Prof. Paul Biran — Algebraic Topology I — HS 2024
ETH Zürich

Problem set 5

Let X be a pointed space (i.e. a space endowed with a given base point $x_0 \in X$). All the spaces in problems 1 - 4 will be assumed to be pointed Hausdorff spaces that are locally compact. Locally compact means that every point $x \in X$ has a compact neighbourhood. (Namely for every $x \in X$ there exists an open subset $U \subset X$ and a compact subset $K \subset X$ such that $x \in U \subset K$.)

1. Recall the wedge product $X \vee Y = (X \sqcup Y)/(x_0 \sim y_0)$ where $x_0 \in X$ and $y_0 \in Y$, endowed with the new base point $* = [x_0] = [y_0]$. (Compare to Problems 2.5 and 3.5.) Prove that $(X \vee Y, *)$ is homeomorphic to $(W, (x_0, y_0)) \subset (X \times Y, (x_0, y_0))$, where

$$W := (X \times \{y_0\}) \cup (\{x_0\} \times Y),$$

   via an obvious homeomorphism that sends $X \subset X \vee Y$ "identically" to $X \times \{y_0\} \subset W$ and $Y \subset X \vee Y$ "identically" to $\{x_0\} \times Y \subset W$. In the following exercises we will view $X \vee Y$ as the space W defined above.

2. Show that the construction $X \vee Y$ is functorial for pointed spaces and maps that preserve base points, i.e. if X', Y' are pointed spaces and $f\colon X \to X'$ and $g\colon Y \to Y'$ are maps that preserve base points, then we get a map $f \vee g\colon X \vee Y \to X' \vee Y'$ that also preserves base points. This assignment satisfies $id_X \vee id_Y = id_{X \vee Y}$ and $(f' \circ f) \vee (g' \circ g) = (f' \vee g') \circ (f \vee g)$ for maps $X \xrightarrow{f} X' \xrightarrow{f'} X''$ and $Y \xrightarrow{g} Y' \xrightarrow{g'} Y''$.

3. Define the smash product of two pointed spaces X, Y as the pointed space $X \wedge Y = (X \times Y)/(X \vee Y)$ endowed with the base point $*$ corresponding to $X \vee Y$.

   (a) Show that the construction $X \wedge Y$ is functorial (in the analogous sense as for $X \vee Y$ in exercise 2).

   (b) Show that there exists a natural homeomorphism

$$\varphi_{X,Y}\colon (X \wedge Y, *) \to (Y \wedge X, *).$$

   By natural we mean that for all maps $f\colon X \to X', g\colon Y \to Y'$ that preserve base points, the following diagram commutes:

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\varphi_{X,Y}} & Y \wedge X \\ \downarrow{\scriptstyle f\wedge g} & & \downarrow{\scriptstyle g\wedge f} \\ X' \wedge Y' & \xrightarrow{\varphi_{X',Y'}} & Y' \wedge X'. \end{array}$$

(c) Show that there exists a natural homeomorphism

$$(X \wedge Y) \wedge Z \approx X \wedge (Y \wedge Z).$$

Naturality has a similar meaning as above, with respect to three maps $X \to X'$, $Y \to Y'$, $Z \to Z'$.

4. Let X, Y be compact pointed spaces.

   (a) Show that $(X \wedge Y, *) \approx (\hat{Q}, q_0)$, where $Q = (X \backslash \{x_0\}) \times (Y \backslash \{y_0\})$, $\hat{Q}$ is the $1-$point compactification of Q, and $q_0 \in \hat{Q}$ is the point corresponding to infinity.

   (b) Deduce that $S^m \wedge S^n \approx S^{m+n}$.

5. Recall that $\mathbb{R}P^n = \left(\mathbb{R}^{n+1} \backslash \{0\}\right) / \sim$, where $\underline{x} \sim \lambda \underline{x}$ for all $0 \neq \lambda \in \mathbb{R}$.

   (a) Find explicit homeomorphisms between $\mathbb{R}P^n$ and the following two spaces:
   $S^n / \sim$, where $\underline{x} \sim -\underline{x}$ for all $\underline{x} \in S^n$,
   $B^n / \sim$, where $x \sim -x$ for all $x \in \partial B^n$.

   (b) Endow $\mathbb{R}P^n$ with the structure of a CW-complex with precisely one k-cell in each dimension $0 \leq k \leq n$ and no cells in dimension higher than n.

   (c) Calculate the cellular homology of $\mathbb{R}P^n$.

6. Let $G \subset \mathbb{R}^2$ be a finite connected planar graph with v vertices, e edges and f faces. (A face is a region in $\mathbb{R}^2$ that is bounded by edges. The infinitely large region outside of the graph is also a face, called the outer face.) Prove the Euler formula:

$$v - e + f = 2.$$

7. The 3-torus is the quotient space $T^3 = \mathbb{R}^3/\mathbb{Z}^3 \approx S^1 \times S^1 \times S^1$. Find a CW-structure on T^3 and use it to compute $H_*(T^3)$.

8. Consider the space X which is the union of the unit sphere $S^2 \subset \mathbb{R}^3$ and the line segment between the north and south poles.

   (a) Give X a CW-structure and use it to compute $H_*(X)$.

   (b) Use that X is homotopy equivalent to $S^2 \vee S^1$ to give an easier computation of $H_*(X)$.

9. Let C be the circle on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is the image, under the covering map $\mathbb{R}^2 \to T^2$, of the line $px = qy$. Define $X = T^2/C$, the quotient space obtained by identifying C to a point. Compute $H_*(X)$.

10. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1 \subset S^1 \times S^1$ to a point is not null-homotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is null-homotopic.

11. Compute $H_*(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$, using cellular homology and equipping $\mathbb{R}P^n$ with the standard CW-structure with $\mathbb{R}P^m$ as its m-skeleton.