D-MATH — Number Theory I — 23/09/2024

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Exercise Sheet 1

1. For a real number $x \geq 0$, we define

$$N(x) = |\{(a,b) \in \mathbb{Z}^2 \mid a^2 + b^2 \leq x\}|.$$

The goal of this exercise is to prove the estimate

$$N(x) = \pi x + O(x^{1/2}),$$

for $x \geq 1$ (recall that $f(x) = O(g(x))$ for $x \in X$ means that there exists a constant $c \geq 0$ such that $|f(x)| \leq cg(x)$ for all $x \in X$).

1. Show that
$$N(x) \leq \pi(\sqrt{x} + \sqrt{2})^2$$
for all $x \geq 0$. (Hint: interpret $N(x)$ as the area of a certain union of squares of side 1, contained in a suitable disc.)
2. Show that
$$N(x) \geq \pi(\sqrt{x} - \sqrt{2})^2$$
for all $x \geq 0$. (Hint: use a similar idea.)
3. Conclude.

2. We order the primes in increasing order $(p_n)_{n\geq 1}$:

$$2 = p_1 < 3 = p_2 < 5 = p_3 < \cdots .$$

The goal of the exercise is to show that there exist positive constants c_1' and c_2' such that

$$c_1' n \log(n) \leq p_n \leq c_2' n \log(n) \tag{1}$$

for all $n \geq 1$.

1. Show that
$$\lim_{n\to+\infty} \frac{\log(n)}{\log(p_n)} = 1.$$
(Hint: observe first that $\pi(p_n) = n$, and then use Chebychev's estimate.)
2. Using again Chebychev's estimate, prove (1).

3. Prove that

$$\sum_{p \leq x} \frac{1}{p} = +\infty.$$

More precisely, how large can you get the partial sums

$$\sum_{p \leq x} \frac{1}{p}$$

to be as $x \to +\infty$?

3. We define the von Mangoldt function $\Lambda(n)$ for integers $n \geq 1$ by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that for $n \geq 1$, we have

$$\sum_{k=1}^{n} \Lambda(k) = O(n).$$

(Hint: split the sum into the sums over primes, squares of primes, etc, and use Chebychev's estimate.)

2. Show that for any integer $n \geq 1$, we have

$$\sum_{k=1}^{n} \log(k) = \sum_{\substack{p,j \\ p^j \leq n}} \left\lfloor \frac{n}{p^j} \right\rfloor \log(p) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \Lambda(k).$$

3. Show that

$$\sum_{k=1}^{n} \log(k) = n \log(n) + O(n)$$

for $n \geq 1$. (Hint: compare the sum with an integral.)

4. Deduce that

$$\sum_{k=1}^{n} \frac{\Lambda(k)}{k} = \log(n) + O(1)$$

for $n \geq 1$, and that

$$\sum_{p \leq n} \frac{\log(p)}{p} = \log(n) + O(1).$$

(Hint: for the first, combine (1), (2) and (3) and $0 \leq x - \lfloor x \rfloor \leq 1$; for the second, show that the contribution to the first sum of squares and higher powers of primes is bounded.)

4. (Optional but recommended) Using a computer, make a table of sums of three and four squares, and try to make suitable guesses or conjectures concerning the numbers that appear, which ones don't, and how many times an integer n might be a sum of three or four squares.

Due date: 30/09/2024