Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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Exam in Algebraic Topology I - Winter 2019

Name:

First Name: **Legi-Nr.:**

Please leave the following spaces blank!

	1. Corr.	2. Corr.	Points	Remarks
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				
Problem 6				
Total				
Grade				
Complete?			☐	

Please read carefully!

- The exam is divided into two parts, **Part A** and **Part B**. Part A consists of four problems (1-4) and part B of two problems (5-6). Each problem is divided into sub-problems.
- For **Part A**: Please choose and solve **three out of the four** problems of Part A. **Only three problems will be graded.** Each problem in part A gives 16 points. You will not get additional points if you solve more than three problems.
- For **Part B**: Please choose and solve only **one out of the two** problems of Part B. **Only one problem will be graded**. Each problem in part B gives 12 points. You will not get additional points if you solve more than one problem.
- In case you hand in too many problems and/or do not clearly indicate which problems you wish to be graded we will only grade the problems that occur first in your work.
- **All answers/statements/counter-examples in your work should be proved.** (It is okay to use theorems/statements proved in class without reproving them.)
- The maximal score of the exam is **60 points** and the duration of the exam is **3 hours**.
- Do not mix sub-problems from different problems.
- The sub-problems of a problem are not necessarily related to each other.
- Please use a separate sheet of paper for each problem (but not sub-problem) and hand in the sheets sorted according to the problem numbers.
- Please do not use red or green pens and do not use pencil.
- Please clearly write your full name on each of the sheets you hand in.
- No auxiliary materials are allowed.

Good Luck!

Part A

1. Let X be a CW-complex of dimension n. Let $c = \sum_{i=1}^{l} n_i \cdot \sigma_i$ be a singular n-dimensional cycle, where $n_i \neq 0$ for all i and $\sigma_i : \Delta^n \to X$. Assume that $0 \neq [c] \in H_n(X)$.
   a) **[10 Points]** Assume that X has exactly one n-dimensional cell K. Prove that $int(K) \subset \bigcup_{i=1}^{l} image(\sigma_i)$.
   b) **[6 Points]** Assume now that X has more than one n-dimensional cell, say $K_1, \ldots, K_r$ for some $r \geq 2$. Is it true that $int(K_j) \subset \bigcup_{i=1}^{l} image(\sigma_i)$ for all $1 \leq j \leq l$? Justify your answer.

2. Consider the topological space M, which is obtained by taking the square in figure 1 and identifying the vertical edges marked by α according to the orientations indicated by the arrows. View M as a CW-complex with two 0-dimensional cells p and q, three 1-dimensional cells α, β and γ and one 2-dimensional cell A. (M is the Möbius strip.)

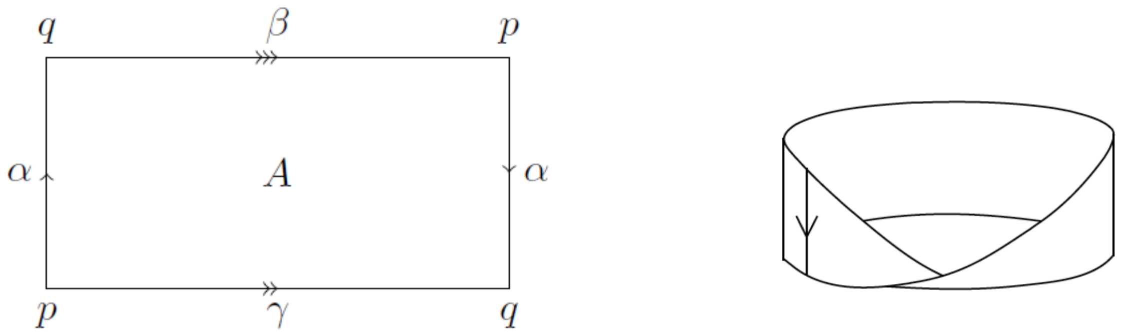

Figure 1: The CW-complex M, which is a Möbius strip.

   a) **[6 Points]** Calculate the cellular homology $H_*^{CW}(M)$ of M.
   b) **[4 Points]** Let $\partial M := \beta \cup \gamma$ be the boundary of M. Describe the map

$$H_1^{CW}(\partial M) \xrightarrow{i_*} H_1^{CW}(M)$$

   induced by the inclusion $i : \partial M \hookrightarrow M$.
   c) **[6 Points]** Calculate the cellular homology $H_*^{CW}(M, \partial M)$ of M relative to its boundary ∂M.

3. Let X be a path-connected topological space. We define the *unreduced suspension* ΣX of X to be the space

$$\Sigma X := X \times I / \{(x,0) \sim (y,0), (x,1) \sim (y,1) \text{ for all } x, y \in X\},$$

i.e. ΣX is obtained from $X \times I$ by collapsing the subspace $X \times \{0\}$ to a point $[X \times \{0\}]$ and $X \times \{1\}$ to another point $[X \times \{1\}]$.

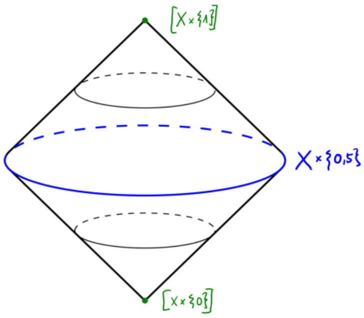

Figure 2: The *unreduced suspension* ΣX.

a) **[4 Points]** Let Y be another path-connected topological space. Explain how a continuous map $f : X \to Y$ induces a continuous map $\Sigma f : \Sigma X \to \Sigma Y$.

b) **[12 Points]** Prove that there exists an isomorphism

$$\tilde{H}_{i+1}(\Sigma X) \xrightarrow{\cong} \tilde{H}_i(X)$$

for every $i \geq 0$, which is natural in the following sense: For every path-connected topological space Y and every continuous map $f : X \to Y$ the diagram

$$\begin{array}{ccc} \tilde{H}_{i+1}(\Sigma X) & \xrightarrow{\cong} & \tilde{H}_i(X) \\ \downarrow{\scriptstyle \Sigma f} & & \downarrow{\scriptstyle f} \\ \tilde{H}_{i+1}(\Sigma Y) & \xrightarrow{\cong} & \tilde{H}_i(Y) \end{array}$$

commutes.

4. a) **[5 points]** Let T be a topological space such that $T = A \cup B$ with A and B open.

   i. Assume now that A, B and $A \cap B$ are acyclic. Prove that also T is acyclic.

   ii. Assume that each set A, B and $A \cap B$ is either acyclic or empty. Prove that $H_n(T) = 0 \quad \forall n \geq 1$.

   Remark: Recall that a space X is acyclic if $\tilde{H}_i(X) = 0$ for every $i \in \mathbb{Z}$. (Note that $X = \emptyset$ is NOT acyclic.)

   b) **[8 Points]** Let X be a topological space and $A, B, C \subset X$ open sets such that $X = A \cup B \cup C$. Assume that each of

   $$A, \quad B, \quad C, \quad A \cap B, \quad B \cap C, \quad A \cap C \quad \text{and} \quad A \cap B \cap C \quad \text{is either acyclic or empty.}$$

   Prove that $H_i(X) = 0 \quad \forall i \geq 2$.

   c) **[3 Points]** Give an example of a space X with open subsets A, B, C as in exercise b) and such $H_1(X) \neq 0$, $H_0(X) \neq 0$ and hence conclude the statement in b) is sharp.

Part B

5. Let $(A_\bullet, d_A)$ and $(B_\bullet, d_B)$ be two chain complexes and $f : A_\bullet \to B_\bullet$ a chain map. Define the mapping cylinder $(Z(f)_\bullet, d_Z)$ by
$$Z(f)_i := A_i \oplus A_{i-1} \oplus B_i$$
with the differential given by
$$d_Z(a', a'', b) = (d_A(a') + a'', -d_A(a''), -f(a'') + d_B(b)) \quad \forall a' \in A_i, a'' \in A_{i-1}, b \in B_i.$$
I.e. in matrix from we can write: $d_Z = \begin{pmatrix} d_A & Id & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix}$

   a) **[3 Points]** Show that $(Z(f)_\bullet, d_Z)$ is a chain complex.

   b) **[5 Points]** Consider the maps
   $$\xi : B_\bullet \to Z(f)_\bullet : b \mapsto (0, 0, b),$$
   $$\eta : Z(f)_\bullet \to B_\bullet : (a', a'', b) \mapsto f(a') + b.$$
   Show that ξ and η are chain maps.

   c) **[4 Points]** Show that ξ is a chain homotopy equivalence between $B_\bullet$ and $Z(f)_\bullet$.

6. Let X be a path-connected topological space and $f : X \to X$ a homeomorphism. Let $\sim$ denote the equivalence relation $(x, 0) \sim (f(x), 1)$. We define the *mapping torus* T_f by
$$T_f := X \times I / \sim .$$
Consider the embedding $i : X \to T_f : x \mapsto [(x, 0)]$ and denote by $X_0 := i(X) \subset T_f$ its image.

   a) **[3 Points]** Consider the quotient map $q : (X \times I, X \times \partial I) \to (T_f, X_0)$. Show that this induces an isomorphism $q_* : H_*(X \times I, X \times \partial I) \to H_*(T_f, X_0)$.

   Hint: Consider the sets $A := (X \times (1-\epsilon, 1]) \cup (X \times [0, \epsilon))/\sim$ and $B := (X \times \{1\}) \cup (X \times \{0\})/\sim$ and use excision.

   b) **[4 Points]** Consider the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{j_*} & H_{n+1}(X \times I, X \times \partial I) & \xrightarrow{\partial_*} & H_n(X \times \partial I) & \xrightarrow{inc_*} & H_n(X \times I) & \xrightarrow{j_*} & H_n(X \times I, X \times \partial I) & \xrightarrow{\partial_*} \cdots \\
& \downarrow q_* \cong & & \downarrow q_* & & \downarrow q_* & & \downarrow q_* \cong & \\
\cdots \xrightarrow{j_*} & H_{n+1}(T_f, X_0) & \xrightarrow{\partial_*} & H_n(X_0) & \xrightarrow{inc_*} & H_n(T_f) & \xrightarrow{j_*} & H_n(T_f, X_0) & \xrightarrow{\partial_*} \cdots ,
\end{array} \tag{1}$$

   where the vertical maps are all induced by the quotient map $q : (X \times I, X \times \partial I) \to (T_f, X_0)$ and the horizontal maps are the standard long exact sequences of a pairs of spaces.

   Show that the map $j_* : H_n(X \times I) \to H_n(X \times I, X \times \partial I)$ in the upper long exact sequence in diagram (1) is zero $\forall n$ and that $H_{n+1}(X \times I, X \times \partial I) \cong H_n(X)$.

   c) **[5 Points]** Use a) and b) to show that there exists a long exact sequence
   $$\cdots \longrightarrow H_n(X) \xrightarrow{Id - f_*} H_n(X) \xrightarrow{i_*} H_n(T_f) \longrightarrow H_{n-1}(X) \xrightarrow{Id - f_*} H_{n-1}(X) \cdots .$$
   Remark: This long exact sequence is called the *Wang sequence*.