \le n$ is given by multiplication with the degree of the map $p_{a_{k-1}} f_{\partial a_k} \colon S^{k-1} \to S^{k-1}$, $1 \le k \le n$. $[0] \in B^{k-1}/\partial B^{k-1} \approx S^{k-1}$ has two preimages under $p_{a_{k-1}} f_{\partial a_k}$: $N = (0, \ldots, 0, 1) \in S^{n-1}$ and $S = (0, \ldots, 0, -1) \in S^{n-1}$. Near N, this map is an orientation-preserving homeomorphism. So the local degree at N is 1. Near S, it is the antipodal map composed with an orientation-preserving homeomorphism. So the local degree near S is $(-1)^k$. Therefore,

$$\deg(p_{a_{k-1}} f_{\partial a_k}) = 1 + (-1)^k = \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even} \end{cases}$$

Suppose n is even. Then the cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{*2} \mathbb{Z} \xrightarrow{0} \ldots \mathbb{Z} \xrightarrow{*2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with non-zero groups exactly in degrees $0, \ldots, n$, and thus we obtain

$$H_k(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 1, 3, \ldots, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

For n being odd, one computes similarly

$$H_k(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, n \\ \mathbb{Z}/2\mathbb{Z}, & k = 1, 3 \ldots, n-2 \\ 0 & \text{otherwise.} \end{cases}$$

An alternative solution can be found in Bredon, Chapter IV. 14.

6. Compactify $\mathbb{R}^2$ and consider the stereographic projection

$$\pi\colon S^2 \to \mathbb{R}^2 \cup \{\infty\}.$$

View the graph G in S^2 by considering $\tilde{G} := \pi^{-1}(G) \subset S^2$. $\tilde{G}$ defines a CW-structure on S^2 with one 0-cell for each vertex of G, one 1-cell for each edge of G and one 2-cell for each face of G.

The Euler characteristic of S^2 therefore is $\xi(S^2) = v - e + f$. On the other hand, $\xi(S^2) = 2$, as can been seen from singular homology. We conclude: $v - e + f = 2$.

7. We view $T^3 = I^3/\sim$ as the quotient space of the cube I^3 under the relation that identifies opposite faces of the boundary. From this description, one sees that T^3 has a CW complex structure with one 0-cell a (any of the corner points—note that these get identified under $I^3 \to T^3$), three 1-cells b_1, b_2, b_3 (the line segments on the coordinate axes), three 2-cells c_1, c_2, c_3 (the squares in the coordinate planes), and one 3-cell d (all of I^3); in all these cases the attaching maps is given by restriction of the quotient map $I^3 \to T^3$.

The corresponding cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

with linear maps ∂_i which we now compute. We have $\partial_1 = 0$ since the attaching maps $f_{b_i} : I \to (T^3)^{(0)} = \{a\}$ take both boundary points $0, 1 \in I$ to the same point (cf. the remark in Bredon after Theorem 10.3). We also have $\partial_2 = 0$, since all maps $p_{b_i} f_{\partial c_j} : \partial I^2 \to S^1$ have degree 0 (by the same argument as for the standard CW complex structure of the 2-torus; see Bredon example 10.5).

As for ∂_3, consider any of the maps $p_{c_i} f_{\partial d} : \partial I^3 \to S^2$. Note that there are two opposite faces of ∂I^3 in whose interiors this map restricts to a homeomorphism, and that the map collapes the rest of ∂I^3 to a point in S^2. The degree of $p_{c_i} f_{\partial d}$ is hence the sum of the two local degrees at any two points q, q' in the two first-mentioned faces which get mapped to the same point in T^3. Now note that the restrictions of $p_{c_i} f_{\partial d}$ to these faces are obtained from one another by precomposing with an orientation-*reversing* map (for orientations induced from an orientation of ∂I^3); therefore the sum of these local degrees vanishes. It follows that also $\partial_3 = 0$.

Summing up, we obtain

$$H_i(T^3) \cong \begin{cases} \mathbb{Z}, & i = 0, 3, \\ \mathbb{Z}^3, & i = 1, 2. \end{cases}$$

8. (a) One possible CW complex structure has two 0-cells a_1, a_2 (the north and south poles), two 1-cells b_1, b_2 (the line segment mentioned in the description of X and another segment on the sphere connecting the poles), and one 2-cell c. We then have

$$\deg(p_{a_2} f_{\partial b_j}) = 1, \quad \deg(p_{a_1} f_{\partial b_j}) = -1$$

for $j = 1, 2$, supposing that the attaching maps $f_{b_j} : I \to X^{(0)}$ are such that both map $0 \in \partial I$ to a_1 and $1 \in \partial I$ to a_2 (cf. the remark in Bredon after Theorem 10.3). Moreover, we have

$$\deg(p_{b_j} f_{\partial c}) = 0$$

for $j = 1, 2$, as both maps $p_{b_j} f_{\partial c}$ are null-homotopic. The cellular chain complex is therefore

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0, \quad \partial_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2.$$

Both the kernel and the cokernel of ∂_1 are 1-dimensional, and therefore

$$H_k(X) \cong \begin{cases} \mathbb{Z}, & k = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that there is an even simpler CW complex structure for X with exactly one k-cell for $k = 0, 1, 2$.)

(b) $X \simeq S^2 \vee S^1$ implies $\widetilde{H}_*(X) = \widetilde{H}_*(S^2 \vee S^1) \cong \widetilde{H}_*(S^2) \oplus \widetilde{H}_*(S^1)$; hence $\widetilde{H}_2(X) = \widetilde{H}_1(X) = \mathbb{Z}$ and $\widetilde{H}_0(X) = 0$, from which the result above follows by the definition of reduced homology.

Alternatively: Excising a neighbourhood of the point joining the two spheres yields $\widetilde{H}_*(X) \cong H_*(D^2, \partial D^2) \oplus H_*(I, \partial I)$ from which the result above again follows easily.

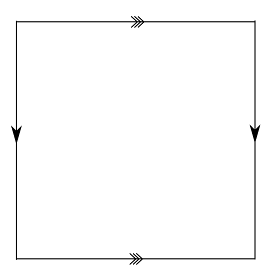
9. We assume wlog that p and q are coprime (otherwise divide by their greatest common divisor), which implies that there exist integers a, b such that $ap - bq = 1$. Hence the matrix

$$\Psi = \begin{pmatrix} a & q \\ b & p \end{pmatrix}$$

lies in $SL(2, \mathbb{Z})$ and therefore induces a homeomorphism $\psi : T^2 \to T^2$ of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Note that $\Psi^{-1} \in SL(2, \mathbb{Z})$ takes the line given by $px = qy$ to the line given by $x = 0$, because Ψ takes $(0, 1)$ to (q, p) (and these vectors generate the two lines). Therefore ψ^{-1} takes C to the curve C' that's the image of $x = 0$ under $\mathbb{R}^2 \to T^2$ and which is the 1-cell of the standard CW complex structure on T^2. Thus T^2/C has a CW complex structure with one cell a_k in dimensions $k = 0, 1, 2$, and the corresponding cellular differential vanishes (by the same reasons as for T^2). Therefore

$$H_k(T^2/C) \cong \begin{cases} \mathbb{Z}, & k = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

10. We view $S^1 \times S^1$ as $I^2/\sim$, the quotient obtained by identifying opposite points on the boundary of ∂I^2 as indicated in the figure below. We endow it with the corresponding obvious CW complex structure with one 0-cell, two 1-cells, and one 2-cell and arrange this to be such that the subspace $S^1 \vee S^1$ that gets collapsed is the union of the two closed 1-cells. Moreover, we equip S^2 with the obvious CW complex structure with one 0-cell and one 2-cell, arranging that the 0-cell is the point to which $S^1 \vee S^1$ gets collapsed.



Our quotient map $g : S^1 \times S^1 \to S^2$ is cellular in this identification. Denoting the 2-cell of $S^1 \times S^1$ by σ and the 2-cell of S^2 by τ, the map $g_\Delta : C_*(S^1 \times S^1) \to C_*(S^2)$ induced by g on cellular chains takes $\sigma \mapsto g_\Delta(\sigma) = \tau$ because $\deg(g_{\tau,\sigma}) = 1$ for the relevant map $g_{\tau,\sigma} : S^2 \to S^2$ (see Bredon chapter IV. 11). The induced map $g_* : H_2(S^1 \times S^1) \times H_2(S^2)$ is hence the identity, and therefore g is not null-homotopic.

Let now $f : S^2 \to S^1 \times S^1$ be a map in the other direction. Consider the covering map $q : \mathbb{R}^2 \to S^1 \times S^1$ (obtained by identifying $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$). As $\pi_1(S^2)$ is trivial, f can be lifted to a map to $\mathbb{R}^2$, i.e., there exists a map $\widetilde{f} : S^2 \to \mathbb{R}^2$ such that $q \circ \widetilde{f} = f$. Since $\mathbb{R}^2$ is contractible, $\widetilde{f}$ is null-homotopic, and hence so is f.

11. As discussed in class, $\mathbb{R}P^n$ has a CW complex structure with exactly one k-cell for every $k = 0, \ldots, n$. Therefore $\mathbb{R}P^n/\mathbb{R}P^m$ has a CW complex structure with one 0-cell a_0 and one k-cell a_k for every $k = m+1, \ldots, n$. As in the case $\mathbb{R}P^n$, we have

$$\deg(p_{a_{k-1}} f_{\partial a_k}) = 1 + (-1)^k \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even}. \end{cases}$$

Thus the cellular chain complex $C_*(\mathbb{R}P^n/\mathbb{R}P^m)$ has one copy of $\mathbb{Z}$ in degrees $k = 0$ and $k = m+1, \ldots, n$, and the cellular differential $C_k(\mathbb{R}P^n/\mathbb{R}P^m) \to C_{k-1}(\mathbb{R}P^n/\mathbb{R}P^m)$ is $1 + (-1)^k$ for all $k = m+2, \ldots, n$ and vanishes in all other cases. The homology is therefore

$$H_k(\mathbb{R}P^n/\mathbb{R}P^m) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}, & k = m+1 \text{ (if } m+1 \text{ is even)}, \\ \mathbb{Z}, & k = n \text{ (if } n \text{ is odd)}, \\ \mathbb{Z}_2, & m+1 \leq k < n \text{ and } k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$