Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Paul Biran HS20

Exam in Algebraic Topology I - Winter 2021

Name:

First Name: **Legi-Nr.:**

Please leave the following spaces blank!

	1. Corr.	2. Corr.	Points	Remarks
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				
Problem 6				
Total				
Grade				
Complete?			☐	

Please read carefully!

- The exam is divided into two parts, **Part A** and **Part B**. Part A consists of four problems (1-4) and part B of two problems (5-6). Each problem is divided into sub-problems.
- For **Part A**: Please choose and solve **three out of the four** problems of Part A. **Only three problems will be graded.** Each problem in part A gives 16 points. You will not get additional points if you solve more than three problems.
- For **Part B**: Please choose and solve only **one out of the two** problems of Part B. **Only one problem will be graded**. Each problem in part B gives 12 points. You will not get additional points if you solve more than one problem.
- In case you hand in too many problems and/or do not clearly indicate which problems you wish to be graded we will only grade the problems that occur first in your work.
- **All answers/statements/counter-examples in your work should be proved.** (It is okay to use theorems/statements proved in class without reproving them.)
- The maximal score of the exam is **60 points** and the duration of the exam is **3 hours**.
- Do not mix sub-problems from different problems.
- The sub-problems of a problem are not necessarily related to each other.
- Please use a separate sheet of paper for each problem (but not sub-problem) and hand in the sheets sorted according to the problem numbers.
- Please do not use red or green pens and do not use pencil.
- Please clearly write your full name on each of the sheets you hand in.
- No auxiliary materials are allowed.

Good Luck!

Part A

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1. a) [**4 Points**] Let X be a space and $A \subset X$. Assume there is a retraction $r\colon X \to A$. Show that the inclusions $A \to X$ and $(X, \emptyset) \to (X, A)$ induce short exact sequences

$$0 \to H_k(A) \to H_k(X) \to H_k(X, A) \to 0,$$

for every $k \in \mathbb{Z}$.

   b) [**2 Points**] Does the sequence in part (a) split? Prove your answer.

   c) [**10 Points**] Let B be a space and $b_0 \in B$ a point. Let $I := [0, 1]$ and define $X := B \times I$ and $A := (B \times \{0\}) \cup (\{b_0\} \times I) \cup (B \times \{1\}) \subset X$. Assume that there exists $j > 0$ such that $H_j(B) \neq 0$. Show that there is no retraction $r\colon X \to A$.

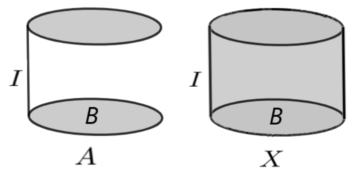


Figure 1: The spaces A and X.

2. For $n \geq 1$ consider an n-dimensional finite CW complex X which is homeomorphic to S^n. Denote by $\mathcal{I}_n$ its n-cells and let $\mathcal{I}'_n \subset \mathcal{I}_n$. Let $a = \sum_{\sigma \in \mathcal{I}'_n} n_\sigma \sigma$, $n_\sigma \in \mathbb{Z}$, be a chain in the cellular chain complex of X. Assume that a is a cycle and $[a] \neq 0 \in H_n^{CW}(X)$.

   a) [**12 Points**] Prove that $\mathcal{I}_n = \mathcal{I}'_n$.
   Hint: Let Z be a space, B^n the closed n-dimensional ball and $f\colon \partial B^n \to Z$ a map. Let

$$Y = Z \cup_f B^n = (Z \sqcup B^n) / (\forall p \in \partial B^n \colon f(p) \sim p)$$

   be the space obtained by attaching B^n to Z along its boundary using the map f. Let $0 \in \mathrm{Int}B^n$ be the origin. Show that $Y \backslash \{0\}$ is homotopy equivalent to Z.

   b) [**4 Points**] Find a counterexample to the statement in (a) above, for every n, in case we only assume that X is homotopy equivalent to S^n.

3. a) [**7 Points**] Let $C_\bullet$ be a bounded chain complex of finitely generated abelian groups. ($C_\bullet$ bounded means that $C_i \neq 0$ for at most finitely many $i \in \mathbb{Z}$.) Prove that

$$\chi(C_\bullet) = \chi(H_*(C_\bullet)).$$

   b) [**2 Points**] Let $0 \to A_N \to A_{N-1} \to \cdots \to A_1 \to A_0 \to 0$ be a long exact sequence of finitely generated abelian groups. Prove that $\chi(A_\bullet) = 0$.

   c) [**7 Points**] Let X be a space of bounded type (i.e. $H_i(X) \neq 0$ for at most finitely many i's and for all i, $H_i(X)$ is finitely generated). Prove that

$$\chi(X \times S^1) = 0.$$

   Hint: Use the Mayer-Vietoris long exact sequence to relate $H_*(X \times S^1)$ to $H_*(X)$.

4. Let X be a space and $f\colon S^n \to X$, $n \geq 1$, a map. Let B^{n+1} be the closed $(n+1)$-dimensional ball. Let X' be the space obtained by attaching B^{n+1} to X along $S^n = \partial B^{n+1}$ via the map f:

$$X' := X \cup_f B^{n+1} = \left(X \sqcup B^{n+1}\right) / (\forall p \in \partial B^{n+1} = S^n \colon f(p) \sim p).$$

a) **[12 Points]** Assume that $f_*\colon H_n(S^n) \to H_n(X)$ is 0. Prove that

$$H_{n+1}(X') \cong H_{n+1}(X) \oplus \mathbb{Z}$$

and

$$H_i(X') \cong H_i(X) \text{ for all } i \neq n+1.$$

b) **[4 Points]** Assume that $f_*\colon H_n(S^n) \to H_n(X)$ is injective. Prove that

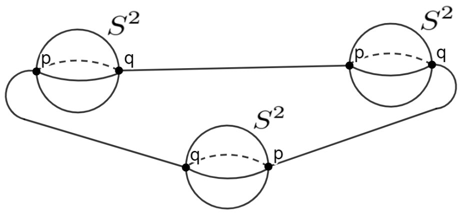
$$H_n(X') \cong H_n(X)/\operatorname{image}(f_*)$$

and

$$H_i(X') \cong H_i(X) \text{ for all } i \neq n.$$

Part B

5. Consider the space X consisting of three disjoint 2-spheres that are joined by three paths as indicated in the picture:



   a) **[2 Points]** Calculate $\chi(X)$.

   b) **[6 Points]** Let Y be a 2-dimensional finite CW complex. Assume that there exists a covering $\pi\colon X \to Y$ and that $\operatorname{rank} H_2(Y) \neq \operatorname{rank} H_1(Y)$. Prove that π is a homeomorphism.

   c) **[4 Points]** Find an example of a covering π which is not 1:1, in case we drop the assumption $\operatorname{rank} H_2(Y) \neq \operatorname{rank} H_1(Y)$.

6. Let $(A_\bullet, d_A)$, $(B_\bullet, d_B)$ be two chain complexes and $f\colon A_\bullet \to B_\bullet$ be a chain map. Define the mapping cylinder $(Z(f)_\bullet, d_Z)$ of f by $Z(f)_i := A_i \oplus A_{i-1} \oplus B_i$ with the differential given by

$$d_Z(a', a'', b) = (d_A(a') + a'', -d_A(a''), -f(a'') + d_B(b)) \ \forall a' \in A_i, a'' \in A_{i-1}, b \in B_i.$$

Define the mapping cone $(K(f)_\bullet, d_K)$ of f by $K(f)_i := A_{i-1} \oplus B_i$ with the differential given by

$$d_K(a'', b) = (-d_A(a''), -f(a'') + d_B(b)) \ \forall a'' \in A_{i-1}, b \in B_i.$$

It is known that the mapping cone and the mapping cylinder are chain complexes, and that they fit into the following short exact sequence of chain complexes

$$0 \to A_\bullet \xrightarrow{j} Z(f)_\bullet \xrightarrow{p} K(f)_\bullet \to 0,$$

where the chain maps j and p are given by $j(a) = (a, 0, 0)$ and $p(a', a'', b) = (a'', b)$.

   a) **[7 Points]** Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ be an exact sequence of chain complexes. Consider the maps

$$\beta\colon Z(f)_\bullet \to B_\bullet, \ (a', a'', b) \mapsto f(a') + b$$

   and

$$\gamma\colon K(f)_\bullet \to C_\bullet, \ (a'', b) \mapsto g(b).$$

   Show that these are chain maps and that the following diagramm commutes:

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "j"] \arrow[d, "id"'] & Z(f)_\bullet \arrow[r, "p"] \arrow[d, "\beta"'] & K(f)_\bullet \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A_\bullet \arrow[r, "f"] & B_\bullet \arrow[r, "g"] & C_\bullet \arrow[r] & 0.
\end{tikzcd}$$

   Show that β is a chain homotopy equivalence by finding a chain homotopy inverse to β.

   b) **[5 Points]** Show that γ induces an isomorphism in homology.